数理解析研究所講究録
1484巻 2006年 105-113

Viscosity approximation methods for countable families of nonexpansive mappings in a Hilbert space

Misako Kikkawa（吉川美佐子）and Wataru Takahashi（髙橋渉）
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
（東京工業大学大学院　数理・計算科学専攻）

1 Introduction

Let H be a Hilbert space and let C be a closed convex subset of H. Then a mapping T from C into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

For a mapping T of C into itself, we denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. Let f be a function of C into itself. Then, f is said to be a-contractive on C if there exists a constant $a \in (0,1)$ such that $\|f(x)-f(y)\| \leq a\|x-y\|$ for all $x, y \in C$. In 1967, Browder [2] obtained the following:

Theorem 1 (Browder [2]) Let H be a Hilbert space and let C be a closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let x_0 be an arbitrary point of C and define $S_n : C \to C$ by

$$S_n x = (1 - \alpha_n)Tx + \alpha_n x_0$$

for all $x \in C$ and $n \in \mathbb{N}$, where $0 < \alpha_n < 1$. Then the following hold:
(i) S_n has a unique fixed point $u_n \in C$;
(ii) if $\alpha_n \to 0$, then the sequence $\{u_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection onto $F(T)$.

After Browder's result, such a problem has been investigated by many authors: see Takahashi and Kim [9]. In 2000, Moudafi [4] proved the following strong convergence theorem:

Theorem 2 (Moudafi [4]) Let H be a Hilbert space and let C be a closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty and let f be a-contractive of C into itself. Let

$$x_n = \frac{1}{1+\epsilon_n} T x_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), \tag{1}$$

where $\{\epsilon_n\}$ is a sequence in $(0,1)$ and $\epsilon_n \to 0$. Then $\{x_n\}$ converges strongly to the unique solution $\hat{x} \in C$ of the variational inequality

$$\hat{x} \in F(T) \text{ such that } \langle (I-f)\hat{x}, \hat{x}-x \rangle \le 0, \quad \forall x \in F(T),$$

i.e., $\hat{x} = P_{F(T)} f(\hat{x})$.

Further, in 2004, Xu [12] extended Moudafi's result in the framework of a Hilbert space to that in a uniformly smooth Banach space.

In this paper, motivated by Moudafi's result, we introduce a sequence for finding a common fixed point of a countable family of nonexpansive mappings in a Hilbert space and prove a strong convergence theorem (Theorem 5) which is a generalization of Browder's theorem.

In chapter 4, using the viscosity approximation method and Theorem 5, we study the problem of find a solution to the equation

$$0 \in Au,$$

where $A \subset H \times H$ is a maximal monotone operator.

2 Preliminaries and Lemmas

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let N be the set of all positive integers. It is known that a Hilbert space H satisfies Opial's condition [5], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for every $y \in H$ with $y \neq x$, where $\rightharpoonup$ denotes the weak convergence. Let C be a nonempty closed convex subset of H. We denote by $P_C(\cdot)$ the metric projection of H onto C. It is known that for $z \in C, z = P_C(x)$ is equivalent to $\langle z - y, x - z\rangle \geq 0$ for every $y \in C$. So, we have $\|x - P_Cx\|^2 \leq \|x - y\|^2 - \|P_Cx - y\|^2$ for every $y \in C$. See [8] for more details.

The function $f : H \to (-\infty, \infty]$ is said to be proper, if $D(f) = \{x \in H : f(x) \in \mathbb{R}\}$ is nonempty. For a proper lower semicontinuous convex function $f : H \to (-\infty, \infty]$, the subdifferential $\partial f(x)$ of f at $x \in H$ is defined by

$$\partial f(x) = \{z \in H : f(x) + \langle y - x, z\rangle \leq f(y), \quad \forall y \in H\}.$$

We know that $\partial f \subset H \times H$ is a monotone operator, that is,

$$\langle x - y, z - w\rangle \geq 0$$

whenever $(x, z), (y, w) \in \partial f$. A monotone operator $A \subset H \times H$ is said to be maximal if the graph of A is not properly containd in the graph of any other monotone operator. We also know that the monotone operator ∂f is maximal. An operator $B : H \to H$ is said to be a strongly monotone if there exists $c > 0$ such that $\langle Bx - By, x - y\rangle \geq c\|x - y\|^2$ for all $x, y \in H$. If A is a maximal monotone operator, then we can define, for any $r > 0$, a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_rx \in AJ_rx$ for all $x \in R(I + rA)$ and $\|A_rx\| \leq \inf\{\|y\| : y \in Ax\}$, for all $x \in D(A) \cap R(I + rA)$. We also know that for a maximal monotone operator A, we have $A^{-1}0 = F(J_r)$ for all $r > 0$.

Let $T_1, T_2, \ldots$ be a infinite family of mappings of C into itself and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, Takahashi [7] (see also [6], [10] and [3]) defined a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I,
\end{aligned}$$

$$\vdots$$

$$U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I,$$
$$W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I.$$

Such a mapping W_n is called the *W-mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Using [6] and [1], we obtain the following two lemmas.

Lemma 3 Let C be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for any $i = 2, 3, \ldots$. Then for every $x \in C$ and $k \in \mathbb{N}$, the $\lim_{n\to\infty} U_{n,k}x$ exists.

Using Lemma 3, for $k \in \mathbb{N}$, we define mappings $U_{\infty,k}$ and U of C into itself as follows:

$$U_{\infty,k}x = \lim_{n\to\infty} U_{n,k}x$$

and

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a U is called the *W-mapping* generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$.

Lemma 4 Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for any $i = 2, 3, \ldots$. Let $W_n (n = 1, 2, \ldots)$ be the *W-mappings* of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ and let U be the *W-mapping* generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Then $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ and $F(U) = \bigcap_{i=1}^{\infty} F(T_i)$.

3 Strong convergence theorem

Next we prove the following strong convergene theorem which generalizes Browder's convergence theorem.

Theorem 5 Let H be a Hilbert space. Let C be a closed convex subset of H and let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let f be an a-contractive mapping of C into itself. Let b be a real number with $0 < b < 1$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_1 \leq 1$ and $0 < \lambda_i \leq b < 1$ for every $i = 2, 3, \ldots$. Let $W_n (n = 1, 2, \ldots)$ be *W-mappings* of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. Let U be the *W-mapping* generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, i.e.,

$$Ux = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x$$

for every $x \in C$. Define $S_n : C \to C$ by

$$S_n x = (1 - \alpha_n) W_n x + \alpha_n f(x)$$

for each $x \in C$ and $n = 1, 2, 3, \ldots$. Then the following hold:
(i) S_n has a unique fixed point u_n in C;
(ii) if $\alpha_n \to 0$, then the sequence $\{u_n\}$ converges strongly to $u = P_{F(U)} f(u)$, where $P_{F(U)}$ is the metric projection onto $F(U)$.

Proof. From Lemma 4, we obtain $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(W_n) = F(U)$.
(i) Let $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \|S_n x - S_n y\| &\leq (1 - \alpha_n)\|W_n x - W_n y\| + \alpha_n \|f(x) - f(y)\| \\ &\leq (1 - \alpha_n)\|x - y\| + a\alpha_n \|x - y\| \\ &= (1 - \alpha_n(1 - a))\|x - y\|. \end{aligned}$$

Then, since S_n is a contraction of C into itself, there exists a unique fixed point u_n of S_n in C.
(ii) Let $z \in F(U)$. Since

$$\begin{aligned} \|u_n - z\| &= \|(1 - \alpha_n)(W_n u_n - z) + \alpha_n (f(u_n) - z)\| \\ &\leq (1 - \alpha_n)\|u_n - z\| + \alpha_n \|f(u_n) - z\| \\ &\leq (1 - \alpha_n)\|u_n - z\| + \alpha_n \{\|f(u_n) - f(z)\| + \|f(z) - z\|\} \\ &\leq (1 - \alpha_n)\|u_n - z\| + a\alpha_n \|u_n - z\| + \alpha_n \|f(z) - z\|, \end{aligned}$$

we have

$$\|u_n - z\| \leq \frac{1}{1 - a} \|f(z) - z\|.$$

Therefore, we obtain $\{u_n\}, \{W_n u_n\}$ and $\{f(u_n)\}$ are bounded. From the definition of u_n, we have

$$\begin{aligned}\|u_n - W_n u_n\| &= \|(1-\alpha_n)W_n u_n + \alpha_n f(u_n) - W_n u_n\| \\ &= \alpha_n \|W_n u_n - f(u_n)\| \\ &\leq \alpha_n \cdot K,\end{aligned}$$

where $K = 2\sup_{x\in C}\|x\|$. Hence we obtain

$$\lim_{n\to\infty} \|u_n - W_n u_n\| = 0. \tag{2}$$

Since $\{u_n\}$ is bounded, we assume that there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $\{u_{n_i}\}$ converges weakly to u. Suppose that $u \neq Uu$. Then, from Opial's theorem, (2) and $\lim_{n\to\infty}\|W_n u - Uu\| = 0$, we have

$$\begin{aligned}&\liminf_{i\to\infty} \|u_{n_i} - u\| \\ &\quad < \liminf_{i\to\infty} \|u_{n_i} - Uu\| \\ &\quad \leq \liminf_{i\to\infty} \{\|u_{n_i} - W_{n_i}u_{n_i}\| + \|W_{n_i}u_{n_i} - W_{n_i}u\| + \|W_{n_i}u - Uu\|\} \\ &\quad \leq \liminf_{i\to\infty} \{\|u_{n_i} - W_{n_i}u_{n_i}\| + \|u_{n_i} - u\| + \|W_{n_i}u - Uu\|\} \\ &\quad = \liminf_{i\to\infty} \|u_{n_i} - u\|.\end{aligned}$$

This is a contradiction. Hence we have $Uu = u$.
Next, we prove $u_{n_i} \to u = P_{F(U)}f(u)$. For each i, we have

$$\alpha_{n_i} f(u_{n_i}) = \alpha_{n_i} u_{n_i} + (1-\alpha_{n_i})(u_{n_i} - W_{n_i}u_{n_i}).$$

Since u is a fixed point of W_{n_i}, we also have

$$\alpha_{n_i} u = \alpha_{n_i} u + (1-\alpha_{n_i})(u - W_{n_i}u).$$

If we substract these two equations and take the inner product of that difference with $u_{n_i} - u$, we obtain

$$\begin{aligned}&(1-\alpha_{n_i})\langle (I-W_{n_i})u_{n_i} - (I-W_{n_i})u, u_{n_i} - u\rangle + \alpha_{n_i}\langle u_{n_i} - u, u_{n_i} - u\rangle \\ &= \alpha_{n_i}\langle f(u_{n_i}) - u, u_{n_i} - u\rangle,\end{aligned}$$

where I is the identity. From $\langle (I-W_{n_i})u_{n_i} - (I-W_{n_i})u, u_{n_i} - u\rangle \geq 0$, we have

$$\|u_{n_i} - u\|^2 \leq \langle f(u_{n_i}) - u, u_{n_i} - u\rangle.$$

Since $\{u_{n_i}\}$ converges weakly to u and

$$\begin{aligned}\|u_{n_i}-u\|^2 &\le \langle f(u_{n_i})-u, u_{n_i}-u\rangle \\ &= \langle f(u_{n_i})-f(u), u_{n_i}-u\rangle + \langle f(u)-u, u_{n_i}-u\rangle \\ &\le a\|u_{n_i}-u\|^2 + \langle f(u)-u, u_{n_i}-u\rangle,\end{aligned}$$

we obtain that $\{u_{n_i}\}$ converges strongly to u. Finally, we show that $\{u_n\}$ converges strongly to u, where $u = P_{F(U)}u$. Since $u_n = (1-\alpha_n)W_n u_n + \alpha_n f(u_n)$, we have

$$(I-f)u_n = -\frac{1-\alpha_n}{\alpha_n}(I-W_n)u_n.$$

Thus, for any $z \in F(U)$, we obtain

$$\begin{aligned}\langle (I-f)u_n, u_n - z\rangle &= -\frac{1-\alpha_n}{\alpha_n}\langle (I-W_n)u_n, u_n-z\rangle \\ &= -\frac{1-\alpha_n}{\alpha_n}\langle (I-W_n)u_n-(I-W_n)z, u_n-z\rangle \\ &\le 0,\end{aligned}$$

and hence $\langle (I-f)u_{n_i}, u_{n_i}-z\rangle \le 0$. Taking the limit, we have

$$\langle (I-f)u, u-z\rangle \le 0$$

for all $z \in F(U)$. This implies $u = P_{F(U)}u$. We assume that $u_{n_k} \to \hat{u}$. Since $\hat{u} \in F(U)$, we have

$$\langle (I-f)u, u-\hat{u}\rangle \le 0.$$

Further we also obtain

$$\langle (I-f)\hat{u}, \hat{u}-u\rangle \le 0.$$

Summing up two inequalities yields

$$\langle (I-f)u-(I-f)\hat{u}, u-\hat{u}\rangle \le 0$$

and hence

$$\|u-\hat{u}\|^2 \le \langle fu - f\hat{u}, u-\hat{u}\rangle \le a\|u-\hat{u}\|^2.$$

This implies that $u = \hat{u}$. So, we obtain that $u_n \to u = P_{F(U)}u$.

4 Applications

Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Next, we consider the problem of finding a point $v \in E$ such that $0 \in Av$, using the viscosity approximation method. For the viscosity approximation method, for instance, see Tikhonov [11]. The abstract setting of the viscosity method is as follows: Let H be a Hilbert space and let $f : H \to (-\infty, \infty]$ be a real-valued function. Let us consider the minimization problem

$$\min\{f(x); x \in H\}. \tag{3}$$

Let $g : H \to [0, \infty]$ be a viscosity function and for any $\epsilon > 0$, consider the approximate minimization problem

$$\min\{f(x) + \epsilon g(x); x \in H\}. \tag{4}$$

The viscosity function g usually has assumptions like strict convexity, continuity and coerciveness with respect to the norm and plays an important role in the existence and uniqueness of the solution sequence $\{u_\epsilon\}$ of (4).

Motivated by this method, we can prove the following theorem:

Theorem 6 Let H be a Hilbert space. Let $A \subset H \times H$ be a maximal monotone operator and let $B \subset H \times H$ be a maximal monotone operator which is strongly monotone with modulus γ.

For $r > 0$, let x_r be an element of H such that

$$0 = A_r(x_r) + rB_r(x_r), \tag{5}$$

where $A_r = \frac{1}{r}(I - J_r^A)$, $B_r = \frac{1}{r}(I - J_r^B)$. Then $\{x_r\} \to \hat{x}$ as $r \to 0$, where $\hat{x} = J_r^A(\hat{x})$.

Proof. The viscosity method (5) can be rewritten as

$$x_r = \frac{1}{1+r} J_r^A x_r + \frac{r}{1+r} J_r^B x_r.$$

Since J_r^A is a nonexpansive mapping and J_r^B is $\frac{1}{1+r\gamma}$-contractive, by Theorem 5, we obtain $x_r \to \hat{x} \in F(J_r^A)$.

References

[1] S. Atsushiba and W. Takahashi, *Strong convergence theorems for a finite family of nonexpansive mappings and applications*, Indian J. Math., 41(1999), 435–453.

[2] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach space*, Archs. Ratio. Mech. Anal., 24(1967), 82–90.

[3] Y. Kimura and W. Takahashi, *Weak convergence to common fixed points of countable nonexpansive mappings and its applications*, J. Korean Math. Soc., 38(2001), 1275–1284.

[4] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., 241(2000), 46–55.

[5] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., 73(1967), 591–597.

[6] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpancive mappings and applications*, Taiwanese J. Math., 5(2001), 387–404.

[7] W. Takahashi, *Weak and strong convergence theorems for families of nonexpansive mappings and their apprications*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 51(1997), 277–292.

[8] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[9] W. Takahashi and G. E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonself-mappings in Banach spaces*, Nonlinear Analysis., 32, (1998), 447–454.

[10] W. Takahashi and K. Shimoji, *Convergence theorems for nonexpansive mappings and feasibility problems*, Math. Comput. Modelling, 32(2000), 1463–1471.

[11] A. N. Tikhonov, *Solution of incorrectly formulated problems and the regularization method*, Soviet Math. Dokl., 4(1963) 1035–1038.

[12] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings* , J. Math. Anal. Appl., 298(2004) 279–291.