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AN INVITATION TO THE SIMILARITY PROBLEMS (AFTER PISIER)

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ABSTRACT. This note is intended as a handout for the minicourse given in RIMS workshop "Operator Space Theory and its Applications" on January 31, 2006.

1. THE SIMILARITY PROBLEMS

1.1. **The similarity problem for continuous homomorphisms.** In this note, we mainly consider *unital* C*-algebras and *unital* (not necessarily $*$-preserving) homomorphisms for the sake of simplicity. Let A be a unital C*-algebra and $\pi\colon A\to\mathbb{B}(\mathcal{H})$ be a unital homomorphism with $\|\pi\|<\infty$. We say that π is *similar* to a $*$-homomorphism if there exists $S\in\mathrm{GL}(\mathcal{H})$ such that $\mathrm{Ad}(S)\circ\pi$ is a $*$-homomorphism. Here, $\mathrm{GL}(\mathcal{H})$ is the set of invertible element in $\mathbb{B}(\mathcal{H})$ and $\mathrm{Ad}(S)(x)=SxS^{-1}$.

Similarity Problem A (Kadison 1955). Is every continuous homomorphism similar to a $*$-homomorphism?

We note that a homomorphism π is a $*$-homomorphism iff $\|\pi\|=1$, since an element $x\in\mathbb{B}(\mathcal{H})$ is unitary iff $\|x\|=\|x^{-1}\|=1$. We say A has the *similarity property* (abbreviated as (SP)) if every unital continuous homomorphism from A into $\mathbb{B}(\mathcal{H})$ is similar to a $*$-homomorphism. Do we really need the assumption that π is continuous? That is another problem. Indeed, the subject of automatic continuity is extensively studied in Banach algebra theory, and it is known that the existence of a *discontinuous* homomorphism from a C*-algebra into some Banach algebra is independent of (ZFC). As far as the author knows, it is not known whether or not the automatic continuity of a homomorphism between C*-algebras (say, with a dense image) is provable within (ZFC).

Similarity Problem A is equivalent to several long-standing problems in C*, von Neumann and operator theories. Among them is the Derivation Problem;

Derivation Problem. Is every derivation $\delta\colon A\to\mathbb{B}(\mathcal{H})$ inner?

Let $A\subset\mathbb{B}(\mathcal{H})$ be a (unital) C*-algebra. A *derivation* $\delta\colon A\to\mathbb{B}(\mathcal{H})$ is a linear map which satisfies the derivative identity $\delta(ab)=\delta(a)b+a\delta(b)$. The celebrated theorem of Kadison and Sakai is that every derivation into A'' is inner. We recall

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that $\delta\colon A \to \mathbb{B}(\mathcal{H})$ is said to be *inner* if there exists $T \in \mathbb{B}(\mathcal{H})$ such that

$$\forall a \in A \quad \delta(a) = \delta_T(a) := Ta - aT.$$

It is known that every derivation is automatically continuous (Ringrose). We say A has the (DP) if any derivation $\delta\colon A \to \mathbb{B}(\mathcal{H})$, for any faithful $*$-representation $A \subset \mathbb{B}(\mathcal{H})$, is inner.

Theorem 1.1 (Kirchberg 1996). *Let A be a unital C^*-algebra. Then A has the* (SP) *iff A has the* (DP).

The easier implication (SP) $\Rightarrow$ (DP) (which precedes Kirchberg) follows from the following lemma.

Lemma 1.2. *Let $A \subset \mathbb{B}(\mathcal{H})$ be a unital C^*-algebra and $\delta\colon A \to \mathbb{B}(\mathcal{H})$ be a derivation. Then the homomorphism $\pi\colon A \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ defined by*

$$\pi(a) = \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}$$

is similar to a $*$-homomorphism iff δ is inner.

Proof. We first observe that π is indeed a homomorphism since δ is a derivation. If $\delta = \delta_T$, then we have

$$\pi(a) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix}$$

and π is similar to a $*$-homomorphism $\mathrm{id}_A \oplus \mathrm{id}_A$. We now suppose that $\sigma(a) = S\pi(a)S^{-1}$ is a $*$-homomorphism. Let $D = S^*S$. Since

$$\|S^{-1}\|^2\langle D\xi, \xi\rangle = \|S^{-1}\|^2\|S\xi\|^2 \geq \|\xi\|^2,$$

we have $D \geq \|S^{-1}\|^{-2}$. Since σ is $*$-preserving, we have

$$D\pi(a) = S^*\sigma(a)S = (S^*\sigma(a^*)S)^* = \pi(a^*)^*D$$

for every $a \in A$. Developing the equation, we get

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ \delta(a^*)^* & a \end{pmatrix}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

Looking at the $(1,1)$-entry, we have $D_{11}a = aD_{11}$ for every $a \in A$. Combined with $D_{11} \geq \|S^{-1}\|^{-2}$, this implies that $D_{11}^{-1} \in A'$ with $\|D_{11}^{-1}\| \leq \|S^{-1}\|^2$. Looking at the $(2,1)$-entry, we have

$$D_{11}\delta(a) + D_{12}a = aD_{12}.$$

It follows that $\delta = \delta_T$ for $T = -D_{11}^{-1}D_{12}$ with $\|T\| \leq \|S\|^2\|S^{-1}\|^2$. □

1.2. **Known cases and open cases.** The important result of Haagerup (1983) is that a continuous homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H})$ admitting a finite cyclic subset (i.e., there exists a finite subset $\mathcal{F} \subset \mathcal{H}$ such that $\mathrm{span}\{\pi(a)\xi : a \in A,\ \xi \in \mathcal{F}\}$ is dense in $\mathcal{H}$), is inner. This does not finish the similarity problem since we cannot decompose a general (non $*$-preserving) representation into a direct sum of cyclic representations.

Theorem 1.3. *The following C^*-algebras have the* (SP).

(1) *Nuclear C^*-algebras.*
(2) *C^*-algebras without tracial states (Haagerup).*
(3) *Type II_1 factors with the property* (Γ) *(Christensen).*

We note that one may reduce Similarity problem A (or derivation problem) for C*-algebras to that for type II_1 factors by considering the second dual, then considering the type decomposition and direct integration. We do not know whether or not the von Neumann algebras $\mathcal{L}\mathbb{F}_2$ and $\prod_{n=1}^{\infty} \mathbb{M}_n$ have the (SP). We suspect that $\prod_{n=1}^{\infty} \mathbb{M}_n$ should be a counterexample.

1.3. **The similarity problem for group representations.** We only consider discrete groups. Let Γ be a discrete group and $C^*\Gamma$ be the full group C*-algebra. We regard Γ as the corresponding subgroup of unitary elements in $C^*\Gamma$. Every continuous homomorphism $\pi\colon C^*\Gamma \to \mathbb{B}(\mathcal{H})$ gives rise to a uniformly bounded (abbreviated as u.b.) representation of Γ on $\mathcal{H}$; $\pi\colon \Gamma \to \mathrm{GL}(\mathcal{H})$ is a group homomorphism such that $\|\pi\| := \sup_{s\in\Gamma} \|\pi(s)\| < \infty$[1]. Obviously, the homomorphism $\pi\colon C^*\Gamma \to \mathbb{B}(\mathcal{H})$ is similar to a $*$-homomorphism iff the representation $\pi_{|\Gamma}$ is *unitarizable* (i.e., $\exists S \in \mathrm{GL}(\mathcal{H})$ such that $\mathrm{Ad}(S) \circ \pi_{|\Gamma}$ is a unitary representation).

Theorem 1.4 (Diximier 1950). *Let Γ be an amenable group. Then, every u.b. representation of Γ is unitarizable. More precisely, if $\pi\colon \Gamma \to \mathrm{GL}(\mathcal{H})$ is a u.b. representation, then there exists $S \in \mathrm{GL}(\mathcal{H}) \cap \mathrm{vN}(\pi(\Gamma))$ with $\|S\|\,\|S^{-1}\| \le \|\pi\|^2$ such that $\mathrm{Ad}(S) \circ \pi$ is unitary.*

Proof. Let Γ be amenable and $\pi\colon \Gamma \to \mathrm{GL}(\mathcal{H})$ be a u.b. representation. Let $F_n \subset \Gamma$ be a Følner net. Since π is u.b., the set $|F_n|^{-1} \sum_{s\in F_n} \pi(s)^*\pi(s) \in \mathrm{vN}(\pi(\Gamma))$ has a weak*-accumulation point. Since the accumulation point is positive, we let S be the the square root of it. Then, we have

$$\|S\xi\|^2 = \lim_n \frac{1}{|F_n|} \sum_{s\in F_n} \|\pi(s)\xi\|^2,$$

and hence $\|\pi\|^{-1} \le S \le \|\pi\|$ and $\|S\pi(s)\xi\| = \|S\xi\|$ for every $s \in \Gamma$ and $\xi \in \mathcal{H}$. It follows that $\|\mathrm{Ad}(S) \circ \pi\| = 1$ and hence $\mathrm{Ad}(S) \circ \pi$ is unitary. □

[1]This notation may cause confusion since the value $\|\pi\|$ is not same as $\|\pi\colon C^*\Gamma \to \mathbb{B}(\mathcal{H})\|$.

If one employ the fact that a nuclear C*-algebra is amenable as a Banach algebra (Haagerup 1983), then we can adopt the above proof to the case of nuclear C*-algebras. We say Γ is *unitarizable* if every u.b. representation of Γ is unitarizable. Pisier (2004, 2005) proved that if Γ is unitarizable and in addition that the similarity S can be chosen so that (i) $S \in \mathrm{GL}(\mathcal{H}) \cap \mathrm{vN}(\pi(\Gamma))$, or (ii) $\|S\|\,\|S^{-1}\| \leq \|\pi\|^2$, then Γ is amenable. However, the following is still open.

Similarity Problem B. Is every unitarizable group amenable?

Theorem 1.5. *The free group $\mathbb{F}_\infty$ on countably many generators is not unitarizable.*

Proof. We denote by $|t|$ the word length of $t \in \mathbb{F}_\infty$, by $\mathbb{C}\mathbb{F}_\infty$ the space of all finitely supported $\mathbb{C}$-valued functions on $\mathbb{F}_\infty$, and by $\lambda(s)$ the left translation operator by s on $\ell_\infty\Gamma$ (and its subspaces). Let $B\colon \mathbb{C}\mathbb{F}_\infty \to \ell_\infty\mathbb{F}_\infty$ be the linear map defined by

$$B\delta_t = \sum\{\delta_{t'} : |t^{-1}t'| = 1,\ |t'| = |t| + 1\},$$

i.e., $B\delta_t$ is the characteristic function of those points which are just one-step ahead of t (looking from e). Then, for every $s \in \mathbb{F}_\infty$, we have

$$(B\lambda(s) - \lambda(s)B)\delta_t = \begin{cases} 0 & \text{if } |s| \neq |st| + |t^{-1}| \\ \delta_{s(|st|+1)} - \delta_{s(|st|-1)} & \text{if } |s| = |st| + |t^{-1}| \end{cases},$$

where $s(k)$ is the unique element such that $|s(k)| = k$ and $|s| = |s(k)| + |s(k)^{-1}s|$. Hopefully, the figures below explain the above equation. It follows that we may

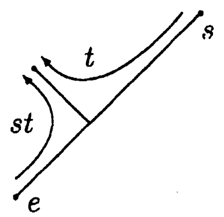

FIGURE 1. $|s| \neq |st| + |t^{-1}|$

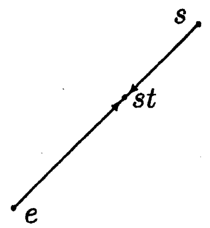

FIGURE 2. $|s| = |st| + |t^{-1}|$

view $D(s) = B\lambda(s) - \lambda(s)B$ as an element in $\mathbb{B}(\ell_2\mathbb{F}_\infty)$ with $\|D(s)\| = 2$. Thus, $D\colon \mathbb{F}_\infty \to \mathbb{B}(\ell_2\mathbb{F}_\infty)$ is a u.b. derivation; $D(st) = D(s)\lambda(t) + \lambda(s)D(t)$. It is not hard to show that D is not inner, i.e., there is no $B_0 \in \mathbb{B}(\ell_2\mathbb{F}_\infty)$ such that $B - B_0$ commutes with every $\lambda(s)$ (in $L(\mathbb{C}\mathbb{F}_\infty, \ell_\infty\mathbb{F}_\infty)$). We define a u.b. representation $\pi\colon \mathbb{F}_\infty \to \mathbb{M}_2(\mathbb{B}(\ell_2\mathbb{F}_\infty))$ by

$$\pi(s) = \begin{pmatrix} \lambda(s) & D(s) \\ 0 & \lambda(s) \end{pmatrix}.$$

We conclude the proof by using the fact, which is proved in the same way to Lemma 1.2, that π is similar to $*$-homomorphism only if D is inner. $\square$

We observe that a subgroup of a unitarizable group is again unitarizable thanks to the fact that the induction of a u.b. representation is again u.b. (and a little more effort). Hence a counterexample (if any) to Similarity Problem B has to be a non-amenable group which does not contain $\mathbb{F}_2$ as a subgroup. Do you think this might be a good time to stop chasing the problem?

2. Isomorphic Characterization of Injectivity

2.1. **A free Khinchine inequality.** Let Γ be a discrete group and $\mathcal{L}\Gamma$ be its group von Neumann algebra. By definition, the map

$$\mathcal{L}\Gamma \ni \lambda(f) \mapsto f = \lambda(f)\delta_e \in \ell_2\Gamma$$

is contractive. For which operator space structure on $\ell_2\Gamma$, does the above map completely bounded? We briefly review the column and row Hilbert space structures. Let $\mathcal{H}$ be a Hilbert space. When it is viewed as a column vector space, we say it is a column Hilbert space and denote it by $\mathcal{H}_C$, i.e., $\mathcal{H}_C = \mathbb{B}(\mathbb{C}, \mathcal{H})$ as an operator space. For any finite sequence[2] $(x_i)_i$ in $\mathbb{B}(H)$ and orthonormal vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have

$$\|(x_i)_i\|_C := \|\sum_i x_i \otimes \xi_i\|_{\mathbb{B}(H)\otimes\mathcal{H}_C} = \left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \right\| = \|\sum_i x_i^* x_i\|^{1/2}.$$

Likewise, we define the row Hilbert space as $\mathcal{H}_R = \mathbb{B}(\overline{\mathcal{H}}, \mathbb{C})$, where $\overline{\mathcal{H}}$ is the conjugate Hilbert space of $\mathcal{H}$. For any finite sequence (x_i) in $\mathbb{B}(H)$ and orthonormal vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have

$$\|(x_i)_i\|_R := \|\sum_i x_i \otimes \xi_i\|_{\mathbb{B}(H)\otimes\mathcal{H}_R} = \| \begin{pmatrix} x_1 & x_2 & \cdots \end{pmatrix} \| = \|\sum_i x_i x_i^*\|^{1/2}.$$

We regard the following lemma trivial and use it without referring it.

Lemma 2.1. *For any finite sequences $(a_i)_i$ and $(b_i)_i$ in $\mathbb{B}(\mathcal{H})$, we have*

$$\|\sum_i a_i b_i\| \le \|(a_i)_i\|_R \|(b_i)_i\|_C.$$

In particular, $\|\sum a_i \otimes b_i\| \le \min\{ \|(a_i)_i\|_R \|(b_i)_i\|_C, \|(a_i)_i\|_C \|(b_i)_i\|_R\}$.

We define $\mathcal{H}_{C\cap R} = \{\xi \oplus \xi \in \mathcal{H}_C \oplus \mathcal{H}_R : \xi \in \mathcal{H}\}$.

Proposition 2.2. *The map*

$$\mathcal{L}\Gamma \ni \lambda(f) \mapsto f \in (\ell_2\Gamma)_{C\cap R}$$

is completely contractive.

[2]A finite sequence is a sequence of vectors such that all but finitely many are zero

Proof. We view $\delta_e \in \mathbb{B}(\mathbb{C}, \ell_2\Gamma)$ and $\delta_e^* \in \mathbb{B}(\overline{\ell_2\Gamma}, \mathbb{C})$. Since $f = \lambda(f)\delta_e \in \mathbb{B}(\mathbb{C}, \ell_2\Gamma)$, the above map is a complete contraction into $\mathcal{H}_C$. Since $f = \delta_e^*\overline{\lambda}(f) \in \mathbb{B}(\overline{\ell_2\Gamma}, \mathbb{C})$, the above map is a complete contraction into $\mathcal{H}_R$ as well. □

We simply write $C \cap R$ for $(\ell_2)_{C\cap R}$ and $\{\theta_i\}$ for a fixed orthonormal basis for $C \cap R$. For instance, we can take $\theta_i = e_{i1} \oplus e_{1i} \in \mathbb{B}(\ell_2) \oplus \mathbb{B}(\ell_2)$. For a finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we set

$$\|(x_i)_i\|_{C\cap R} = \|\sum_i x_i \otimes \theta_i\|_{\mathbb{B}(\mathcal{H})\otimes(C\cap R)} = \max\{\|(x_i)_i\|_C, \|(x_i)_i\|_R\}.$$

The following is the rudiment of free Khinchine inequalities.

Theorem 2.3 (Haagerup and Pisier 1993). *Let $\mathbb{F}_\infty$ be the free group on countable generators, $\mathcal{S} = \{s_i\} \subset \mathbb{F}_\infty$ be the standard set of free generators and*

$$E_\lambda = \overline{\mathrm{span}}\{s_i\} \subset \mathcal{L}\mathbb{F}_\infty$$

be an operator subspace. Then, the map

$$\Phi\colon C \cap R \ni \theta_i \mapsto \lambda(s_i) \in \mathcal{L}\mathbb{F}_\infty$$

is completely bounded with $\|\Phi\|_{\mathrm{cb}} \leq 2$. In particular, the projection Q from $\mathcal{L}\mathbb{F}_\infty$ onto E_λ, defined by

$$Q\colon \mathcal{L}\mathbb{F}_\infty \ni \lambda(s) \mapsto \begin{cases} \lambda(s) & \text{if } s \in \mathcal{S} \\ 0 & \text{if } s \notin \mathcal{S} \end{cases},$$

is completely bounded with $\|Q\|_{\mathrm{cb}} \leq 2$.

Proof. For each i, let $\Omega_i^\pm \subset \mathbb{F}_\infty$ be the subsets of all reduced words which begins with respectively $s_i^{\pm 1}$, and $P_i^\pm \in \mathbb{B}(\ell_2\mathbb{F}_\infty)$ be the orthogonal projection onto $\ell_2\Omega_i^\pm$. Then, for each i, we have

$$\lambda(s_i) = \lambda(s_i)P_i^- + \lambda(s_i)(1 - P_i^-) = \lambda(s_i)P_i^- + P_i^+\lambda(s_i).$$

Therefore for any finite sequence $(x_i)_i \subset \mathbb{B}(H)$, we have

$$\|\sum_i x_i \otimes \lambda(s_i)P_i^-\|_{\mathbb{B}(H\otimes\ell_2\mathbb{F}_\infty)} \leq \|(x_i)_i\|_R \|(\lambda(s_i)P_i^-)_i\|_C \leq \|(x_i)_i\|_R$$

since $\|(\lambda(s_i)P_i^-)_i\|_C = \|\sum_i P_i^-\|^{1/2} = 1$. Likewise, we have

$$\|\sum_i x_i \otimes P_i^+\lambda(s_i)\|_{\mathbb{B}(H\otimes\ell_2\mathbb{F}_\infty)} \leq \|(x_i)_i\|_C \|(P_i^+\lambda(s_i))_i\|_R \leq \|(x_i)_i\|_C.$$

It follows that

$$\|\sum_i x_i \otimes \lambda(s_i)\|_{\mathbb{B}(H\otimes\ell_2\mathbb{F}_\infty)} \leq 2\|(x_i)_i\|_{C\cap R} = 2\|\sum_i x_i \otimes \theta_i\|.$$

This means that $\|\Phi\|_{\mathrm{cb}} \leq 2$. The second assertion follows from Proposition 2.2. □

Remark 2.4. The above property of $\mathcal{L}\mathbb{F}_\infty$ is related to the fact that $\mathcal{L}\mathbb{F}_\infty$ is not injective. We simply write E_n for $(\ell_2^n)_{C\cap R}$. Thus

$$E_n = \mathrm{span}\{e_{i1} \oplus e_{1i} : i = 1, \dots, n\} \subset \mathbb{M}_n \oplus \mathbb{M}_n.$$

It is known that E_n is far from injective, i.e., any projection from $\mathbb{M}_n \oplus \mathbb{M}_n$ onto E_n has cb-norm $\geq \frac{1}{2}(\sqrt{n}+1)$. It follows that if M is an injective von Neumann algebra, then any maps $\alpha\colon E_n \to M$ and $\beta\colon M \to E_n$ with $\beta\circ\alpha = \mathrm{id}_{E_n}$ satisfy $\|\alpha\|_{\mathrm{cb}}\|\beta\|_{cb} \geq \frac{1}{2}(\sqrt{n}+1)$. It is conjectured by Pisier(?) that for any non-injective von Neumann algebra M, there exist sequences of maps $\alpha_n\colon E_n \to M$ and $\beta_n\colon M \to E_n$ such that $\beta_n \circ \alpha_n = \mathrm{id}_{E_n}$ and $\sup \|\alpha_n\|_{\mathrm{cb}}\|\beta_n\|_{cb} < \infty$. An affirmative answer would solve several problems around operator spaces (e.g., whether existence of a bounded linear projection from $\mathbb{B}(\mathcal{H})$ onto M implies injectivity of M.) A negative answer would lead to a non-injective type II_1 factor which does not contain $\mathcal{L}\mathbb{F}_2$.

2.2. **Isomorphic characterization of injective von Neumann algebras.** For a finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we set

$$\|(x_i)_i\|_{C+R} = \|\Phi\colon C\cap R \ni \theta_i \mapsto x_i \in \mathbb{B}(\mathcal{H})\|_{\mathrm{cb}}.$$

We say that a von Neumann algebra M has the *property* (P)[3] if there exists a constant $C_M > 0$ with the following property; For any finite sequence $(x_i)_i$ in M with $\|(x_i)_i\|_{C+R} \leq 1$, there exist finite sequences $(a_i)_i$ and $(b_i)_i$ in M such that

$$\|(a_i)_i\|_C \leq C_M,\ \|(b_i)_i\|_R \leq C_M \text{ and } x_i = a_i + b_i \text{ for every } i.$$

Theorem 2.5 (Pisier 1994). *A von Neumann algebra M is injective iff it has the property* (P).

The "if" part requires several lemmas, and we first prove the "only if" part. Let M be an injective von Neumann algebra and consider a complete contraction $\Phi\colon C\cap R \ni \theta_i \mapsto x_i \in M$. Since M is injective, this map extends to a complete contraction $\tilde{\Phi}\colon C \oplus R \to M$, where $C = \overline{\mathrm{span}}\{e_{i1}\}$ and $R = \overline{\mathrm{span}}\{e_{1i}\}$. Then $a_i = \tilde{\Phi}(0 \oplus e_{1i})$ and $b_i = \tilde{\Phi}(e_{i1} \oplus 0)$ satisfies the required condition with $C_M = 1$. We note that $\|(\varphi(a_i))_i\|_C \leq \|\varphi\|_{\mathrm{cb}}\|(a_i)_i\|_C$ for any cb-map φ and any finite sequence $(a_i)_i$. Hence the following is trivial.

Lemma 2.6. *The property* (P) *inherits to a von Neumann subalgebra which is the range of a completely bounded projection.*

As a corollary to Theorem 2.5, we see that a von Neumann subalgebra $M \subset \mathbb{B}(\mathcal{H})$ which is the range of a completely bounded projection is in fact injective. We observe that by the type decomposition and the Takesaki duality, it suffices to show Theorem 2.5 for a von Neumann algebra of type II_1.

Let $M \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra. An *M-central state* is a state φ on $\mathbb{B}(\mathcal{H})$ such that $\varphi(uxu^*) = \varphi(x)$ for $u \in M$ and $x \in \mathbb{B}(\mathcal{H})$ (or equivalently

[3]This nomenclature is nonstandard.

$\varphi(ax) = \varphi(xa)$ for $a \in M$ and $x \in \mathbb{B}(\mathcal{H})$). Recall that the celebrated theorem of Connes states that a finite von Neumann algebra M is injective iff there exists an M-central state φ such that $\varphi_{|M}$ is a faithful normal tracial state.

Lemma 2.7. *Let $M \subset \mathbb{B}(\mathcal{H})$. Then, there exists an M-central state if*

$$\|\sum_{i=1}^{n} u_i \otimes \overline{u_i}\|_{\mathbb{B}(\mathcal{H}\otimes\overline{\mathcal{H}})} = n$$

for every n and unitary elements $u_1, \ldots, u_n \in M$.

Proof. We first recall that $\overline{\mathcal{H}}$ is the complex conjugate Hilbert space of $\mathcal{H}$ and $\overline{x} \in \mathbb{B}(\overline{\mathcal{H}})$ means the element associated with $x \in \mathbb{B}(\mathcal{H})$. We have the canonical identification between the Hilbert space $\mathcal{H}\otimes\overline{\mathcal{H}}$ and the space $\mathcal{S}_2(\mathcal{H})$ of the Hilbert-Schmidt class operators on $\mathcal{H}$, given by $\xi \otimes \overline{\eta} \leftrightarrow \langle\cdot,\eta\rangle\xi \in \mathcal{S}_2(\mathcal{H})$. Under this identification, $\sum a_i \otimes \overline{b_i}$ acts on $\mathcal{S}_2(\mathcal{H})$ as $\mathcal{S}_2(\mathcal{H}) \ni h \mapsto \sum a_i h b_i^* \in \mathcal{S}_2(\mathcal{H})$.

Let $u_1, \ldots, u_n \in M$ be unitary elements such that $u_1 = 1$. If $\|\sum_{i=1}^{n} u_i \otimes \overline{u_i}\| = n$, then there exists a unit vector $h \in \mathcal{S}_2(\mathcal{H})$ such that $\|\sum_{i=1}^{n} u_i h u_i^*\|_2 \approx n$. By uniform convexity, we must have $\|u_i h u_i^* - h\|_2 \approx 0$ for every i. This implies that $\|u_i h^* h u_i - h^* h\|_1 \approx 0$ for every i. It follows that $\varphi(x) = \mathrm{Tr}(h^* h x)$ defines a state on $\mathbb{B}(\mathcal{H})$ such that $\|\varphi\circ \mathrm{Ad}(u_i) - \varphi\|_{\mathbb{B}(\mathcal{H})_*} \approx 0$ for every i. Therefore, taking appropriate limit, we can obtain an M-central state. □

Lemma 2.8 (Haagerup 1985). *Let M be a von Neumann algebra. Assume that there exists a constant $c > 0$ with the following property; For every n, unitary elements $u_1, \ldots, u_n \in M$ and every non-zero central projection $p \in M$, we have*

$$\|\sum_{i=1}^{n} pu_i \otimes \overline{pu_i}\|_{\mathbb{B}(p\mathcal{H}\otimes\overline{p\mathcal{H}})} \geq cn.$$

Then, M is injective.

Proof. Let $u_1, \ldots, u_n \in M$ be unitary elements and $p \in M$ be a non-zero central projection. By assumption, we have

$$\|(\sum_{i=1}^{n} pu_i \otimes \overline{pu_i})^k\|_{\mathbb{B}(p\mathcal{H}\otimes\overline{p\mathcal{H}})} \geq cn^k$$

for every positive integer k. Therefore, we actually have that

$$\|\sum_{i=1}^{n} pu_i \otimes \overline{pu_i}\|_{\mathbb{B}(p\mathcal{H}\otimes\overline{p\mathcal{H}})} \geq \lim_{k\to\infty} c^{1/k} n = n.$$

By Lemma 2.7, there exists a pM-central state φ_p on $\mathbb{B}(pM)$ for every non-zero central projection $p \in M$. Fix a normal faithful tracial state τ on M. For any finite partition $\mathcal{P} = \{p_i\}_i$ of unity by central projections in M, we define the M-central state $\varphi_{\mathcal{P}}$ on $\mathbb{B}(\mathcal{H})$ by

$$\varphi_{\mathcal{P}}(x) = \sum_i \tau(p_i)\varphi_{p_i}(p_i x p_i).$$

Taking appropriate limit of $\varphi_{\mathcal{P}}$, we obtain an M-central state φ on $\mathbb{B}(\mathcal{H})$ such that $\varphi_{|M} = \tau$. We conclude that M is injective by Connes's theorem. □

For a finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we set

$$\|(x_i)_i\|_{OH} = \|\sum_i x_i \otimes \overline{x_i}\|^{1/2}_{\mathbb{B}(\mathcal{H}\otimes\overline{\mathcal{H}})}.$$

We note that $\|(x_i)_i\|_{OH} \leq \|(x_i)_i\|_R^{1/2}\|(x_i)_i\|_C^{1/2} \leq \|(x_i)_i\|_{C\cap R}$. Besides those appearing in Lemma 2.1, we have the following mysterious inequality (which manifests the self-dual property of the operator Hilbert spaces).

Lemma 2.9. *For every finite sequences $(a_i)_i$ in $\mathbb{B}(\mathcal{H})$ and $(b_i)_i$ in $\mathbb{B}(\mathcal{K})$, we have*

$$\|\sum_i a_i \otimes b_i\|_{\mathbb{B}(\mathcal{H}\otimes\mathcal{K})} \leq \|(a_i)_i\|_{OH}\|(b_i)_i\|_{OH}$$

Proof. We may assume that $\mathcal{K} = \overline{\mathcal{H}}$ and use $\overline{b_i}$ in the place of b_i. Identifying $\mathcal{H}\otimes\overline{\mathcal{H}}$ with $\mathcal{S}_2(\mathcal{H})$ as in the proof of Lemma 2.7, we see

$$\|\sum_i a_i \otimes \overline{b_i}\|_{\mathbb{B}(\mathcal{H}\otimes\overline{\mathcal{H}})} = \sup\{|\sum_i \mathrm{Tr}(ha_ikb_i^*)| : h, k \in \mathcal{S}_2(\mathcal{H}) \text{ with norm } 1\}.$$

Let $h, k \in \mathcal{S}_2(\mathcal{H})$ with norm 1 be given. Then, we can find decompositions $h = h_1h_2$ and $k = k_1k_2$ such that $h_j, k_j \in \mathcal{S}_4(\mathcal{H})$ with norm 1. It follows that

$$\begin{aligned}|\sum_i \mathrm{Tr}(ha_ikb_i^*)| &= |\sum_i \mathrm{Tr}((h_2a_ik_1)(k_2b_i^*h_1))| \\ &\leq \mathrm{Tr}(\sum_i h_2a_ik_1k_1^*a_i^*h_2^*)^{1/2}\,\mathrm{Tr}(\sum_i h_1^*b_ik_2^*k_2b_i^*h_1)^{1/2} \\ &\leq \|\sum_i a_i\otimes\overline{a_i}\|^{1/2}_{\mathbb{B}(\mathcal{H}\otimes\overline{\mathcal{H}})}\|\sum_i b_i\otimes\overline{b_i}\|^{1/2}_{\mathbb{B}(\mathcal{H}\otimes\overline{\mathcal{H}})}.\end{aligned}$$

This proves the assertion. □

Lemma 2.10. *For every finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we have*

$$\|(x_i)_i\|_{C+R} \leq \|(x_i)_i\|_{OH}.$$

Proof. Let $\Phi\colon C\cap R \ni \theta_i \mapsto x_i \in \mathbb{B}(\mathcal{H})$ and take $z = \sum_i a_i \otimes \theta_i \in \mathbb{B}(\mathcal{H}) \otimes (C\cap R)$. We note that $\|z\| = \|(a_i)_i\|_{C\cap R} \geq \|(a_i)_i\|_{OH}$. Hence, by Lemma 2.9, we have

$$\|(\mathrm{id}\otimes\Phi)(z)\| = \|\sum a_i \otimes x_i\| \leq \|(a_i)_i\|_{OH}\|(x_i)_i\|_{OH} \leq \|(x_i)_i\|_{OH}\|z\|.$$

This implies that $\|(x_i)_i\|_{C+R} = \|\Phi\|_{\mathrm{cb}} \leq \|(x_i)_i\|_{OH}$. □

We have prepared enough lemmas for the proof of Theorem 2.5.

Proof of Theorem 2.5. It is left to show that a finite von Neumann algebra M with the property (P) is injective. To verify the assumption of Lemma 2.8, we give ourselves unitary elements $u_1, \dots, u_n \in M$, a non-zero central projection $p \in M$ and a constant $c > 0$ such that

$$\|(pu_i)_i\|_{OH}^2 \leq cn.$$

Then, by Lemma 2.10 and the property (P), there exist $(a_i)_i$ and $(b_i)_i$ in M such that $\|(a_i)_i\|_C \leq C_M\sqrt{cn}$, $\|(b_i)_i\|_R \leq C_M\sqrt{cn}$ and $pu_i = a_i + b_i$ for every i. We fix a tracial state on pM and denote by $\|\cdot\|_2$ the corresponding 2-norm. It follows that

$$n = \sum_{i=1}^{n} \|pu_i\|_2^2 \leq 2\sum_{i=1}^{n}(\|a_i\|_2^2 + \|b_i\|_2^2) \leq 2(\|(a_i)_i\|_C^2 + \|(b_i)_i\|_R^2) \leq 2C_M^2 cn.$$

Therefore, we have $c \geq (2C_M^2)^{-1}$ and we are done. □

2.3. **A characterization of nuclearity.** Let A be a (unital) C*-algebra. We say A has the *strong similarity property* (abbreviated as (SSP)) if for every unital continuous homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H})$, there exists $S \in \mathrm{GL}(\mathcal{H}) \cap \mathrm{vN}(\pi(A))$ such that $\mathrm{Ad}(S) \circ \pi$ is a $*$-homomorphism.

Theorem 2.11 (Pisier 2005). *A C*-algebra A is nuclear iff it has the* (SSP).

Proof. As we remarked, the "only if" part follows from Diximier's proof + the amenability of nuclear C*-algebra. To prove the "if" part, let A be a C*-algebra with the (SSP). By a standard direct sum argument, it is not hard to see that there exists a constant $C > 0$ with the following property; Every unital continuous homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H})$ with $\|\pi\| \leq 5$[4], there exists $S \in \mathrm{GL}(\mathcal{H}) \cap \mathrm{vN}(\pi(A))$ with $\|S\|\,\|S^{-1}\| \leq C$ such that $\mathrm{Ad}(S) \circ \pi$ is a $*$-homomorphism. Let $A \subset \mathbb{B}(\mathcal{H})$ be a universal $*$-representation. It suffices to show that A' is injective. Let $(x_i)_i$ be a finite sequence in A' with $\|(x_i)_i\|_{C+R} \leq 1$. Since $\mathbb{B}(\mathcal{H})$ is injective, there exist $(c_i)_i$ and $(d_i)_i$ in $\mathbb{B}(\mathcal{H})$ such that $\|(c_i)_i\|_C \leq 1$, $\|(d_i)_i\|_R \leq 1$ and $x_i = c_i + d_i$ for every i. We define a derivation $\delta\colon A \to \mathbb{B}(\mathcal{H})\bar{\otimes}\mathcal{L}\mathbb{F}_\infty$ by

$$\delta(a) = \delta_{\sum c_i \otimes \lambda(s_i)}(a \otimes 1) = \sum_i \delta_{c_i}(a) \otimes \lambda(s_i) \in \mathbb{B}(\mathcal{H}) \otimes E_\lambda \subset \mathbb{B}(\mathcal{H})\bar{\otimes}\mathcal{L}\mathbb{F}_\infty.$$

We recall from the proof of Theorem 2.3 that $\lambda(s_i) = u_i + v_i$ with $\|(u_i)\|_C \leq 1$ and $\|(v_i)\|_R \leq 1$. Since $\delta_{c_i} = \delta_{-d_i}$ on A, we have $\delta = \delta_B$, where $B = \sum(c_i \otimes v_i - d_i \otimes u_i)$

[4]We can choose any other number that is strictly greater than 1 by scaling the δ later.

with $\|B\| \le \|(c_i)_i\|_C\|(v_i)\|_R + \|(d_i)_i\|_R\|(u_i)\|_C \le 2$. Hence, we have $\|\delta\|_{\mathrm{cb}} \le 4$. We define a homomorphism $\pi\colon A \to \mathbb{M}_2(\mathbb{B}(\mathcal{H})\bar{\otimes}\mathcal{L}\mathbb{F}_\infty)$ by

$$\pi(a) = \begin{pmatrix} a\otimes 1 & \delta(a) \\ 0 & a\otimes 1 \end{pmatrix}.$$

By the assumption on the (SSP), there exists an invertible element $S \in \mathrm{vN}(\pi(A))$ with $\|S\|\,\|S^{-1}\| \le C$ such that $\mathrm{Ad}(S)\circ\pi$ is a $*$-homomorphism. By the proof of Lemma 1.2, there exists $T \in \mathbb{B}(\mathcal{H})\bar{\otimes}\mathcal{L}\mathbb{F}_\infty$ with $\|T\| \le C^2$ such that $\delta(a) = \delta_T(a\otimes 1)$. Let $Q\colon \mathcal{L}\mathbb{F}_\infty \to E_\lambda$ be the projection appearing in Theorem 2.3. Since $\delta(A) \subset \mathbb{B}(\mathcal{H}) \otimes E_\lambda$ and $\mathrm{id}\otimes Q$ is A-linear, we have

$$\delta(a) = (\mathrm{id}\otimes Q)(\delta(a)) = \delta_{(\mathrm{id}\otimes Q)(T)}(a\otimes 1)$$

for every $a \in A$. We write $(\mathrm{id}\otimes Q)(T) = \sum z_i \otimes \lambda(s_i)$. Then, by Lemma 2.1 and Theorem 2.3, we have

$$\|(z_i)_i\|_{C\cap R} \le \|(\mathrm{id}\otimes Q)(T)\| \le \|Q\|_{\mathrm{cb}}\|T\| \le 2C^2.$$

Since $\lambda(s_i)$'s are linearly independent, we have $\delta_{c_i} = \delta_{z_i}$, or equivalently $c_i - z_i \in A'$. Therefore, we have $a_i = c_i - z_i \in A'$ with

$$\|(a_i)_i\|_C \le \|(c_i)_i\|_C + \|(z_i)_i\|_C \le 1 + 2C^2,$$

and likewise $b_i = x_i - a_i = d_i + z_i \in A'$ with $\|(b_i)_i\|_R \le 1 + 2C^2$. We conclude the injectivity of A' by Theorem 2.5. □

We say a group Γ has the (SSP) if for every u.b. representation $\pi\colon \Gamma \to \mathrm{GL}(\mathcal{H})$, there exists $S \in \mathrm{GL}(\mathcal{H})\cap\mathrm{vN}(\pi(\Gamma))$ such that $\mathrm{Ad}(S)\circ\pi$ is a unitary representation.

Corollary 2.12. *A discrete group Γ is amenable iff it has the* (SSP).

Proof. This follows from the fact that Γ is amenable iff $C^*\Gamma$ is nuclear. □

3. Similarity Length of C*-algebras

The following is the fundamental characterization of a homomorphism which is similar to a $*$-homomorphism. This has several applications to dilation theory.

Theorem 3.1 (Haagerup, Paulsen). *Let A be a unital C*-algebra (or just a unital operator algebra), $\pi\colon A \to \mathbb{B}(\mathcal{H})$ be a unital homomorphism and $C > 0$ be a constant. Then, $\|\pi\|_{\mathrm{cb}} \le C$ iff there exists $S \in \mathrm{GL}(\mathcal{H})$ with $\|S\|\,\|S^{-1}\| \le C$ such that $\|\mathrm{Ad}(S)\circ\pi\|_{\mathrm{cb}} = 1$.*

Proof. The "if" part is obvious. To prove the "only if" part, let $A \subset \mathbb{B}(H)$ and $\pi\colon A \to \mathbb{B}(\mathcal{H})$ be a homomorphism with $\|\pi\|_{\mathrm{cb}} \le C$. By a Stinespring type theorem, there exist a Hilbert space $\widehat{\mathcal{H}}$, a $*$-homomorphism $\sigma\colon \mathbb{B}(H) \to \mathbb{B}(\widehat{\mathcal{H}})$, and operators $V \in \mathbb{B}(\mathcal{H},\widehat{\mathcal{H}})$, $W \in \mathbb{B}(\widehat{\mathcal{H}},\mathcal{H})$ with $\|V\|\,\|W\| \le \|\pi\|_{\mathrm{cb}}$ such that

$$\forall a\in A \quad \pi(a) = V\sigma(a)W.$$

Let $\mathcal{K}_1 = \overline{\text{span}}(\sigma(A)W\mathcal{H})$. The subspace $\mathcal{K}_1$ is $\sigma(A)$-invariant and we may assume that $V = VP_{\mathcal{K}_1}$. Since

$$V\sigma(a)\big(\sigma(x)W\xi\big) = \pi(ax)\xi = \pi(a)V\sigma(x)W\xi,$$

we have $V\sigma(a)P_{\mathcal{K}_1} = \pi(a)V$ for every $a \in A$. It follows that $\mathcal{K}_2 = \ker V \subset \mathcal{K}_1$ is also $\sigma(A)$-invariant. Hence $\mathcal{L} = \mathcal{K}_1 \ominus \mathcal{K}_2$ is "semi-invariant" under $\sigma(A)$, i.e.,

$$\forall a \in A \quad P_{\mathcal{L}}\sigma(a) = P_{\mathcal{L}}\sigma(a)P_{\mathcal{L}}.$$

Consequently, we have

$$\forall a \in A \quad \pi(a) = VP_{\mathcal{L}}\sigma(a)W = VP_{\mathcal{L}}\sigma(a)P_{\mathcal{L}}W.$$

Since $VP_{\mathcal{L}}$ is injective on $\mathcal{L}$ and $VP_{\mathcal{L}}W = \pi(1) = 1$, the operator $S = VP_{\mathcal{L}}$ is a linear isomorphism from $\mathcal{L}$ onto $\mathcal{H}$ with $S^{-1} = P_{\mathcal{L}}W$. We have $\pi = \mathrm{Ad}(S) \circ \sigma$ with $\|S\|\,\|S^{-1}\| \le C$ and, since $\mathcal{L} \cong \mathcal{H}$, we are done. □

Corollary 3.2. *A derivation δ is inner iff it is completely bounded.*

By a standard direct sum argument, we obtain the following.

Corollary 3.3. *Let A be a unital C*-algebra with the* (SP). *Then, there exists a function f on $[1,\infty)$ such that*

$$\|\pi\|_{\mathrm{cb}} \le f(\|\pi\|)$$

for every unital continuous homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H})$.

Definition 3.4. Let A be a unital C*-algebra (or a unital operator algebra). The *similarity length* of A, denoted by $l(A)$, is the smallest integer l with the following property; There exists a constant $C > 0$ such that for any $x \in \mathbb{M}_\infty(A)$, there exist $\alpha_0, \alpha_1, \dots \alpha_l \in \mathbb{M}_\infty(\mathbb{C})$ and $D_1, \dots, D_l \in \mathrm{Diag}_\infty(A)$ satisfying

$$x = \alpha_0 D_1 \alpha_1 \cdots D_l \alpha_l$$

and

$$\prod_{m=0}^{l} \|\alpha_m\| \prod_{m=1}^{l} \|D_m\| \le C\|x\|.$$

Here, $\mathbb{M}_\infty(A) = \bigcup_{n=1}^\infty \mathbb{M}_n(A)$ and $\mathrm{Diag}_\infty(A) \subset \mathbb{M}_\infty(A)$ is the set of diagonal matrices with entries in A. If there is no l satisfying the above condition, then we set $l(A) = \infty$ by convention.

Theorem 3.5 (Pisier 1999). *Let A be a unital C*-algebra (or a unital operator algebra) with* $\dim(A) > 1$. *The following are equivalent.*

(1) *A has the* (SP).
(2) *There exist $d > 0$ and $C > 0$ such that $\|\pi\|_{\mathrm{cb}} \le C\|\pi\|^d$ for every unital continuous homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H})$.*
(3) *$l(A) \le d$.*

The constant d appearing in the conditions (2) and (3) are taken to be same and are possibly non-integer. It follows that the "optimal" function f appearing in Corollary 3.3 is a polynomial of degree $l(A)$. The implication (2) $\Rightarrow$ (1) follows from Theorem 3.1. We do not prove the hard implication (1) $\Rightarrow$ (3), but explain (3) $\Rightarrow$ (2);

$$\|\pi(x)\| = \|\alpha_0\pi(D_1)\alpha_1\cdots\pi(D_l)\alpha_l\| \le \|\pi\|^l \prod_{m=0}^{l}\|\alpha_m\| \prod_{m=1}^{l}\|D_m\| \le C\|\pi\|^l\|x\|$$

for $x = \alpha_0 D_1\alpha_1\cdots D_l\alpha_l \in \mathbb{M}_\infty(A)$.

For a unital C*-algebra A with $\dim(A) > 1$, it is known that

(1) $l(A) = 1 \Leftrightarrow \dim(A) < \infty$ (Exercise),
(2) $l(A) = 2 \Leftrightarrow A$ is nuclear with $\dim(A) = \infty$ (Pisier 2004),
(3) $l(A) \le 3$ if A has no tracial state,
(4) $l(M) = 3$ if M is a type II_1 factor with the property (Γ) (Christensen 2002),
(5) $l(A) = \max\{l(I), l(A/I)\}$ for every closed 2-sided ideal $I \triangleleft A$ (Exercise).

It is not known whether there exists a unital C*-algebra with $l(A) > 3$. We note that an affirmative answer to Similarity Problem A would imply that there exists l_0 such that $l(A) \le l_0$ for every C*-algebra A. We close this note by showing $l(A) \le 3$ for any C*-algebra A which contains a unital copy of the Cuntz algebra $\mathcal{O}_\infty$. (The case where A has no tracial state is then dealt by passing to the second dual.)

Let $x \in \mathbb{M}_n(A)$ be given. We choose unitary matrices $W_1, W_2 \in \mathbb{M}_n(\mathbb{C})$ with $|W_1(i,j)| = |W_2(i,j)| = n^{-1/2}$ for all i,j (e.g., $W_k(i,j) = n^{-1/2}\exp(2\pi\sqrt{-1}ij/n)$). Let $D_1(i) = S_i^*$ and $D_3(j) = S_j$ for every i,j, where S_i's are isometries satisfying $S_i^*S_j = \delta_{i,j}I$. For every k, we set

$$\begin{aligned} D_2(k) &= n\sum_{i,j}\overline{W_1(i,k)}S_i x_{i,j}S_j^*\overline{W_2(k,j)} \\ &= n\begin{pmatrix}\overline{W_1(1,k)}S_1 & \cdots & \overline{W_1(n,k)}S_n\end{pmatrix}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n}\end{pmatrix}\begin{pmatrix}\overline{W_2(k,1)}S_1^* \\ \vdots \\ \overline{W_2(k,n)}S_n^*\end{pmatrix}. \end{aligned}$$

From the latter expression, we see that $\|D_2(k)\| \le \|x\|$. We obtained $W_1, W_2 \in \mathbb{M}_n(\mathbb{C})$ and $D_1, D_2, D_3 \in \mathrm{Diag}_n(A) \subset \mathbb{M}_n(A)$ such that

$$\|D_1\|\,\|W_1\|\,\|D_2\|\,\|W_2\|\,\|D_3\| \le \|x\|$$

and

$$x = D_1W_1D_2W_2D_3.$$

Indeed, we have

$$(D_1W_1D_2W_2D_3)_{i,j} = \sum_{k=1}^{n} S_i^* W_1(i,k)D_2(k)W_2(k,j)S_j$$
$$= n\sum_{k=1}^{n} |W_1(i,k)|^2|W_2(k,j)|^2 x_{i,j} = x_{i,j}.$$

REFERENCES

[1] G. Pisier, *Introduction to operator space theory.* London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003.

[2] ———, *Similarity problems and completely bounded maps.* Second, expanded edition. Includes the solution to "The Halmos problem". Lecture Notes in Mathematics, 1618. Springer-Verlag, Berlin, 2001.

[3] ———, *A similarity degree characterization of nuclear C^*-algebras.* Preprint. math.OA/0409091

[4] ———, *Simultaneous similarity, bounded generation and amenability.* Preprint. math.OA/0508223

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