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Hanner type inequalities and duality

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We shall first discuss two kinds of Hanner type inequalities with a weight in a Banach space X in connection with sharp uniform smoothness and convexity: the first kind of inequalities will characterize the 2-uniform smoothness and 2-uniform convexity of X, and the other the p-uniform smoothness and q-uniform convexity of X. Next we shall present a duality theorem on a "general" Hanner type inequality with "several weights", which is valid for both kinds of the above inequalities. Finally the best value of the weight constant in these inequalities for L_p-spaces will be determined.

Let X be a Banach space and X^* its dual space. Let S_X be the unit sphere of X. Let $1 \leq p, q, r, ... \leq \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1/r + 1/r' = ... = 1$.

1. Hanner's inequalities for L_p (Hanner [3], 1956)

(i) If $1 < p \leq 2$, for all f, g in L_p

$$\|f+g\|_p^p + \|f-g\|_p^p \geq \Big|\|f\|_p + \|g\|_p\Big|^p + \Big|\|f\|_p - \|g\|_p\Big|^p \tag{H1}$$

(ii) If $2 \leq p < \infty$, for all f, g in L_p

$$\|f+g\|_p^p + \|f-g\|_p^p \leq \Big|\|f\|_p + \|g\|_p\Big|^p + \Big|\|f\|_p - \|g\|_p\Big|^p \tag{H2}$$

2. Definition (i) The modulus of convexity of X:

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in S_X, \|x-y\| = \varepsilon\right\} \quad \text{for } 0 \leq \varepsilon \leq 2.$$

(ii) X is uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$.

(iii) X is q-uniformly convex ($2 \leq q < \infty$) if there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for all $\varepsilon > 0$.

3. Remark (i) If $1 \leq q < 2$ no Banach space is q-uniformly convex (cf. [2]; for a proof see e.g., [11, esp., p. 268]).

(ii) Let $2 \leq q \leq q_1 < \infty$. Then if X is q-uniformly convex, X is q_1-uniformly convex.

(iii) L_q ($2 \leq q < \infty$) is q-uniformly convex (by Clarkson's inequality of (q,q)-type).

(iv) L_p ($1 < p \leq 2$) is p'-uniformly convex ($p' \geq 2$) (by Clarkson's inequality of (p,p')-type). But in fact, L_p ($1 < p \leq 2$) is 2-uniformly convex by Hanner's inequality (H1).

For convenience of the reader we see (iii) and the latter statement of (iv) in the general Banach space setting.

Proof of (iii). Let $2 \leq q < \infty$. Assume that Clarkson's inequality of (q,q)-type holds in X:

$$(\|x+y\|^q + \|x-y\|^q)^{1/q} \leq 2^{1/q'}(\|x\|^q + \|y\|^q)^{1/q}.$$

Let $x, y \in S_X$ and $\|x-y\| = \epsilon$. Then

$$\|x+y\|^q + \epsilon^q \leq 2^{q/q'}2 = 2^{q(1/q'+1/q)} = 2^q,$$

whence

$$\left\|\frac{x+y}{2}\right\|^q + \left(\frac{\epsilon}{2}\right)^q \leq 1.$$

Therefore

$$\left(\frac{\epsilon}{2}\right)^q \leq 1 - \left\|\frac{x+y}{2}\right\|^q \leq q\left(1 - \left\|\frac{x+y}{2}\right\|\right).$$

Consequently we have

$$1 - \left\|\frac{x+y}{2}\right\| \geq \frac{1}{q}\left(\frac{\epsilon}{2}\right)^q,$$

from which it follows that

$$\delta_X(\epsilon) \geq \frac{1}{q2^q}\epsilon^q,$$

or X is q-uniformly convex.

Proof of the latter assertion of (iv). Let $1 < p \leq 2$. We have to show the following: If Hanner's inequality (H1),

$$\|x+y\|^p + \|x-y\|^p \geq \Big|\|x\| + \|y\|\Big|^p + \Big|\|x\| - \|y\|\Big|^p,$$

holds in X, then X is 2-uniformly convex. Assume (H1). Then

$$\begin{aligned}\left(\frac{\|x+y\|^2+\|x-y\|^2}{2}\right)^{1/2} &\geq \left(\frac{\|x+y\|^p+\|x-y\|^p}{2}\right)^{1/p} \\ &\geq \left(\frac{\left|\|x\|+\|y\|\right|^p+\left|\|x\|-\|y\|\right|^p}{2}\right)^{1/p} \\ &\geq \left(\frac{\left|\|x\|+\gamma\|y\|\right|^2+\left|\|x\|-\gamma\|y\|\right|^2}{2}\right)^{1/2},\end{aligned}$$

where $\gamma = \sqrt{(p-1)/(2-1)} = \sqrt{p-1}$ ([6, Corollary 1.e.15]). Therefore

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &\geq \left|\|x\|+\gamma\|y\|\right|^2+\left|\|x\|-\gamma\|y\|\right|^2 \\ &= 2[\|x\|^2+\gamma^2\|y\|^2].\end{aligned}$$

Put here $x+y=u, x-y=v$. Then

$$\|u\|^2+\|v\|^2 \geq 2\left[\left\|\frac{u+v}{2}\right\|^2+(p-1)\left\|\frac{u-v}{2}\right\|^2\right].$$

Now let $u, v \in S_X$ and $\|u-v\| = \epsilon$. Then

$$2 \geq 2\left[\left\|\frac{u+v}{2}\right\|^2+(p-1)\left(\frac{\epsilon}{2}\right)^2\right],$$

whence

$$(p-1)\left(\frac{\epsilon}{2}\right)^2 \leq 1-\left\|\frac{u+v}{2}\right\|^2 \leq 2\left(1-\left\|\frac{u+v}{2}\right\|\right).$$

Therefore

$$\frac{p-1}{8}\epsilon^2 \leq 1-\left\|\frac{u+v}{2}\right\|.$$

Consequently we have

$$\delta_X(\epsilon) \geq \frac{p-1}{8}\epsilon^2,$$

or X is 2-uniformly convex, as is desired.

4. Definition (i) The modulus of smoothness of X is defined by

$$\rho_X(\tau) := \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1 : x, y \in S_X\right\} \quad \text{for } \tau > 0$$

(ii) X is uniformly smooth if $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$.

(iii) X is p-uniformly smooth $(1 < p \le 2)$ if there exists $K > 0$ such that $\rho_X(\tau) \le K\tau^p$ for all $\tau > 0$.

5. Remark (i) No Banach space is p-uniformly smooth for $2 < p < \infty$.

(ii) Let $1 < p_1 \le p \le 2$. Then if X is p-uniformly smooth, X is p_1-uniformly smooth.

(iii) L_p $(1 < p \le 2)$ is p-uniformly smooth.

(iv) L_q $(2 \le q < \infty)$ is 2-uniformly smooth.

The first kind of Hanner type inequalities

6. Theorem (Yamada-Takahashi-Kato [13]) Let $1 < p, s, t < \infty$. Then the following are equivalent.

(i) X is 2-uniformly convex.

(ii) There exists $\gamma > 0$ for which

$$\|x+y\|^p + \|x-y\|^p \ge \Big|\|x\| + \|\gamma y\|\Big|^p + \Big|\|x\| - \|\gamma y\|\Big|^p \tag{1}$$

holds in X.

(iii) There exists $\gamma > 0$ for which

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \ge \left(\frac{\Big|\|x\| + \|\gamma y\|\Big|^t + \Big|\|x\| - \|\gamma y\|\Big|^t}{2}\right)^{1/t} \tag{2}$$

holds in X.

According to Remark 3 (iv) the Hanner type inequalities (1) and (2) hold in L_r, $1 < r \le 2$.

7. Theorem ([13]) Let $1 < p, s, t < \infty$. Then the following are equivalent.

(i) X is 2-uniformly smooth.

(ii) There exists $\gamma > 0$ for which

$$\|x+y\|^p + \|x-y\|^p \le \Big|\|x\| + \|\gamma y\|\Big|^p + \Big|\|x\| - \|\gamma y\|\Big|^p \tag{3}$$

holds in X.

(iii) There exists $\gamma > 0$ for which

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \le \left(\frac{\Big|\|x\| + \|\gamma y\|\Big|^t + \Big|\|x\| - \|\gamma y\|\Big|^t}{2}\right)^{1/t} \tag{4}$$

holds in X.

The above Hanner type inequalities (3) and (4) hold in L_r, $2 \leq r < \infty$.

The second kind of Hanner type inequalities

8. Theorem ([13]) Let $2 \leq q < \infty$, $1 \leq t \leq q$. Then the following are equivalent.
(i) X is q-uniformly convex.
(ii) There exists $\gamma > 0$ such that

$$\left(\|x+y\|^q + \|\gamma(x-y)\|^q\right)^{1/q} \leq \left(\Big|\|x\| + \|y\|\Big|^t + \Big|\|x\| - \|y\|\Big|^t\right)^{1/t} \tag{5}$$

for all $x, y \in X$.

The Hanner type inequality (5) holds in L_q $(2 \leq q < \infty)$.

9. Theorem ([13]) Let $1 < p \leq 2$ and $p \leq s \leq \infty$. Then the following are equivalent.
(i) X is p-uniformly smooth.
(ii) There exists $\gamma > 0$ such that

$$\left(\|x+y\|^p + \|\gamma(x-y)\|^p\right)^{1/p} \geq \left(\Big|\|x\| + \|y\|\Big|^s + \Big|\|x\| - \|y\|\Big|^s\right)^{1/s} \tag{6}$$

for all $x, y \in X$.

The Hanner type inequality (6) holds in L_p $(1 < p \leq 2)$.

Duality between Hanner type inequalities

According to Ball-Carlen-Lieb [1] Hanner's inequalities (H1) and (H2) are equivalent. This is extended as follows.

10. Theorem ([13]) Let $1 < s, t < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1$ and $\alpha, \beta, \gamma > 0$. Then the following are equivalent.
(i) For all $x, y \in X$

$$\left(\|\alpha(x+y)\|^s + \|\beta(x-y)\|^s\right)^{1/s} \geq \left(\Big|\|x\| + \|\gamma y\|\Big| + \Big|\|x\| - \|\gamma y\|\Big|^t\right)^{1/t} \tag{7}$$

(ii) For all $x^*, y^* \in X^*$

$$\left(\|\alpha^{-1}(x^*+y^*)\|^{s'} + \|\beta^{-1}(x^*-y^*)\|^{s'}\right)^{1/s'} \le \left(\left|\|x^*\| + \|\gamma^{-1}y^*\|\right|^{t'} + \left|\|x^*\| - \|\gamma^{-1}y^*\|\right|^{t'}\right)^{1/t'} \tag{8}$$

11. Corollary Let $1 < s, t, p < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1/p + 1/q = 1$ and $\gamma > 0$.

(i) The inequality

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \ge \left(\frac{\left|\|x\| + \|\gamma y\|\right|^t + \left|\|x\| - \|\gamma y\|\right|^t}{2}\right)^{1/t} \tag{2}$$

holds in X if and only if

$$\left(\frac{\|x^*+y^*\|^{s'} + \|x^*-y^*\|^{s'}}{2}\right)^{1/s'} \le \left(\frac{\left|\|x^*\| + \|\gamma^{-1}y^*\|\right|^{t'} + \left|\|x^*\| - \|\gamma^{-1}y^*\|\right|^{t'}}{2}\right)^{1/t'} \tag{4*}$$

holds in X^*.

(ii) The inequality

$$\|x+y\|^p + \|x-y\|^p \ge \left|\|x\| + \|\gamma y\|\right|^p + \left|\|x\| - \|\gamma y\|\right|^p \tag{1}$$

holds in X if and only if

$$\|x^*+y^*\|^q + \|x^*-y^*\|^q \le \left|\|x^*\| + \|\gamma^{-1}y^*\|\right|^q + \left|\|x^*\| - \|\gamma^{-1}y^*\|\right|^q \tag{3*}$$

holds in X^*.

The best value of the weights for L_p-spaces

12. Theorem ([13]) Let $1 < p \le 2$ and $1 < s, t < \infty$. Then the Hanner type inequality (2) holds in L_p:

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \ge \left(\frac{\left|\|x\| + \|\gamma y\right|^t + \left|\|x\| - \|\gamma y\|\right|^t}{2}\right)^{1/t}$$

The best value of γ is

$$\gamma = \min\left\{1, \sqrt{\frac{p-1}{t-1}}, \sqrt{\frac{s-1}{t-1}}\right\}$$

13. Theorem ([13]) Let $2 \leq p < \infty$ and $1 < s, t < \infty$. Then the Hanner type inequality (4) holds in L_p:

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq \left(\frac{\left|\|x\| + \|\gamma y\|\right|^t + \left|\|x\| - \|\gamma y\|\right|^t}{2}\right)^{1/t}$$

The best value of γ is

$$\gamma = \max\left\{1, \sqrt{\frac{p-1}{t-1}}, \sqrt{\frac{s-1}{t-1}}\right\}$$

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