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A SURVEY ON FIXED POINT THEOREMS IN GENERALIZED CONVEX SPACES

SEHIE PARK

ABSTRACT. We review some fixed point theorems which have appeared in our previous works [P1-11] on generalized convex spaces.

The concept of generalized convex spaces is a common generalization of various abstract convexities with or without linear structure and includes those of convex subsets of topological vector spaces, convex spaces of Lassonde, C-spaces due to Horvath, and many others. In the present paper, we review some fixed point theorems which have appeared mainly in our previous works [P1-11] on generalized convex spaces. Most of them are generalizations of well-known corresponding ones for topological vector spaces (t.v.s.).

1. Generalized convex spaces

A *generalized convex space* or a *G-convex space* $(Y, D; \Gamma)$ consists of a topological space Y, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap Y$ such that for each $A \in \langle D\rangle$ with cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where $\langle D\rangle$ is the class of all nonempty finite subsets of D, Δ_n denotes the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$.

We may write $\Gamma_A = \Gamma(A)$ and it is possible to assume $\Gamma_A = \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$. A G-convex space $(X, D; \Gamma)$ with $X \supset D$ is denoted by $(X \supset D; \Gamma)$ and $(X; \Gamma) := (X, X; \Gamma)$. For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$. For details on G-convex spaces and examples, see [P1,4,5, PK1-6], where basic theory was extensively developed.

A G-convex space $(X, D; \Gamma)$ is called a C-*space* if each Γ_A is contractible (or more generally, n-connected for all $n \geq 0$) and, for each $A, B \in \langle D\rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. For $X = D$, this concept reduces to the one due to Horvath [H1,2].

We give here only a few examples of G-convex spaces:

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Examples 1. [PM] Let $X = D = [0,1)$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi it} : t \in [0,1)\}$ in the complex plane $\mathbb{C}$. Let $f : X \to Y$ be a continuous function defined by $f(t) = e^{2\pi it}$. Define $\Gamma : \langle D' \rangle \multimap Y$ by

$$\Gamma_A = f(\mathrm{co}(f^{-1}(A))) \quad \text{for} \quad A \in \langle D' \rangle.$$

Then $(Y \supset D'; \Gamma)$ is a compact G-convex space. (More generally, it is known that any continuous image of a G-convex space is a G-convex space.) We note the following:

(1) $\mathbb{S}^1$ lacks the fixed point property. Moreover, $\mathbb{S}^1$ is an example of a compact C-space since each Γ_A is contractible. Therefore, it shows that the Schauder conjecture (that is, any compact convex subset of a t.v.s. has the fixed point property) does not hold for G-convex spaces.

(2) Note that, in $(Y \supset D'; \Gamma)$, singletons are Γ-convex; that is, $\Gamma_{\{y\}} = \{y\}$ for each $y \in D'$.

(3) $(Y, D; \Gamma)$ with $\Gamma : \langle D \rangle \multimap Y$ defined by

$$\Gamma_A = f(\mathrm{co}\, A) \quad \text{for} \quad A \in \langle D \rangle$$

is an example of a G-convex space satisfying $D \not\subset Y$.

Examples 2. Let $X = D = [0,1]$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi it} : t \in [0,1]\}$. Define f and Γ_A as in Examples 1. Then $(Y \supset D'; \Gamma)$ is a compact G-convex space.

(1) Note that $1 \in \mathbb{S}^1$ and that $\Gamma_{\{1\}} = \mathbb{S}^1$ is not contractible. Hence, $(Y \supset D'; \Gamma)$ is not a C-space.

(2) Moreover $\Gamma_{\{1\}} \neq \{1\}$. Therefore, in general, $\Gamma_{\{y\}} \neq \{y\}$ in a G-convex space.

Examples 3. Similarly, for $X = [0,1) \times [0,1)$ or $X = [0,1] \times [0,1]$, we can made the torus, the Möbius band, and the Klein bottle into compact G-convex spaces, as was noted by Horvath [H1].

Several authors modified our definition of G-convex spaces and claimed that theirs are general than ours. All of them failed to give any proper meaningful example justifying their claims.

The following is known:

Theorem 1. [PM] *Let X be a compact Hausdorff uniform space with a basis $\mathcal{U}$ of the uniformity and $f : X \to X$ a continuous map. Then f has a fixed point if and only if for any $V \in \mathcal{U}$, $\mathrm{Gr}(f) \cap \overline{V} \neq \emptyset$.*

2. Fan-Browder maps

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from X into the power set 2^Y of Y. $T(x)$ is called the *value* of T at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the *fiber* of T at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its *graph* $\mathrm{Gr}(T) := \{(x,y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if its range $T(X)$ is contained in a compact subset of Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that a compact closed multimap is u.s.c. and compact-valued; and that every u.s.c. map with closed values is closed.

A multimap with nonempty convex values and open fibers is called a *Browder map.* The well-known *Fan-Browder fixed point theorem* states that a Browder map T from a compact convex subset X of a t.v.s. into itself has a fixed point [Br].

From the celebrated KKM theorem, we obtained the following general form of the Fan-Browder fixed point theorem:

Theorem 2. [P4,8] *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ multimaps. Suppose that*

(1) *$S(z)$ is open [resp. closed] for each $z \in D$;*
(2) *for each $y \in X$, $M \in \langle S^-(y)\rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$X = S(N)$ for some $N \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

In [P8], this is applied to obtain various forms of known Fan-Browder type theorems, the Ky Fan intersection theorem, and the Nash equilibrium theorem.

The following is the dual form of Theorem 2:

Theorem 3. [P7] *Let $(X, D; \Gamma)$ be a G-convex space and $S : X \multimap D$, $T : X \multimap X$ maps such that*

(1) *for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$;*
(2) *$S^-(z)$ is open [resp. closed] for each $z \in D$; and*
(3) *$X = \bigcup\{S^-(z) : z \in N\}$ for some $N \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in X$.

From Theorem 3, we have the following:

Theorem 4. [P10] *Let $(X \supset D; \Gamma)$ be a G-convex space and $A : X \multimap X$ a multimap such that $A(x)$ is Γ-convex for each $x \in X$. If there exist $z_1, z_2, \cdots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

Theorem 5. [P7] *Let $(X, D; \Gamma)$ be a G-convex space and $S : X \multimap D$, $T : X \multimap X$ maps such that*

(1) *for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$; and*
(2) *$X = \bigcup\{\mathrm{Int}\, S^-(z) : z \in N\}$ for some $N \in \langle D\rangle$.*

Then T has a fixed point.

From Theorems 2-5, most of popular variations or generalizatons of the Fan-Browder theorem (in the forms of the compact or so-called non-compact versions) can be deduced; see [P7,8,10].

3. Φ-spaces and compact Φ-maps

For a topological space X and a G-convex space $(Y, D; \Gamma)$, a multimap $T : X \multimap Y$ is called a *Φ-map* provided that there exists a multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$; and
(b) $X = \bigcup \{\operatorname{Int} S^-(y) : y \in D\}$.

A G-convex space $(Y, D; \Gamma)$ is called a *Φ-space* if Y is a Hausdorff uniform space and for each entourage V there exists a Φ-map $T : Y \multimap Y$ such that $\operatorname{Gr}(T) \subset V$. This concept is originated from Horvath [H1], where a number of examples are given.

Theorem 6. [P1] *If $(Y, D; \Gamma)$ is a Φ-space, then any compact continuous function $g : Y \to Y$ has a fixed point.*

Recall that a nonempty convex subset X of a t.v.s. E is said to be *locally convex* (in the sense of Krauthausen) if for every $x \in X$ there exists a basis $\mathcal{V}(x)$ of neighborhoods of x such that every $V \in \mathcal{V}(x)$ is convex.

It is easily checked that every locally convex subset X is a Φ-space $(X; \Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. Therefore, Theorem 6 works when X is a locally convex subset of a Hausdorff t.v.s. or X is a convex subset of a locally convex Hausdorff t.v.s.

For C-spaces, Theorem 6 reduces to Horvath [H1, Theorem 4.4], where some examples of Φ-spaces and applications of Theorem 6 were given.

A *G-convex uniform space* $(X \supset D; \Gamma)$ is a G-convex space such that D is dense in X and $(X, \mathcal{U})$ is a Hausdorff uniform space, where $\mathcal{U}$ is a basis of the uniformity consisting of symmetric entourages.

A *locally G-convex space* is a G-convex uniform space $(X \supset D; \Gamma)$ with a basis $\mathcal{U}$ such that for each $U \in \mathcal{U}$ and each $x \in X$,

$$U[x] = \{x' \in X : (x, x') \in U\}$$

is Γ-convex.

Lemma 1. [P6] *A locally G-convex space $(X \supset D; \Gamma)$ is a Φ-space.*

An *LG-space* is a G-convex uniform space $(X \supset D; \Gamma)$ with a basis $\mathcal{U}$ such that for each $U \in \mathcal{U}$, $U[C] := \{x \in X : C \cap U[x] \neq \emptyset\}$ is Γ-convex whenever $C \subset X$ is Γ-convex.

For a C-space $(X; \Gamma)$, the concept of LG-spaces reduces to that of *LC-spaces* due to Horvath [H1,2].

Lemma 2. [P6] *Every LG-space $(X \supset D; \Gamma)$ is a locally G-convex space if $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$.*

A C-space $(X; \Gamma)$ is an *LC-metric space* if X is equipped with a metric d such that for any $\varepsilon > 0$, the set $\{x \in X : d(x, A) < \varepsilon\}$ is Γ-convex whenever A is Γ-convex in X and open balls in (X, d) are Γ-convex.

Examples 4. The G-convex spaces $(Y \supset D'; \Gamma)$ in Examples 1 and 2 are not Φ-spaces because of Theorem 6. Moreover, in view of Lemmas 1 and 2, they are neither locally G-convex nor an LG-space. Note that in these examples, a neighborhood of $1 \in \mathbb{S}^1$ is not Γ-convex.

In 1990, Ben-El-Mechaiekh raised the following problem (see [P2]): Does the Fan-Browder fixed point theorem hold if we assume the map T is compact instead of the compactness of its domain X?

This is still open. The following are general forms of partial solutions:

Theorem 7. [P2] *Let E be a Hausdorff t.v.s. whose nonempty convex subsets have the fixed point property for compact continuous single-valued selfmaps. Let X be a nonempty convex subset of E and $T : X \multimap X$ a Φ-map. If T is compact, then T has a fixed point.*

Theorem 8. [P1,2] *Let $(Y, D; \Gamma)$ be a paracompact C-space. If it is also a Φ-space, then any compact Φ-map $T : Y \multimap Y$ has a fixed point.*

Theorem 9. [P2] *Let $(X; \Gamma)$ be a Hausdorff G-convex space, and $T : X \multimap X$ a Φ-map. If T is compact, then T^n has a fixed point for $n \geq 2$.*

Theorem 10. [P2,9] *Let $(X \supset D; \Gamma)$ be a paracompact LC-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in D$. Then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Theorem 11. [P9] *Let $(X; \Gamma)$ be a paracompact LC-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, Y a compact LC-metric subset of X, and $Z \subset X$ with $\dim_X Z \leq 0$. Let $T : X \multimap Y$ be a l.s.c. map with closed values such that $T(x)$ is Γ-convex for $x \notin Z$. Then T has a fixed point.*

In [P3], further fixed point theorems for l.s.c. multimaps in LC-metric spaces are given.

4. Kakutani maps

Usually, an u.s.c. multimap with nonempty closed convex values is called a *Kakutani map* within the category of t.v.s.

We have the following fixed point theorem for general Kakutani type maps defined on particular types of G-convex spaces:

Theorem 12. [P9] *Let $(X \supset D; \Gamma)$ be an LG-space and $T : X \multimap X$ a compact u.s.c. multimap with closed Γ-convex values. Then T has a fixed point $x_0 \in X$.*

For a single-valued map, Theorem 12 reduces to the following:

Corollary 12.1. [P9] *Let $(X \supset D; \Gamma)$ be an LG-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in D$. Then any compact continuous function $f : X \to X$ has a fixed point.*

In view of Lemmas 1 and 2, this is also a simple consequence of Theorem 6.

Let $(X \supset D; \Gamma)$ be a G-convex uniform space with a basis $\mathcal{U}$ and K a nonempty subset of X. We say that K is *of the Zima type* [PK6] whenever for every $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for every $A \in \langle D \rangle$ and every Γ-convex subset M of K the following implication holds:

$$M \cap U[z] \neq \emptyset, \ \forall z \in A \ \Rightarrow \ M \cap V[u] \neq \emptyset, \ \forall u \in \Gamma_A,$$

where $U[z] = \{x \in X : (z, x) \in U\}$.

Lemma 3. [PK6] *For an LG-space* $(X \supset D; \Gamma)$, *any nonempty subset* K *of* X *is of the Zima type.*

In view of Lemma 3, the following generalizes Theorem 12.

Theorem 13. [PK6] *Let* $(X \supset D; \Gamma)$ *be a G-convex uniform space and* $T : X \multimap X$ *a compact u.s.c. map with nonempty closed* Γ*-convex values. If* $T(X)$ *is of the Zima type, then* T *has a fixed point* $x_* \in X$.

5. Better admissible maps

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows [P4]:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that for any $N \in \langle D \rangle$ with $|N| = n+1$ and any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|\Gamma_N}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n)$.

We give some subclasses of $\mathfrak{B}$ as follows [P4, PK1,3]:

For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^\kappa(X,Y)$ of maps $F : X \multimap Y$ is one such that, for each nonempty compact subset K of X, there exists a map $G \in \mathfrak{A}_c(K,Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope P is a homeomorphic image of a standard simplex. There are lots of examples of $\mathfrak{A}$ and $\mathfrak{A}_c^\kappa$.

Subclasses of the admissible class $\mathfrak{A}_c^\kappa$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), Browder maps, Φ-maps, selectionable maps, locally selectionable maps having convex values, the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the u.s.c. approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^+$ of Lassonde, the class $\mathbb{V}_c^+$ of Park et al., u.s.c. approximable maps of Ben-El-Mechaiekh and Idizk, and many others.

Note that for a subset X of a t.v.s. and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X,Y)$ is a subclass of $\mathfrak{B}(X,Y)$. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were known. Note that the connectivity map due to Nash and Girolo is such an example.

For a particular type of G-convex spaces, we established fixed point theorems for the class $\mathfrak{B}$:

Theorem 14. [P4] *Let $(X, D; \Gamma)$ be a Φ-space and $F \in \mathfrak{B}(X, X)$. If F is closed and compact, then F has a fixed point.*

Note that Theorem 14 generalizes Theorem 6.

For a non-closed map, we have the following:

Corollary 14.1. *Let $(X, D; \Gamma)$ be a compact Φ-space and $F \in \mathfrak{A}_c^\kappa(X, X)$. Then F has a fixed point.*

Since a locally G-convex space is a Φ-space by Lemma 1, we have

Corollary 14.2. *Let $(X \supset D; \Gamma)$ be a locally G-convex space. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Similarly, by Lemma 2, we have

Corollary 14.3. *Let $(X \supset D; \Gamma)$ be an LG-space such that $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

For topological spaces X and Y, we adopt the following [PK1]:

$F \in \mathbb{V}(X, Y) \iff F : X \multimap Y$ is an acyclic map; that is, an u.s.c. multimap with compact acyclic values.

$F \in \mathbb{V}_c(X, Y) \iff F : X \multimap Y$ is a finite composition of acyclic maps where the intermediate spaces are topological.

It is known that $\mathbb{V}_c(X, Y) \subset \mathfrak{B}(X, Y)$ whenever X is a G-convex space, and that any map in $\mathbb{V}_c$ is closed.

Corollary 14.4. *Let $(X, D; \Gamma)$ be a Φ-space. Then any compact map $F \in \mathbb{V}_c(X, X)$ has a fixed point.*

6. Approximable maps

In this section, all spaces are assumed to be Hausdorff.

Recently, Ben-El-Mechaiekh *et al.* [B, BC] introduced the class $\mathbb{A}$ of approachable multimaps as follows:

Let X and Y be uniform spaces (with respective bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages). A multimap $T : X \multimap Y$ is said to be *approachable* whenever T admits a continuous W-approximative selection $s : X \to Y$ for each W in the basis $\mathcal{W}$ of the product uniformity on $X \times Y$; that is, $\mathrm{Gr}(s) \subset W[\mathrm{Gr}(F)]$, where

$$W[A] := \bigcup_{z \in A} W[z] = \{z' \in X \times Y : W[z'] \cap A \neq \emptyset\}$$

for any $A \subset X \times Y$, and

$$W[z] := \{z' \in X \times Y : (z, z') \in W]$$

for $z \in X \times Y$.

A multimap $T : X \multimap Y$ is said to be *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable.

Note that an approachable map is always approximable. Recall that Ben-El-Mechaiekh et al. [B, BC] established a large number of properties and examples of approachable or approximable maps.

We denote $F \in \mathbb{A}(X,Y)$ if $F : X \multimap Y$ is approachable.

The following two lemmas are [BC, Lemmas 2.4 and 4.1], respectively.

Lemma 4. *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$, $(Z,\mathcal{W})$ be three uniform spaces, with Z compact, and let $\Psi : Z \multimap X$, $\Phi : X \multimap Y$ be two u.s.c. closed-valued approachable maps. Then so is their composition $\Phi \circ \Psi$.*

Lemma 5. *If X is a nonempty convex subset of a locally convex t.v.s. and if $\Phi \in \mathbb{A}(X,X)$ is closed and compact, then Φ has a fixed point.*

From Lemmas 4 and 5, we show that certain approachable maps are better admissible if their domains are G-convex spaces as follows:

Lemma 6. [P6] *Let $(X \supset D;\Gamma)$ be a G-convex uniform space and $(Y,\mathcal{V})$ a uniform space. If $F \in \mathbb{A}(X,Y)$ is closed and compact, then $F \in \mathfrak{B}(X,Y)$.*

From Theorem 14 and Lemma 6, we have

Theorem 15. [P6] *Let $(X \supset D;\Gamma)$ be a Φ-space and $F \in \mathbb{A}(X,X)$. If F is closed and compact, then F has a fixed point.*

Examples 5. We give some examples of approachable maps $T : X \to Y$ as follows:

(1) Any selectionable multimap is approximable.

(2) A locally selectionable map T with convex values is approximable whenever Y is a convex subset of a t.v.s.

(3) An u.s.c. map T with nonempty convex values is approachable whenever X is paracompact and Y is a convex subset of a locally convex t.v.s.

(4) An u.s.c. map T with nonempty compact contractible values is approachable whenever X is a finite polyhedron.

(5) An u.s.c. map T with nonempty compact values having trivial shape (that is, contractible in each neighborhood in Y) is approachable whenever X is a finite polyhedron.

For (1) and (2), see [P11]; and for (3)-(5), see [B].

The following is due to Ben-El-Mechaiekh et al. [BC, Proposition 3.9]:

Lemma 7. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces. If either*

(i) *X is paracompact and $(Y;\Gamma)$ is an LC-space; or*

(ii) *X is compact and $(Y;\Gamma)$ is an LG-space,*

then every u.s.c. map $F : X \multimap Y$ with nonempty Γ-convex values is approachable; that is, $F \in \mathbb{A}(X,Y)$.

Note that Lemma 7(i) generalizes Examples 5(3).

In our previous work [P6], (ii) is incorrectly stated and causes some incorrect statements. For example, [P6, Theorem 4] should be stated for LG-spaces as in Theorem 12.

From Lemmas 6 and 7, we have the following correction of [P4, Lemma 4.5]:

Lemma 8. *Let $(X \supset D; \Gamma)$ be a compact LG-space. Then any u.s.c. map $F : X \multimap X$ with nonempty closed Γ-convex values belongs to $\mathfrak{B}(X, X)$.*

Consequently, correct forms of [P4, Corollary 4.7 and Theorem 4.8] are Theorem 12 and Corollary 14.1, respectively, in the present paper.

We add two types of new multimaps in the class $\mathfrak{B}$:

Lemma 9. *Let $(X, D; \Gamma)$ be a G-convex space and $F : X \multimap X$ be an u.s.c. map such that either*

(i) *F has nonempty compact contractible values; or*

(ii) *F has nonempty compact values having trivial shape,*

then $F \in \mathfrak{B}(X, X)$.

Proof. For any $N \in \langle D \rangle$ with $|N| = n + 1$ and any continuous function $p : F(\Gamma_N) \to \Delta_n$, consider the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n.$$

Note that $(F|_{\Gamma_N}) \circ \phi_N$ is an u.s.c. multimap having values of the type (i) or (ii) and defined on a finite polyhedron Δ_n. Therefore $p \circ (F|_{\Gamma_N}) \circ \phi_N$ is approachable by Lemma 4, and has a fixed point by Lemma 5. This completes our proof.

From Theorem 14 and Lemma 9, we have

Theorem 16. [P4] *Let $(X, D; \Gamma)$ be a Φ-space and $F : X \multimap X$ be a map such that all of its values are either (i) nonempty contractible or (ii) nonempty and of trivial shape. If F is closed and compact, then F has a fixed point.*

Note that Case (i) of Theorem 16 is a consequence of Corollary 14.4. In the category of t.v.s., Theorem 16(i) holds for Kakutani maps since convex values are contractible. But, for G-convex spaces, Γ-convex values are only known to be connected and that is why we need Lemma 7.

References

[B] H. Ben-El-Mechaiekh, *Spaces and maps approximation and fixed points*, J. Comp. Appl. Math. **113** (2000), 283–308.

[BC] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–151.

[Br] F.E. Browder, *The fixed point theory of multivalued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[H1] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.

[H2] ———, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.

[P1] ———, *Continuous selection theorems in generalized convex spaces*, Numer. Funct. Anal. and Optimiz. **25** (1999), 567–583.

[P2] ———, *Remarks on a fixed point problem of Ben-El-Mechaiekh*, Nonlinear Analysis and Convex Analysis (Proc. NACA'98, Niigata, Japan, July 28–31, 1998), 79–86, World Sci., Singapore, 1999.

[P3] ———, *Fixed points of lower semicontinuous multimaps in LC-metric spaces*, J. Math. Anal. Appl. **235** (1999), 142–150.

[P4] ———, *Fixed points of better admissible multimaps on generalized convex spaces*, J. Korean Math. Soc. **37** (2000), 885–899.

[P5] ———, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. & Appl. Math. **7** (2000), 1–28.

[P6] ———, *Remarks on fixed point theorems for generalized convex spaces*, Fixed Point Theory and Applications (Y.J. Cho, ed.), 135–144, Nova Sci. Publ., New York, 2000.

[P7] ———, *Remarks on topologies of generalized convex spaces*, Nonlinear Funct. Anal. Appl. **5** (2000), 67–79.

[P8] ———, *New topological versions of the Fan–Browder fixed point theorem*, Nonlinear Anal. **47** (2001), 595–606.

[P9] ———, *Fixed point theorems in locally G-convex spaces*, Nonlinear Anal. **48** (2002), 869–879.

[P10] ———, *Coincidence, almost fixed point, and minimax theorems on generalized convex spaces*, J. Nonlinear Convex Anal. **4** (2003), 151–164.

[P11] ———, *Fixed points of approximable or Kakutani maps in generalized convex spaces*, Preprint.

[PK1] S. Park and H. Kim, *Admissible classes of multifunctons on generalized convex spaces*, Proc. Coll. Natur. Sci. Seoul National University **18** (1993), 1–21.

[PK2] ———, *Coincidence theorems for admissible multifunctions on generalized convex spaces*, J. Math. Anal. Appl. **197** (1996), 173–187.

[PK3] ———, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209** (1997), 551–571.

[PK4] ———, *Generalizations of the KKM type theorems on generalized convex spaces*, Ind. J. Pure Appl. Math. **29** (1998), 121–132.

[PK5] ———, *Coincidence theorems in a product of generalized convex spaces and applications to equilibria*, J. Korean Math. Soc. **36** (1999), 813–828.

[PK6] ———, *Generalized KKM maps, maximal elements and almost fixed points*, to appear.

[PM] S. Park and K.B. Moon, *Comments on a coincidence theorem in generalized convex spaces*, Soochow J. Math. **25** (1999), 387–393.

National Academy of Sciences, Republic of Korea, and
School of Mathematical Sciences, Seoul National University, Seoul 151–747, Korea
E-mail address: shpark@math.snu.ac.kr