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# Hyperbolicity of positively expansive $C^r$ maps on compact smooth manifolds which are $C^r$ structurally stable

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Let $X$ be a metric space with metric $d$, and let $f : X \to X$ be a continuous map. We say that $f$ is *positively expansive* if there is a constant $e > 0$, called a *expansive constant*, such that for $x, y \in X$ if $d(f^n(x), f^n(y)) \leq e$ for all $n \geq 0$ then $x = y$. If $X$ is compact, the property that $f : X \to X$ is positively expansive does not depend on the choice of metrics for $X$ compatible with the topology of $X$, although so is not the expansive constant. Also, for continuous maps of compact metric spaces, positive expansiveness is preserved under topological conjugacy.

Reddy [20] proved that if $X$ is compact and $f : X \to X$ is positively expansive then $f : X \to X$ is *topologically expanding*, i.e. there are constants $\lambda > 1$ and $\delta > 0$ and a metric $D$ for $X$, called the *hyperbolic metric*, compatible with the topology of $X$ such that for $x, y \in X$ if $D(x, y) < \delta$ then $D(f(x), f(y)) \geq \lambda D(x, y)$. As an application of this result, it is easily obtained that if a compact metric space $X$ admits a positively expansive homeomorphism then $X$ must be a finite set (for example, see [1, Theorem 2.2.12]).

If a positively expansive map $f : X \to X$ is an open map, obviously $f$ is a local homeomorphism. Let $X$ be compact. Then, using the hyperbolic metric, we can show that a positively expansive map $f : X \to X$ is an open map if and only if $f$ has the shadowing property (for example, see [1, Theorem 2.3.10]). From this fact it follows that if a positively expansive map $f : X \to X$ is an open map then the dynamics of $f$ behaves like Axiom A differetiable dynamics in topological viewpoint and, especially, $X$ has Markov partitions. For details the readers can refer to [1].

Let $M$ be a compact connected manifold. If $M$ admits a positively expansive map then the boundary $\partial M$ must be empty ([11]). Hence, every positively expansive map $f : M \to M$ is an open map, by Brouwer's theorem on invariance of domain, and it is a self-covering map with the covering degree greater than one. After the studies of expanding differetiable maps by Shub [21], Franks [5] and so on (see below for the definition), Coven-Reddy [3] showed that if $f : M \to M$ is positively expansive then the set $\mathrm{Fix}(f)$ of all fixed points is not empty, the set $\mathrm{Per}(f)$ of all periodic points is dense in $M$, the universal covering space of $M$ is homeomorphic to the Euclidean space, and if another positively expansive $g : M \to M$ is homotopic to $f$ then $f$ and $g$ are topologically conjugate. The author [9] proved that $M$ admits a positively expansive map then the fundamental group $\pi_1(M)$ has polynomial growth. Combining these facts with results of Franks [5] and Gromov [7], we have that a positively expansive map $f : M \to M$ is topologically conjugate to an expanding infra-nilmanifold endomorphism, in the same way as expanding differetiable maps. See also [10]. Thus, the dynamics of positively expansive maps on compact manifolds is well-understood in topological viewpoint.

The purpose of this paper is to study the dynamics of positively expansive map form differetiable viewpoint.

Let $M$ be a closed Riemannian smooth (= $C^\infty$) manifold, and let $f : M \to M$ be a $C^1$ map. We recall that $f$ is *expanding* if there are constants $C > 0$ and $\lambda > 1$ such that the derivative $Df : TM \to TM$ has the following property; for all $v \in TM$ and $n \geq 0$

$$\|Df^n(v)\| \geq C\lambda^n\|v\|,$$

where $\|\cdot\|$ is the Riemannian metric. It is not difficult to check that an expanding $C^1$ map $f : M \to M$ is positively expansive.

Let $1 \leq r \leq \infty$, and denote by $C^r(M, M)$ the space of all $C^r$ maps of $M$ with the $C^r$ topology. We let

$$PE^r(M) = \{f \in C^r(M, M) \mid f \text{ is positively expansive }\},$$

and denote by int$PE^r(M)$ the interior of $PE^r(M)$ in $C^r(M, M)$ with respect to the $C^r$ topology.

**Theorem 1.** *Let $f : M \to M$ be a $C^r$ map, $1 \leq r \leq \infty$. Then*

$$f \in \text{int}PE^r(M) \iff f : M \to M \text{ is expanding.}$$

The implication $\Longleftarrow$ in Theorem 1 is clear because the set of all expanding $C^1$ maps on $M$ is an open subset of $C^1(M, M)$ with respect the $C^1$ topology (see [21], and also Lemma 3.1). The case of $r = 1$ for the implication $\Longrightarrow$ in Theorem 1 can be shown in the same method as the proof given by Mañé [16] whose result says that the interior int$E^1(M)$ of the set $E^1(M)$ of all expansive $C^1$ diffeomorphisms in the space Diff$^1(M)$ of all $C^1$ diffeomorphisms endowed with the $C^1$ topology is consistent with the set of all Axiom A $C^1$ diffeomorphisms satisfying the condition that $T_xW^s(x) \cap T_xW^u(x) = \{0\}$ for all $x \in M$, where $W^s(x)$ and $W^u(x)$ are stable and unstable manifolds of $x$. However, our proof of the implication $\Longrightarrow$ in Theorem 1 will be different from the one given by Mañé, because we handle the $C^r$ cases, $1 \leq r \leq \infty$, and can not use well-known methods such as Pugh's closing lemmma ([19]), Franks' lemma ([6]) and Hayashi's connecting lemma ([8]) which work only for the $C^1$ case.

From Theorem 1 the following corollary is obtained immediately.

**Corollary 2.** *Let $1 \leq r \leq \infty$. Then*

$$\text{int}PE^r(M) = \text{int}PE^1(M) \cap C^r(M, M).$$

We say that a $C^r$ map $f : M \to M$ is $C^r$ *structurally stable* if there is a neighborhood $\mathcal{N}$ of $f$ in $C^r(M, M)$ such that any $g \in \mathcal{N}$ is topologically conjugate to $f$. Since positive expansiveness is preserved under topological conjugacy, we also obtain the following corollary.

**Corollary 3.** *Let $1 \le r \le \infty$. If a $C^r$ map $f : M \to M$ is positively expansive and $C^r$ structurally stable, then $f : M \to M$ is expanding.*

For $f \in C^r(M, M)$ we denote by Sing($f$) the set of all singularities of $f$, i.e.

$$\text{Sing}(f) = \{x \in M \mid D_x f : T_x M \to T_{f(x)} M \text{ is not an isomorphism } \}.$$

If Sing$(f) = \emptyset$, then $f : M \to M$ is called *regular*, which is a self-covering map. It is evident that any expanding $C^1$ map is regular.

We say that $p \in \text{Per}(f)$ is *repelling* if the absolute value of any eigenvalue of $Df^n : T_pM \to T_pM$ is greater than one, where $n$ is the period of $p$. Using our idea of the proof of Theorem 1, we will also obtain the following theorem.

**Theorem 4.** *Let $f : S^1 \to S^1$ be a $C^r$ map of the circle, $1 \le r \le \infty$. Suppose that $f : S^1 \to S^1$ is positively expansive. Then $f$ belongs to $PE^r(S^1) \setminus \text{int}PE^r(S^1)$ if and only if* Sing$(f) \neq \emptyset$ *or there exists a periodic point of $f$ which is not repelling.*

**Corollary 5.** *Suppose that a $C^1$ map $f : S^1 \to S^1$ of the circle is positively expansive and regular. If all periodic points of $f$ are repelling, then $f : S^1 \to S^1$ is expanding.*

We remark that the $C^2$ version of Corollary 5 is obtained from a result of Mañé [18, Theorem A].

It remains a problem of whether or not there is $f \in PE^r(M) \setminus \text{int}PE^r(M)$, in the case where $\dim(M) \ge 2$, such that $f$ is regular and all periodic points of $f$ are repelling, where $1 \le r \le \infty$. Compare with a result of Bonatti-Díaz-Vuillemin [2] which says that there are expansive $C^3$ diffeomorphisms on the two-dimensional torus $T^2$ with the property that all periodic points are hyperbolic but the diffeomorphisms do not belong to the interior int$E^3(T^2)$ of the set $E^3(M)$ of all expansive $C^3$ diffeomorphisms in the space Diff$^3(T^2)$ of all $C^3$ diffeomorphisms with the $C^3$ topology. See also Enrich [4].

## §1 Positively expansive $C^r$ maps with singularities

In this section we first show the following Lemma 1.1.

**Lemma 1.1.** *Let $f : M \to M$ be a $C^r$ map, $1 \le r \le \infty$. If $f : M \to M$ is a self-covering map and there is a neighborhood $\mathcal{N}$ of $f$ in $C^r(M, M)$ with respect to the $C^r$ topology such that any $g : M \to M$ belonging to $\mathcal{N}$ is a self-covering map, then $f : M \to M$ is regular.*

*Proof.* Let $\{(U_i, \varphi_i)\}_{i=1}^k$ be an atlas of $M$ with a finite number of charts such that each chart $\varphi_i : U_i \to D$ is a $C^\infty$ diffeomorphism, where $D$ is the unit open disc in $\mathbb{R}^n$, $n = \dim(M)$. Since $f : M \to M$ is a $C^r$ covering map and each $U_i$ is an open disc in $M$, it follows that $U_i$ is evenly covered by $f$, i.e. $f^{-1}(U_i)$ is expressed as a finite disjoint union $f^{-1}(U_i) = \cup_j^d V_j^i$ of open discs in $M$, where $d$ is the covering degree of $f$, such that each restriction $f : V_j^i \to U_i$ is a $C^r$ bijection. Let $2\delta > 0$ be the Lebesgue number of the covering $\{V_j^i \mid i = 1, \cdots, k, j = 1, \cdots, d\}$ of $M$. For $x \in M$ denote by

$D_\delta(x)$ the open disc of radius $\delta$ centered at $x$. Then the closure $\overline{D_\delta(x)}$ is contained in some $V_j^i$, which is homeomorphically mapped by $f$ onto $U_i$. Therefore, there is a path connected neighborhood $\mathcal{V}$ of $f$ in $C^r(M,M)$, with $\mathcal{V} \subset \mathcal{N}$, such that for any $g \in \mathcal{V}$ and any $x \in M$, $g(\overline{D_\delta(x)})$ is contained in some $U_i$. Let $g \in \mathcal{V}$. By assumption, $g : M \to M$ is a covering map. Since $U_i$ is an open disc, $U_i$ is evenly covered by $g$, which implies that $D_\delta(x)$ is homeomorphically mapped by $g$ onto an open subset of $U_i$.

Fix $x \in M$. Choose orientations

$$\{1_y \in H_n(D_\delta(x), D_\delta(x) \setminus \{y\}) \mid y \in D_\delta(x)\} \quad \text{and} \quad \{1_z \in H_n(U_i, U_i \setminus \{z\}) \mid z \in U_i\}$$

of $D_\delta(x)$ and $U_i$ respectively. Since $\mathcal{V}$ is path connected, there is a constant $\tau = \pm 1$ such that for any $g \in \mathcal{V}$ and $y \in D_\delta(x)$, $g_*(1_y) = \tau 1_{g(y)}$, where $g_* : H_n(D_\delta(x), D_\delta(x)\setminus\{y\}) \to H_n(U_i, U_i\setminus\{g(y)\})$ is the induced isomorphism. Since $\delta > 0$ is chosen to be small, we can take a $C^\infty$ diffeomorphism $\phi_x : D_\delta(x) \to D$. For $y \in D_\delta(x)$ let $A_y = D_{\phi_x(y)}(\varphi_i \circ f \circ \phi_x^{-1})$ be the derivative. Without loss of generality, we may assume that $\varphi_i : U_i \to D$ and $\phi_x : D_\delta(x) \to D$ send the orientations of $U_i$ and of $D_\delta(x)$ to the standard orientation of $D$. Then, if the determinant $\det(A_y)$ is not zero, the sign of the constant $\tau$ is consistent with that of $\det(A_y)$.

For given $y \in D_\delta(x)$ assume $\det(A_y) = 0$, and choose regular matrices $P$ and $Q$ such that the signs of $\det(P)$ and $\det(Q)$ are both positive, and

$$PA_yQ = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & B_{22} \end{pmatrix},$$

where $O_{11}$, $O_{12}$ and $O_{21}$ are zero matrices, and $B_{22}$ is a regular matrix. Let

$$B_{11}^\varepsilon = \begin{pmatrix} \varepsilon_1 & & O \\ & \ddots & \\ O & & \varepsilon_m \end{pmatrix}$$

be a regular diagonal matrix, where $m$ is the size of the matrix $O_{11}$, such that the absolute values $|\varepsilon_1|, \cdots, |\varepsilon_m|$ are small enough and the sign of $\det(B_{11}^\varepsilon) \cdot \det(B_{22})$ is different from that of $\tau$. Then

$$A_y^\varepsilon = P^{-1} \begin{pmatrix} B_{11}^\varepsilon & O_{12} \\ O_{21} & B_{22} \end{pmatrix} Q^{-1}$$

is a regular matrix and the norm $\|A_y - A_y^\varepsilon\|$ is small enough. Let $W_1$ and $W_2$ be open neighborhoods of $\phi_x(y)$ in $D$ such that $\overline{W}_1 \subset W_2$ and $\overline{W}_2 \subset D$, and choose a $C^\infty$ function $b : D \to \mathbb{R}$ satisfying the condition that $b(z) = 1$ for $z \in W_1$ and $b(z) = 0$ for $z \in D \setminus W_2$. Define $g : M \to M$ by

$$\varphi_i \circ g \circ \phi_x^{-1}(z) = b(z)(A_y - A_y^\varepsilon)(z - \phi_x(y)) + \varphi_i \circ f \circ \phi_x^{-1}(z)$$

for $z \in D$, and $g = f$ otherwise. Since each element of $A_y - A_y^\varepsilon$ can be chosen to be approximately zero, we have that $g \in \mathcal{V}$. On the other hand, $D_{\phi_x(y)}(\varphi_i \circ g \circ \phi_x^{-1}) = A_y^\varepsilon$, whose determinant has a different sign from $\tau$, a contradiction.

We proved that $\det(A_y) \neq 0$ for all $y \in D_\delta(x)$. Since $x$ is arbitrary, it follows that $f$ is regular. The proof is complete.

From Lemma 1.1 the followng Proposition 1.2 is obtained immediately.

**Proposition 1.2.** *Let $f : M \to M$ be a $C^r$ map, $1 \le r \le \infty$. Suppose that $f : M \to M$ is positively expansive. If* $\mathrm{Sing}(f) \neq \emptyset$, *then $f$ belongs to $PE^r(M) \setminus \mathrm{int}PE^r(M)$.*

In the rest of this section we give an example of a positively expansive $C^\infty$ map $f : S^1 \to S^1$ on the circle such that $\mathrm{Sing}(f) \neq \emptyset$.

Take $\ell \ge 1$ an integer. Let $\tilde{h} : \mathbb{R} \to \mathbb{R}$ be a strictly monotone increasing $C^\infty$ function having the property that $\tilde{h}(x+1) = \tilde{h}(x) + 1$ for all $x \in \mathbb{R}$, the derivative $\tilde{h}'(x)$ is positive whenever $x$ is not an integer, $\tilde{h}(x) = x^{2\ell+1}$ on a small neighborhood of $x = 0$ ,and $\tilde{h}(x) = 2x - \frac{1}{2}$ on a small neighborhood of $x = \frac{1}{2}$. We choose $\tilde{g} : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$, and define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f} = \tilde{h} \circ \tilde{g} \circ \tilde{h}^{-1}$. Then $\tilde{f}(x) = 2^{2\ell+1}x$ if $x$ is in a neighborhood of 0, and $\tilde{f}(x) = (4x-2)^{2\ell+1} + 1$ if $x$ is in a neighborhood of $\frac{1}{2}$. Let $p : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the covering projection, and define $f : S^1 \to S^1$ as the projection of $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $p$. Then $f : S^1 \to S^1$ is positively expansive and of class $C^\infty$, and $\mathrm{Sing}(f) = \{p(\frac{1}{2})\} \neq \emptyset$.

## §2 Invariant manifolds

Let $f : X \to X$ be a continuous map of a compact metric space, and denote the set of all orbits of $f$ by

$$\lim_{\leftarrow}(X, f) = \{(x_i) \in \Pi_{-\infty}^{\infty} X \mid f(x_i) = x_{i+1}, \forall i \in \mathbb{Z}\},$$

which is called the *inverse limit* of $f$. Let $d$ be the metric for $X$, and define a metric $\tilde{d}$ for $\Pi_{-\infty}^{\infty} X$ by

$$\tilde{d}((x_i), (y_i)) = \sum_{i \in \mathbb{Z}} \frac{d(x_i, y_i)}{2^{|i|}}$$

and the *shift* $\sigma : \Pi_{-\infty}^{\infty} X \to \Pi_{-\infty}^{\infty} X$ by $\sigma((x_i)) = (x_{i+1})$. Then $\lim_{\leftarrow}(X, f)$ is a $\sigma$-invariant closed subset. The homeomorphism $\sigma : \lim_{\leftarrow}(X, f) \to \lim_{\leftarrow}(X, f)$ is called the *inverse limit system* for $f$, which is a natural extension of $f$. Define $p_0 : \lim_{\leftarrow}(X, f) \to X$ by $p_0((x_i)) = x_0$. Then, $f \circ p_0 = p_0 \circ \sigma$ holds.

Let $f : M \to M$ be a regular $C^r$ map, and let $\Lambda \subset M$ be an $f$-invariant closed set (i.e. $f(\Lambda) = \Lambda$). Then $\lim_{\leftarrow}(\Lambda, f)$ is a $\sigma$-invariant closed subset of $\lim_{\leftarrow}(M, f)$. We say that $\Lambda$ is a *hyperbolic set* if there there are constants $C > 0$ and $0 < \lambda < 1$ such that for any $(x_i) \in \lim_{\leftarrow}(\Lambda, f)$ there is a splitting

$$\coprod_{i \in \mathbb{Z}} T_{x_i} M = \coprod_{i \in \mathbb{Z}} E^s_{x_i} \oplus E^u_{x_i} = E^s \oplus E^u,$$

which is left invariant by $Df$, such that for all $n \ge 0$,

$$\|Df^n(v)\| \le C\lambda^n \|v\| \text{ if } v \in E^s \quad \text{and} \quad \|Df^n(v)\| \ge C^{-1}\lambda^{-n}\|v\| \text{ if } v \in E^u.$$

When $(x_i) \neq (y_i)$ and $x_0 = y_0$, we have $E^u_{x_0} \neq E^u_{y_0}$ in most cases. Hence, we will sometimes write $E^u_{x_0} = E^u_{x_0}((x_i))$. On the other hand, even if $(x_i) \neq (y_i)$, it follows that $E^s_{x_0} = E^s_{y_0}$ whenever $x_0 = y_0$.

For $x \in \Lambda$ and $\varepsilon > 0$ we define the *local stable set*

$$W^s_\varepsilon(x) = \{y \in M \mid d(f^i(x), f^i(y)) \leq \varepsilon, \forall i \geq 0\},$$

and for $(x_i) \in \lim_{\leftarrow}(\Lambda, f)$ and $0 < \varepsilon \leq \varepsilon_0$, the *local unstable set* is defined by

$$W^u_\varepsilon((x_i)) = \{y \in M \mid \text{ there exists } (y_i) \in \varprojlim(M, f) \text{ such that } y_0 = y \text{ and } d(x_i, y_i) \leq \varepsilon, \forall i \leq 0\}.$$

Let $Y$ be a subset of $\lim_{\leftarrow}(M, f)$. For $\delta > 0$ denote by $L_\delta(Y)$ the set of points $\mathbf{x} \in \lim_{\leftarrow}(M, f)$ such that there is a path $w$, contained in a $\delta$-neighborhood of $\tilde{\Lambda}$ in $\lim_{\leftarrow}(M, f)$, jointing $\mathbf{x}$ and some point of $Y$.

**Stable manifold theorem.** *Let $f : M \to M$ be a regular $C^r$ map, $1 \leq r \leq \infty$, and let $\Lambda$ be a hyperbolic set. Then there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, $\{W^s_\varepsilon(x) \mid x \in \Lambda\}$ and $\{W^u_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)\}$ are families of discs of class $C^r$ which varify continuously on $x \in \Lambda$ and $\mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)$ respectively. Furthermore, there is $\delta > 0$ such that $\{W^s_\varepsilon(x) \mid x \in \Lambda\}$ and $\{W^u_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)\}$ are extended to families $\{D^s_\varepsilon(x) \mid x \in p_0(L_\delta(\lim_{\leftarrow}(\Lambda, f)))\}$ and $\{D^u_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in L_\delta(\lim_{\leftarrow}(\Lambda, f))\}$ of discs of class $C^r$, respectively, which are semi-invariant under $f$ and have the local product structure.*

Let $\Lambda$ be an $f$-invariant closed set of $M$. We say that $\Lambda$ has the *dominated splitting* if there are constants $C > 0$ and $0 < \lambda < 1$ such that for any $(x_i) \in \lim_{\leftarrow}(\Lambda, f)$ there is a splitting

$$\coprod_{i \in \mathbb{Z}} T_{x_i}M = \coprod_{i \in \mathbb{Z}} E_{x_i} \oplus F_{x_i},$$

which is left invariant by $Df$, such that for all $n \geq 0$ and $i \in \mathbb{Z}$,

$$\frac{\|Df^n_{|E_i}\|_M}{\|Df^n_{|F_i}\|_m} \leq C\lambda^n,$$

where $\| \cdot \|_M$ is the maximum norm and $\| \cdot \|_m$ is the minimum norm, and the correspondances $(x_i) \in \lim_{\leftarrow}(\Lambda, f) \mapsto E_{x_0} = E_{x_0}((x_i))$ and $(x_i) \in \lim_{\leftarrow}(\Lambda, f) \mapsto F_{x_0} = F_{x_0}((x_i))$ are continuous.

**Invariant manifold theorem.** *Let $f : M \to M$ be a regular $C^r$ map, $1 \leq r \leq \infty$, and let $\Lambda$ be an $f$-invariant closed set having the dominatted splitting. Then there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ there are families $\{D_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)\}$ and $\{D'_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)\}$ of discs of class $C^r$ which are semi-invariant under $f$ and varify continuously on $\mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)$ respectively. Furthermore, there is $\delta > 0$ such that $\{D_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)\}$ and $\{D'_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in \lim_{\leftarrow}(\Lambda, f)\}$ are extended to families $\{D_\varepsilon(x) \mid \mathbf{x} \in L_\delta(\lim_{\leftarrow}(\Lambda, f))\}$ and $\{D'_\varepsilon(\mathbf{x}) \mid \mathbf{x} \in L_\delta(\lim_{\leftarrow}(\Lambda, f))\}$ of discs of class $C^r$, respectively, which are semi-invariant under $f$ and have the local product structure.*

## §3 Proofs of Theorems 1 and 4

Let $f : M \to M$ be a regular $C^r$ map, $1 \le r \le \infty$. For $b > 1$ we define

$$\Lambda_b = \{x \in M \mid \text{ there is } v \in T_xM, v \neq 0, \text{ such that } \|Df^v(v)\| \le b\|v\| \text{ for all } n \ge 0\}.$$

It is evident that $\Lambda_b$ is a closed subset of $M$.

**Lemma 3.1.** *If there is $b > 1$ such that $\Lambda_b = \emptyset$, then $f : M \to M$ is expanding.*

*Proof.* By assumption, for any $x \in M$ and $v \in T_xM$ with $v \neq 0$ there is $n > 0$ such that $\|Df^n(v)\| > b\|v\|$. Let $S^1(M) = \{v \in TM \mid \|v\| = 1\}$. Since $S^1(M)$ is compact, there are a finite open cover $\{U_1, \cdots, U_k\}$ of $S^1(M)$ and a sequence $\{n_1, \cdots, n_k\}$ of positive integers such that for each $v \in U_i$, $1 \le i \le k$, $\|Df^{n_i}(v)\| \le b\|v\|$. Let $N_0 = \max\{n_1, \cdots, n_k\}$, and choose $c > 0$ such that for all $v \in TM$ and $0 \le n \le N_0$, $\|Df^n(v)\| \ge c\|v\|$. Since $b > 1$, there is $\ell > 0$ such that $\lambda = b^\ell c > 1$. Take $N > 0$ such that $N/N_0 \ge \ell$. Then, for any $v \in TM$ there is $m \ge \ell$ such that

$$v \in U_{i_1}, Df^{n_{i_1}}(v) \in U_{i_2}, \cdots, Df^{n_{i_1}+n_{i_2}+\cdots+n_{i_{m-1}}}(v) \in U_{i_m},$$

and $0 \le n = N - (n_{i_1} + n_{i_2} + \cdots + n_{i_m}) \le N_0$. Hence, we have

$$\begin{aligned}\|Df^N(v)\| &= \|Df^n \circ Df^{n_{i_m}} \circ \cdots Df^{n_{i_1}}(v)\| \\ &= cb^m\|v\| \ge \lambda\|v\|,\end{aligned}$$

which means that $f^N : M \to M$ is expanding. The proof is complete.

By Lemma 3.1, if $f : M \to M$ is not expanding, then $\Lambda_b \neq \emptyset$ for all $b > 1$. In this case, for $b > 1$ given we define

$$E_x^{sc}(0) = \{v \in T_xM \mid \text{ there is } K > 0 \text{ such that } \|Df^n(v)\| \le K\|v\| \text{ for all } n \ge 0\}, \quad x \in \Lambda_b.$$

It is easy to see that $E_x^{sc}(0)$ is a subspace of $T_xM$. Since $x \in \Lambda_b$, it follows that $1 \le \dim E_x^{sc}(0) \le \dim M$. Let $\Lambda(b) = \cap_{n=0}^{\infty} f^{-n}(\Lambda_b)$. If $x \in \Lambda(b)$ then $f^n(x) \in \Lambda_b$ for all $n \ge 0$, and so $f(x) \in \Lambda_b$, which implies that $f(\Lambda(b)) \subset \Lambda(b)$. Hence, $\Lambda_\infty(b) = \cap_{n=0}^{\infty} f^n(\Lambda(b))$ is an $f$-invariant closed set.

We consider the following two cases.

**Bounded case.** $\Lambda(b) \neq \emptyset$ *for some* $b > 1$.

In this case, $\Lambda_\infty(b) \neq \emptyset$. Thus, we can choose a minimal set, say $\Lambda_{min}(b)$, for $f : \Lambda_\infty(b) \to \Lambda_\infty(b)$.

**Unbounded case.** $\Lambda(b) = \emptyset$ *for all* $b > 1$.

In this case, we take $b > 1$ sufficently large, and define $\Lambda_{exit}(b)$ as the set of points $x \in \Lambda_b$ such that $f(x) \notin \Lambda_b$. Then, $\Lambda_{exist}(b)$ is an open subset of $\Lambda_b$.

Let $x \in \Lambda_{exist}(b)$. Then, there is $v \in E^{sc}_x(0)$ with $v \neq 0$ such that $\|Df^n(v)\| \leq b\|v\|$ for all $n \geq 0$. If $f(x), \cdots, f^j(x) \notin \Lambda_b$, for $1 \leq i \leq j$ there is $n_i \geq 1$ such that $\|Df^{n_i}(Df^i(v))\| > b\|Df^i(v)\|$. Since $\|Df^{n_i}(Df^i(v))\| \leq b\|v\|$, we have $\|v\| > \|Df^i(v)\|$ for $1 \leq i \leq j$. Hence, if $f^i(x) \notin \Lambda_b$ for all $i \geq 1$ then, since $b > 1$ is taken large, $f(x) \in \Lambda_b$, a contradiction. Therefore, there is $j_x \geq 2$ such that $f(x), \cdots, f^{j_x-1}(x) \notin \Lambda_b$ and $f^{j_x}(x) \in \Lambda_b$. Since $b > 1$ is taken sufficiently large, it follows that $\{j_x \mid x \in \Lambda_{exit}(b)\}$ is unbounded.

We define $r : \Lambda_b \to \Lambda_b$ by $r(x) = f(x)$ if $x \in \Lambda_b \setminus \Lambda_{exit}(b)$ and $r(x) = f^{j_x}(x)$ if $x \in \Lambda_{exit}(b)$. Then, we can choose a minimal set, say $\Lambda_{min}(b) = \Lambda_{min}(b; r)$, for $r : \Lambda_b \to \Lambda_b$, i.e. if $\Lambda$ is a nonempty closed subset of $\Lambda_b$, $r(\Lambda) \subset \Lambda$, and $\Lambda \subset \Lambda_{min}$, then $\Lambda = \Lambda_{min}$. Note that $\overline{r(\Lambda_{min})} = \Lambda_{min}$. Let $\Lambda_{min}(b; f) = \overline{\cup_{n=0}^{\infty} f^n(\Lambda_{min}(b))}$.

**Lemma 3.2.**

(1) *If the bounded case happens then* $\dim \Lambda_{min}(b) = 0$.
(2) *If the unbounded case happens then* $\dim \Lambda_{min}(b; f) = 0$.

**Proposition 3.3.** *Let* $f : M \to M$ *be a regular* $C^r$ *map,* $1 \leq r \leq \infty$. *Suppose that* $f : M \to M$ *is positively expansive and not expanding. Let* $\Lambda_{min} = \Lambda_{min}(b)$ *for the bounded case, and* $\Lambda_{min} = \Lambda_{min}(b; f)$ *for the unbounded case. Then in the both cases the following holds. There are a* $Df$*-invariant continuous subbundle* $E^{sc}(i_0) = \cup_{x \in \Lambda_{min}} E^{sc}_x(i_0)$ *of* $T_{\Lambda_{min}}M$ *with* $\dim E^{sc}(i_0) \geq 1$, *where* $i_0 \geq 0$ *is an integer, and finite fimilies* $\{D^u_i\}_{i=1}^{\ell}$ *and* $\{D^{u\prime}_i\}_{i=1}^{\ell}$ *of* $m$*-discs of class* $C^r$, $m = \dim M - \dim E^{sc}(i_0)$, *such that*

(1) *there is a constant* $C_{i_0} > 0$ *such that if* $v \in E^{sc}(i_0)$ *then* $\|Df^n(v) \leq C_{i_0} n^{i_0} \|v\|$ *for all* $n \geq 0$,
(2) $D^u_i \subset \text{int} D^{u\prime}_i$ *for* $i = 1, \cdots, \ell$,
(3) $\Lambda_{min} \subset \cup_{i=1}^{\ell} \text{int} D^u_i$,
(4) *if* $x \in D^u_i \cap D^u_j \cap \Lambda_{min}$ *then there is a neighborhood* $\Lambda_x$ *of* $x$ *in* $\Lambda_{min}$ *such that* $\Lambda_x \subset D^{u\prime}_i \cap D^{u\prime}_j$, *and*
(5) *if* $x \in D^u_i \cap \Lambda_{min}$ *then* $E^{sc}_x(i_0) \oplus T_x D^{u\prime}_i = T_x M$ *and there are constant* $C > 0$ *and* $\lambda > 1$ *such that if* $v \in T_x D^{u\prime}_i$ *then* $\|Df^n(v)\| \geq C\lambda^n \|v\|$ *for all* $n \geq 0$.

*Proof of Theorem 1.* Let $f \in \text{int}PE^r(M)$. By Proposition 1.2, $f : M \to M$ is regular. We assume that $f : M \to M$ is not expanding, and derive a contradiction. Let $\Lambda_{min} = \Lambda_{min}(b)$ for the bounded case, and $\Lambda_{min} = \Lambda_{min}(b; f)$ for the unbounded case, as in Proposition 3.3

By Proposition 3.3 there are a $Df$-invariant continuous subbundle $E^{sc}(i_0)$ of $T_{\Lambda_{min}}M$, and finite fimilies $\{D^u_i\}_{i=1}^{\ell}$ and $\{D^{u\prime}_i\}_{i=1}^{\ell}$ of $m$-discs of class $C^r$ such that the properties in Proposition 3.3 hold. Let $D_m = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_m^2 \leq 1, x_{m+1} = \cdots = x_n = 0\}$, where $n = \dim M$. Choose charts $\varphi_i : U_i \to V_i$, $i = 1, \cdots, \ell$, of $M$ such that $U_i$ is an open neighborhood of $D^{u\prime}_i$ in $M$, $V_i$ is an open neighborhood of $D_m$ in $\mathbb{R}^n$, and

$\varphi_i(D^{u'}_i) = D_m$. By Lemma 3.2 and Proposition 3.3 (4) we can decompose $\Lambda_{min}$ into a disjoint union $\Lambda_{min} = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ of open and closed subsets such that $\Lambda_i \subset \text{int}D^{u'}_i$ for $i = 1, \cdots, \ell$. Fix $i$ with $1 \leq i \leq \ell$. Choose $W^i_1, W^i_2 \subset V_i$, which are neighborhoods of $\varphi_i(\Lambda_i)$ in $M$, such that $\overline{W^i_1} \subset W^i_2$, $\overline{W^i_2} \subset V_i$, and $W^i_2 \cap \varphi_i(\Lambda_{min} \setminus \Lambda_i) = \emptyset$. Let $\varepsilon > 0$ be sufficiently small. Let $E_m$ is the identity matrix of size $m$, and let $B$ be a diagonal matrix of size $n - m$ defined by

$$B = \begin{pmatrix} 1 - \varepsilon g(x) & & O \\ & \ddots & \\ O & & 1 - \varepsilon g(x) \end{pmatrix},$$

where $g : V_i \to \mathbb{R}$ is a $C^\infty$ function satisfying $g(x) = 1$ on $\overline{W^i_1}$ and $g(x) = 0$ on $V_i \setminus W^i_2$. Define $g_i : V_i \to V_i$ by

$$x \mapsto \begin{pmatrix} E_m & O \\ O & B \end{pmatrix} x,$$

where $O$ is the zero matrix. Then $g_i : V_i \to V_i$ is a $C^\infty$ diffeomorphism. If $x \in \varphi_i(\Lambda_i)$ then

$$D_x g_i = \begin{pmatrix} 1 & & & & & O \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1-\varepsilon & & \\ & & & & \ddots & \\ O & & & & & 1-\varepsilon \end{pmatrix},$$

and $g_i = id$ on $D_m$.

Define $g : M \to M$ by

$$g = \begin{cases} \varphi_i^{-1} \circ g_i \circ \varphi_i & \text{on } V_i \quad (i = 1, \cdots, \ell) \\ id & \text{ortherwise} . \end{cases}$$

Then we have

(1) $g = id$ on $\Lambda_{min}$,
(2) there is $0 < \tau < 1$ such that if $x \in \Lambda_i$, $1 \leq i \leq \ell$, and $v \in (T_x D^{u'}_i)^\perp$ then $\|Dg(v)\| \leq \tau \|v\|$, and
(3) $g : M \to M$ is sufficiently close to $id : M \to M$ with respect to the $C^r$ topology.

By (3), $g \circ f : M \to M$ is sufficiently close to $f : M \to M$ with respect to the $C^r$ topology, and so $g \circ f \in \text{int}PE^r(M)$. Therefore, $g \circ f : M \to M$ is positively expansive. By (1), $\Lambda_{min}$ is $g \circ f$-invariant. From (2) it follows that $\Lambda_{min}$ is a hyperbolic set of $g \circ f$ with contracting direction. Hence, by the stable manifold theorem all points in $\Lambda_{min}$ have non-trivial local stable manifolds with sufficiently small diameter, a contradiction. The proof is complete.

*Proof of Theorem 4.* If Sing$(f) \neq \emptyset$ or there exists a non-repelling periodic point of $f$, then by Proposition 1.2 and the discussion in the proof of Theorem 1 it follows that

$f$ belongs to $PE^r(M) \setminus \text{int}PE^r(M)$. Conversely, if $f \in PE^r(M) \setminus \text{int}PE^r(M)$ and $f : M \to M$ is regular, then by Theorem 1, $f : M \to M$ is not expanding. Since $\dim S^1 = 1$, from Proposition 3.3 it follows that $m = 0$, and so $\Lambda_{min}$ is a finite set, which implies that there is a non-repelling periodic point. The proof is complete.

For the details of this paper, the author hope to appear elsewhere.

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