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Weakly amenable group and CBAP

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In this note, we discuss weak amenability of groups which is a generalization of amenability in some sense.

Definition 1. A discrete group Γ is said to be *weakly amenable* if there exists a net (φ_i) of finitely supported functions on Γ such that $\varphi_i \to 1$ pointwise and $\limsup \|m_{\varphi_i}\|_{\mathrm{cb}} \leq C$. Recall that the Herz-Schur norm [1] $\|m_\varphi\|_{\mathrm{cb}}$ is $\leq C$ iff there exist families of vectors$(\xi_s)_{s\in\Gamma}$ and $(\eta_t)_{t\in\Gamma}$ in a Hilbert space H such that $\varphi(st^{-1}) = \langle \eta_t, \xi_s\rangle$ for every $s \in \Gamma$ and $\sup_{s,t\in\Gamma}\|\xi_s\|\|\eta_t\| \leq C$. Since $\|m_\varphi\|_{\mathrm{cb}} \leq \|\varphi\|_2$ in general, we may replace the term "finitely supported" in the above definition with "square summable". The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(\Gamma)$ is the infimum of all such C for which such a net (φ_i) exists. We set $\Lambda_{\mathrm{cb}}(\Gamma) = \infty$ if Γ is not weakly amenable.

Example 1. The following are easy from the definition.

1. If Γ is an amenable group, then $\Lambda_{\mathrm{cb}}(\Gamma) = 1$.

2. If $\Lambda \leq \Gamma$ is a subgroup, then $\Lambda_{\mathrm{cb}}(\Lambda) \leq \Lambda_{\mathrm{cb}}(\Gamma)$.

3. If $\{\Gamma_i\}$ is a directed family of groups, then $\Lambda_{\mathrm{cb}}(\cup\Gamma_i) = \sup_i \Lambda_{\mathrm{cb}}(\Gamma)$.

Here are some non-amenable examples.

Theorem 1. *A group Γ which acts properly on a tree weakly amenable with $\Lambda_{\mathrm{cb}}(\Gamma) = 1$. In particular, free groups and SL(2,$\mathbb{Z}$) are weakly amenable with their Cowling-Haagerup constant 1.*

[1] The Herz-Schur multiplier m_φ is the Schur multiplier that is associated with the kernel$(s,t) \mapsto \varphi(st^{-1})$ so that $m_\varphi(\lambda(s)) = \varphi(s)\lambda(s)$ for $s \in \Gamma$.

Let T be a tree and we identify its vertex set with T. We denote by $\mathrm{dist}(x,y)$ the graph distance of vertices $x, y \in \mathrm{T}$. Let χ_n be the characteristic function on $\{(x,y) \in \mathrm{T}^2; \mathrm{dist}(x,y) = n\}$ and denote by m_{χ_n} the corresponding Schur multiplier on $\mathbb{B}(\ell^2(\mathrm{T}))$.

Lemma 1. *We have* $\|m_{\chi_n}\|_{\mathrm{cb}} \leq 2n$ for every n.

Proof . Fix a geodesic ray ω in T, i.e. ω is an isometric function from $\mathbb{Z}_+$ into T. For every $x \in \mathrm{T}$, there exists a unique geodesic ray ω_x which starts at x and eventually flows into ω. It is not hard to see

$$\theta_n(x,y) = \sum_{k=0}^{n} \langle \delta_{\omega_y(k)}, \delta_{\omega_x(n-k)} \rangle = \sum_{m=0}^{[n/2]} \chi_{n-2m}(x,y)$$

for any $x, y \in \mathrm{T}$. In particular, $\chi_n = \theta_n - \theta_{n-2}$ for every $n \geq 2$. Since we have $\|m_{\theta_n}\|_{\mathrm{cb}} \leq n+1$, we are done.

Lemma 2. *Let* T *be a tree which is identified with its vertex set. Then, there exists a sequence of finitely supported functions* $\varphi_n : \mathbb{Z}_+ \mapsto [0,1]$ *such that* $\varphi_n \to 1$ *pointwise and if we define kernels* $k_n : \mathrm{T} \times \mathrm{T} \mapsto [0,1]$ *by* $k_n = \varphi_n(\mathrm{dist}(x,y))$ *for* $x, y \in \mathrm{T}$, *then* $\limsup \|m_{k_n}\|_{\mathrm{cb}} \leq 1$.

Proof . For any $n \geq 1$, the kernel

$$\psi_n(x,y) = \exp(-\frac{1}{n}\mathrm{dist}(x,y))$$

is positive definite and hence m_{ψ_n} is a u.c.p map on $\mathbb{B}(\ell^2(\mathrm{T}))$. On the other hand, since $\chi_k\psi_n = e^{-\frac{k}{n}}\chi_k$ for every n and k, Lemma 1 implies that

$$\|\sum_{k\leq K} m_{\chi_k\psi_n}\|_{\mathrm{cb}} \leq \|m_{\psi_n}\|_{\mathrm{cb}} + \|\sum_{k>K} m_{\chi_k\psi_n}\|_{\mathrm{cb}} \leq 1 + \sum_{k>K} 2ke^{-\frac{k}{n}}.$$

Thus, if K_n is chosen sufficiently large, then the kernel $\varphi_n = \sum_{k\leq K_n} \chi_k\psi_n$ satisfies $\|m_{\varphi_n}\|_{\mathrm{cb}} \leq 1 + n^{-1}$. Since $\varphi_n(x,y)$ depends only on $\mathrm{dist}(x,y)$ and $\varphi_n \to 1$ pointwise, we are done.

Proof of Theorem . Suppose that a group Γ acts on a tree T and take functions φ_n as in Lemma 2. Fix a base point o in T and consider the pseudo-length function $l(s) = \mathrm{dist}(o, so)$ on Γ. Then, the functions ψ_n on Γ, defined by $\psi_n(s) = \varphi_n(l(s))$ for $s \in \Gamma$, satisfy that $\varphi_n \to 1$ pointwise and $\limsup_n \|m_{\psi_n}\|_{\mathrm{cb}} \leq 1$. Moreover, the functions ψ_n are finitely supported if the Γ-action on T is proper and we are done.

Their exist weakly amenable groups whose Cowling-Haagerup constants are greater than 1. They are lattices in real simple Lie groups of rank one. Before stating the theorem, we note that the definition of weak amenability extends to a locally compact group G. Moreover, for a lattice $\Gamma \leq G$, one has $\Lambda_{\text{cb}}(G) = \Lambda_{\text{cb}}(\Gamma)$. The result is summarized in the following theorem whose proof is beyond the scope of this note and we refer the reader to [CH].

Theorem 2. *We have the following.*

1. $\Lambda_{\text{cb}}(SO(1,n)) = 1$ and $\Lambda_{\text{cb}}(SU(1,n)) = 1$.
2. $\Lambda_{\text{cb}}(Sp(1,n)) = 2n-1$ and $\Lambda_{\text{cb}}(F_{4(-20)}) = 21$.
3. *If the real rank of G is greater than or equal to 2, then $\Lambda_{\text{cb}}(G) = \infty$. In particular, $\Lambda_{\text{cb}}(SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2) = \infty$.*

Definition 2. We say a C^*-algebra A has the *CBAP* (completely bounded approximation property) if there exists a net of finite rank maps $\theta_i : A \mapsto A$ such that $\theta_i \to \text{id}_A$ in the point-norm topology and $\sup \|\theta_i\|_{\text{cb}} \leq C$. The Haagerup constant $\Lambda_{\text{cb}}(A)$ is the infimum of all such C for which such a net (θ_i) exists. We say $\Lambda_{\text{cb}}(A) = \infty$ if A does not have the CBAP.
We say a von Neumann algebra M has the *weak*CBAP* if there exists a net of (weak*-continuous) finite rank maps $\theta_i : M \mapsto M$ such that $\theta_i \to \text{id}_M$ in the point-weak*-topology and $\sup \|\theta_i\|_{\text{cb}} \leq C$. The Haagerup constant $\Lambda_{\text{cb}}(M)$ is the infimum of all such C for which such a net (θ_i) exists. We set $\Lambda_{\text{cb}}(M) = \infty$ if M does not have the weak* CBAP.
We trust that Λ_{cb} for C^*-algebras and von Neumann algebras are not mixed up. We remark that in the definition of weak* CBAP, it does not matter whether weak*-continuity of finite rank maps is required or not. This fact is non-trivial, but we do not give a proof because it requires the "local reflexivity" of the operator space predual of M.

Theorem 3. *Let Γ be a discrete group. Then, we have*

$$\Lambda_{\text{cb}}(\Gamma) = \Lambda_{\text{cb}}(C^*_{\text{r}}(\Gamma)) = \Lambda_{\text{cb}}(L(\Gamma))$$

Proof. We trivially have $\Lambda_{\text{cb}}(\Gamma) \geq \Lambda_{\text{cb}}(C^*_{\text{r}}(\Gamma))$ and $\Lambda_{\text{cb}}(\Gamma) \geq \Lambda_{\text{cb}}(L(\Gamma))$. To prove the reverse inequalities at once, let a finite subset $E \subset \Gamma$ and $\varepsilon > 0$ be given, and choose a finite rank map $\theta : C^*_{\text{r}}(\Gamma) \mapsto L(\Gamma)$ such that $\|\theta\|_{\text{cb}} = C$ and $|1 - \tau(\lambda(s))^*\theta(\lambda(s)))| < \varepsilon$ for $s \in E$. It suffices to show that the function

$\varphi(s) = \tau(\lambda(s)^*\theta(\lambda(s)))$ is in $\ell^2(\Gamma)$ and $\|(m_\varphi)_{|C_r^*(\Gamma)}\| \leq C$. Since θ is finite rank, there exist finite sequences $\omega_1, \ldots, \omega_n \in C_r^*(\Gamma)$ and $x_1, \ldots, x_n \in L(\Gamma)$ such that $\theta(a) = \sum_{k=1}^n \omega_k(a)x_k$ for all $a \in C_r^*(\Gamma)$. It follows that

$$\varphi(s) = \tau(\lambda(s)^*\theta(\lambda(s))) = \sum_{k=1}^n \omega_k(\lambda(s))\tau(\lambda(s)^*x_k)$$

Since $\sup_{s\in\Gamma} |\omega_k(\lambda(s))| \leq \|\omega_k\|$ and $\sum_{s\in\Gamma} |\tau(\lambda(s)^*x_k|^2 = \|x_k\delta_e\|^2 < \infty$ for every k, the function φ is in $\ell^2(\Gamma)$. We denote by π by the *-homomorphism from $C_r^*(\Gamma)$ into $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ given by $\pi(\lambda(s)) = \lambda(s) \otimes \lambda(s)$ for every $s \in \Gamma$. (We note that $\lambda \otimes 1$ and $\lambda \otimes \lambda$ are unitarily equivalent.) Let V be the isometry from $\ell^2(\Gamma)$ into $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$ given by $V\delta_s = \delta_s \otimes \delta_s$ for $s \in \Gamma$. It is not hard to check that

$$m_\varphi(a) = V^*(\mathrm{id}_{|C_r^*(\Gamma)} \otimes \theta)(\pi(a))V$$

for $a \in C_r^*(\Gamma)$, and hence $\|(m_\varphi)_{|C_r^*(\Gamma)}\|_{\mathrm{cb}} \leq \|\theta\|_{\mathrm{cb}}$.

There are some permanence properties of CBAP below. We does not prove them and refer the reader to [AD],[BP],[SS1] and [SS2].

Example 2. We have the following.

1. If $\Gamma_i, i = 1, 2$ are groups with $\Lambda_{\mathrm{cb}}(\Gamma_i) = 1$, then $\Lambda_{\mathrm{cb}}(\Gamma_1 * \Gamma_2) = 1$.

2. If $\Lambda \leq \Gamma$ is a subgroup such that the homogeneous space Γ/Λ is amenable, then $\Lambda_{\mathrm{cb}}(\Lambda) = \Lambda_{\mathrm{cb}}(\Gamma)$.

3. If Γ is an amenable groups actiong on a C^*-algebra A(resp. a von Neumann algebra M), then $\Lambda_{\mathrm{cb}}(A \rtimes \Gamma) = \Lambda_{\mathrm{cb}}(A)$ (resp.$\Lambda_{\mathrm{cb}}(M \rtimes \Gamma) = \Lambda_{\mathrm{cb}}(M)$).

4. We have $\Lambda_{\mathrm{cb}}(A \otimes_{\min} B) = \Lambda_{\mathrm{cb}}(A)\Lambda_{\mathrm{cb}}(B)$ for any C^*-algebras A and B. A similar statement holds for von Neumann algebras. In particular, $\Lambda_{\mathrm{cb}}(\Gamma_1 \times \Gamma_2) = \Lambda_{\mathrm{cb}}(\Gamma_1)\Lambda_{\mathrm{cb}}(\Gamma_2)$ for any groups Γ_1 and Γ_2.

Remark . Here we consider some counterexamples to basic constructions. First, an extension of weakly amenable groups need not be weakly amenable. (e.g.$\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$) Also, $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ serves as a counterexample to weak amenability for amalgamated free products. Indeed, $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ can be written as an amalgamated free product $\mathbb{Z}^2 \rtimes \mathbb{Z}_4 *_{\mathbb{Z}^2 \rtimes \mathbb{Z}_2} \mathbb{Z}^2 \rtimes \mathbb{Z}_6$, whose factors are amenable groups. So, the class of C^*-algebras with CBAP is not closed under taking arbitrary amalgamated free products.

Next, it is well-known that the reduced group C^*-algebra of exact group is embeddable into a nuclear C^*-algebra and, from the definition, nuclear C^*-algebras have CBAP(with Cowling-Haagerup constant 1), so a C^*-subalgebra of a C^*-algebra with CBAP need not have CBAP (Consider $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$.).

Regarding to extensions, there are some observations in [DS]. But in [TH], a non-exact C^*-algebra which is an extension of $C^*_r(\mathbb{F}_2)$ by compact operators was constructed. On the other hand, it is easy to see that a C^*-algebra with CBAP is exact. So the class of C^*-algebras with CBAP is not closed under taking arbitrary extensitons.

Lastly, though the class of C^*-algebras with CBAP is closed under minimal tensor products as cited above, the same does not hold for maximal tensor products. Indeed, it is well-known that $C^*(\Gamma)$(full group C^*-algebra) can be embedded into $C^*_r(\Gamma) \otimes_{\max} C^*_r(\Gamma)$ as "diagonals" (See [Pi].). So, if Γ is $\mathbb{F}_2$, $C^*_r(\Gamma) \otimes_{\max} C^*_r(\Gamma)$ is not exact and hence does not have CBAP.

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