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Invariant sets associated with critical orbits for holomorphic endomorphisms of $\mathbb{P}^2$

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This note is the abstract of my talk in the conference held at RIMS, 20-24 June 2005. The study concerns the dynamics of rational self-maps of $\mathbb{P}^2$ (the complex projective plane), mainly focusing on the case of holomorphic maps. We proceed on the basis of [U] and refer to [S] for the general theory. A forthcoming paper [M2] will contain more details.

1 Steinness of Fatou sets

Let f be a rational self-map of $\mathbb{P}^2$ of degree at least 2 which is dominant, i.e. $f(\mathbb{P}^2) = \mathbb{P}^2$. Denote by $I(f)$ the set of indeterminacy points for f, i.e. the set of points where f cannot extend to be holomorphic. The set $I(f)$ is a finite set. In case of dimension 1, every rational map has no indeterminacy points, so, when we extend the Fatou-Julia theory to a higher dimensional setting, we must consider how we deal with $I(f)$. The following regularity condition was introduced by Fornæss-Sibony.

Definition 1.1. ([S]) We say that f is *algebraically stable (AS)* if for all $n \geq 1$, the set $f^{-1}(I(f^n))$ contains no compact complex curve in $\mathbb{P}^2$. This is equivalent to $\deg(f^n) = (\deg(f))^n$ for all $n \geq 1$.

In the sequel, we suppose that f is AS. For any $m \geq 1$, the map f^m is holomorphic in $\mathbb{P}^2 \setminus \overline{\bigcup_{n\geq 1} I(f^n)}$. From a dynamical viewpoint, we can define the Fatou set to be the set of Lyapunov stable points.

Definition 1.2. We define the *Fatou set* $\mathcal{F}$ to be the maximal open subset of $\mathbb{P}^2 \setminus \overline{\bigcup_{n\geq 1} I(f^n)}$ in which $\{f^n\}_{n\geq 1}$ is locally equicontinuous. A connected component of $\mathcal{F}$ is called a *Fatou component*. The complement $\mathcal{J}$ of $\mathcal{F}$ is called the *Julia set*.

On the other hand, from a viewpoint of complex analysis, we can define Fatou sets using several notions of convergence for a sequence of meromorphic maps.

Definition 1.3. Let $\{g_n\}_{n\geq 1}$ be a sequence of meromorphic maps from an open set $D \subset \mathbb{P}^2$ to $\mathbb{P}^2$. Let $\Gamma_n \subset D \times \mathbb{P}^2$ denote the graph of g_n. Let $g : D \to \mathbb{P}^2$ be a meromorphic map and $\Gamma \subset D \times \mathbb{P}^2$ be the graph of g.

(i) We say that $\{g_n\}_{n\geq 1}$ *strongly converges* to g in D if for any compact set $K \subset D$

$$\lim_{n\to\infty} \Gamma_n \cap (K \times \mathbb{P}^2) = \Gamma \cap (K \times \mathbb{P}^2)$$

with respect to the Hausdorff metric.

(ii) We say that $\{g_n\}_{n\geq 1}$ *weakly converges* to g in D if there is an analytic subset $A \subset D$ of $\operatorname{codim}_{\mathbb{C}} A \geq 2$ such that $\{g_n\}_{n\geq 1}$ strongly converges to g in $D \setminus A$.

By (i) and (ii) above, we may introduce notions of normality for a sequence of meromorphic maps in strong and weak senses. Thus, in case of the iterates $\{f^n\}_{n\geq 1}$, we define the strong (resp. weak) Fatou set $\mathcal{F}_s$ (resp. $\mathcal{F}_w$) as the maximal open subset of $\mathbb{P}^2$ in which $\{f^n\}_{n\geq 1}$ is strongly (resp. weakly) normal.

By definition, it follows that $\mathcal{F} \subset \mathcal{F}_s \subset \mathcal{F}_w$. By combining Ivashkovich's results on the convergence of meromorphic maps to a compact Kähler manifold and Sibony's results on Green currents, the following theorem is verified.

Theorem A. *If f is a dominant AS rational self-map of $\mathbb{P}^2$ of degree at least 2,*

$$\mathcal{F} = \mathcal{F}_s = \mathcal{F}_w.$$

In particular, each Fatou component is Stein, hence, the Julia set $\mathcal{J}$ is connected.

Concerning the dynamics inside Fatou sets, we can find an interesting dynamical phenomenon which is related with indeterminacy points ([M1]).

2 Critically hyperbolic maps

Suppose that f is a holomorphic self-map of $\mathbb{P}^2$ of degree $d \geq 2$. Then, f is a d^2 to 1 branched covering. We denote by $C = C(f)$ the critical set for f. We define the critical limit set $E = E(f)$ by

$$E := \bigcap_{j\geq 1} \overline{\bigcup_{i\geq j} f^i(C)}.$$

We denote the Green (1,1) current for f by T. Since f is holomorphic, it follows that

$$\mathcal{J} = \mathcal{J}_1 := \mathrm{supp}(T).$$

Further, it is known that $T \wedge T$ is a unique invariant probability measure of maximal entropy. We set $\mathcal{J}_2 := \mathrm{supp}(T \wedge T)$.

Throughout this section, we consider a set $\Lambda = \Lambda(f)$ defined by

$$\Lambda := \bigcap_{n \geq 0} f^n(\mathcal{J}_1 \cap E \cap \Omega),$$

where Ω is the nonwandering set for f. Since $\mathcal{F}$ is Stein, the critical set C **always** intersects $\mathcal{J}_1$. This implies that Λ is nonempty.

Proposition 2.1. *The set Λ is a nonempty compact set such that $f(\Lambda) = \Lambda$. All saddle periodic points for f are contained in Λ.*

We consider the situation in which f is hyperbolic on Λ. (Concerning hyperbolic sets for non-invertible maps, see [BJ] for instance.) We are going to study the global dynamics assuming some condition on the critical orbit. Critically finite maps have been studied by several authors (Fornæss-Sibony, Ueda, Jonsson, de Thelin, ...), so here we introduce a new condition. Let $\hat{\Lambda}$ denote the space of histories of points in Λ for $f|_\Lambda : \Lambda \to \Lambda$.

Definition 2.2. We say that f is *critically hyperbolic* if Λ is a hyperbolic set for f and $\hat{\Lambda}$ has local product structure.

We find examples of critically hyperbolic maps in the class of Axiom A. In case when f satisfies Axiom A, we denote by

$$\Omega = S_0 \cup S_1 \cup S_2$$

the decomposition of the nonwandering set Ω for f according to the unstable dimensions.

Proposition 2.3. *Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree at least 2. If f satisfies Axiom A and $f^{-1}(S_2) = S_2$, then f is a critically hyperbolic map such that*

$$S_1 = \Lambda, \; S_2 = \mathcal{J}_2.$$

Remark 2.4. If f is a direct product of two hyperbolic polynomials in one variable, then f and its perturbated maps satisfy this condition.

For critically hyperbolic maps, we can establish the following theorems.

Theorem B says that each Fatou component is eventually mapped to the immediate basin of an attracting periodic orbit and the number of attracting periodic orbits is finite.

Theorem B. *Suppose f is critically hyperbolic. Then, the Fatou set $\mathcal{F}$ for f consists of the basins of attraction for finitely many attracting periodic orbits.*

For a history $\hat{p} \in \hat{\Lambda}$, we denote the unstable manifold by $W^u(\hat{p})$. Then, the critical limit set E can be discribed as follows.

Theorem C. *Suppose that f is critically hyperbolic and Λ has pure unstable dimension 1. Then,*

$$E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p}\in\hat{\Lambda}} W^u(\hat{p}).$$

The dynamics inside the Julia sets for critically hyperbolic maps will be investigated in a future article.

References

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