Spherical Fourier transforms of the Berezin kernels on symmetric Siegel domains

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ABSTRACT. This manuscript was written out in December, 1998. After submission, we noticed the paper

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and recognized that our computation techniques overlap with theirs significantly. Thus we have withdrawn our paper. However, since we believe that our paper still has its meaning in the point that it directly relates the spherical Fourier transform of the Berezin kernel to the spectral expression of the Berezin transform via Helgason's theory, and since RIMS kōkyūroku is of somewhat informal character, we decide to publish the original manuscript here. Of course we claim no priority concerning the contents of this manuscript. — The abstract of the original paper is the following.

This paper presents a direct computation of the spherical Fourier transforms of the Berezin kernels on symmetric Siegel domains. Our computation is based on the Jordan-theoretic structure in terms of which symmetric Siegel domains are described, and involves evaluations of some integrals related to symmetric cones.

INTRODUCTION

Berezin transforms on symmetric Siegel domains are studied by Unterberger-Upmeier [9], and a proof is supplied for the spectral-theoretic expression previously announced by Berezin himself [1] for the case of classical domains only. However, the approach of [9] necessitates a number of inequalities which are not quite trivial. The aim of this paper is to show a shorter way by computing directly the spherical Fourier transforms of the integral kernels (the *Berezin kernels*) of the Berezin transforms. Since the Berezin transforms are invariant integral operators for the holomorphic transformation group G on the domain, Helgason's theory [5] of Fourier transform on Riemannian symmetric spaces then yields the spectral expression.

The basic standpoint that we take is the Jordan algebraic one like [9], because symmetric Siegel domains are described in terms of Jordan triple systems (JTS), see [7] and [8]. With the Jordan structure everything that we need acquires a concrete expression. In particular, by [9], the Berezin kernels $A_\lambda(z_1, z_2)$ are expressed by using

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the determinant function of a Jordan algebra (see (1.12) and (1.13)). Moreover it turns out that the zonal spherical function is an average over a maximal compact subgroup K of G of a function defined through a product of Jordan algebra principal minors (see (2.2) and (2.5)) as in the case of symmetric cones (cf. [2, Theorem XIV.3.1]). These facts make it possible to carry out a direct explicit computation of the spherical Fourier transform of $a_\lambda := A_\lambda(\cdot, e)$, where e is the reference point. It should be noted in passing that by a simple observation using the property of reproducing kernel, a_λ is an integrable function (Lemma 1.1). Our computation has its own interests, because it involves evaluations of some integrals related to symmetric cones. Among them, a decisive role is played by a generalization of the beta integral stated much earlier in [3, Proposition 2.6], to which we shall give a corrected formula in Theorem 3.4. Furthermore, our proof of Theorem 3.4 is simpler in that it does not require the introduction of a group structure on the cone unlike [3]. The main theorem of this paper is Theorem 4.1, which shows that the spherical Fourier transform of a_λ is expressed in terms of the gamma function Γ_Ω of the symmetric cone Ω. Although the result is expected by interpreting the paper [9], what we would like to emphasize here is the fact that it is directly computable as an explicit harmonic analysis on symmetric cones, thereby giving a simplification.

1. PRELIMINARIES

1.1. Symmetric Siegel domains.
We describe here symmetric Siegel domains by using JTS following the presentation in [9]. Let Z be a Hermitian JTS. This means that Z is a finite-dimensional complex vector space endowed with a *real* trilinear map $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z$ such that

(1) $\{x, y, z\}$ is complex linear in x, z and antilinear in y,

(2) $\{x, y, z\} = \{z, y, x\}$,

(3) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$.

We put $(x \,\square\, y)z := \{x, y, z\}$ and $Q(y)z := \{y, z, y\}$. We suppose further that the trace form $\mathrm{tr}(x \,\square\, y)$ defines a positive definite Hermitian inner product on Z.

An element $e \in Z$ is said to be a *tripotent* if $\{e, e, e\} = e$. Let r be the rank of the JTS Z and fix a JTS frame $\{e_1, \ldots, e_r\}$, that is, a maximal system of primitive tripotents in Z such that $e_i \,\square\, e_j = 0$ if $i \neq j$. Put $e := e_1 + \cdots + e_r$. Then e is a maximal tripotent, so that the selfadjoint operator $e \,\square\, e$ has only eigenvalues $1/2$ and 1. Denoting by U (resp. W) the $1/2$-eigenspace (resp. 1-eigenspace), we have

$$(1.1) \qquad\qquad Z = U \oplus W.$$

We do not exclude the possibility $U = \{0\}$. The product $z_1 \circ z_2 := \{z_1, e, z_2\}$ turns Z into a complex Jordan algebra. W is a Jordan subalgebra of $(Z, \circ)$. Moreover the anti-linear operator $Q(e)$ induces an involutive real Jordan algebra automorphism on W. For $w \in W$, we shall write w^* for $Q(e)w$. Let $V := W^{Q(e)}$, the fixed points of $Q(e)$. Then V is a real form of W, and is in fact a euclidean Jordan algebra of rank r with unit element e. Furthermore, $\{e_1, \ldots, e_r\}$ becomes a Jordan algebra frame of V.

Let Ω be the interior of the set $\{x \circ x \; ; \; x \in V\}$ in V. We know that Ω is a symmetric cone [2]. Define a sesqui-linear map $\Phi : U \times U \to W$ by

$$(1.2) \qquad\qquad \Phi(u, u') := 2\{u, u', e\}.$$

It turns out that Φ is an Ω-positive Hermitian map. With these data we form a Siegel domain $D = D(\Omega, \Phi)$:

$$D = \{(u, w) \in U \times W \; ; \; w + w^* - \Phi(u, u) \in \Omega\}.$$

We suppose from now on that the Siegel domain D is irreducible. Thus Z is a simple JTS and V a simple Jordan algebra. The frame $\{e_1, \ldots, e_r\}$ gives the Peirce decompositions $U = \sum_{1 \le j \le r}^{\oplus} U_j$ and $V = \sum_{1 \le i \le j \le r}^{\oplus} V_{ij}$ of U and V respectively, where

$$(1.3) \qquad U_j := \left\{u \in U \; ; \; (e_k \square e_k)u = \tfrac{1}{2}\delta_{jk}\, u \quad (1 \le k \le r)\right\},$$

$$(1.4) \qquad V_{ij} := \left\{v \in V \; ; \; (e_k \square e_k)v = \tfrac{1}{2}\left(\delta_{ik} + \delta_{jk}\right)v \quad (1 \le k \le r)\right\}.$$

The dimension $b := \dim U_j$ is independent of j, and similarly $d := \dim V_{ij}$ does not depend on i, j.

1.2. Gamma functions of Ω. Let Δ_j $(j = 1, 2, \ldots, r)$ be the principal minors $(\Delta_r := \Delta$, the Jordan algebra determinant of V) of the euclidean Jordan algebra V defined by the Jordan algebra frame $\{e_1, \ldots, e_r\}$ (cf. [2, p. 114]). For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we set

$$(1.5) \qquad \Delta_{\mathbf{s}}(x) := \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}.$$

Let $\langle \cdot \,|\, \cdot \rangle$ be the real inner product on V defined by the Jordan algebra trace form of V: $\langle x \,|\, y \rangle := \mathrm{tr}(x \circ y)$. We know that if dx denotes the euclidean measure on V, then $\Delta(x)^{-n/r}\, dx$ is the $GL(\Omega)$-invariant measure on Ω, where $n := \dim_{\mathbb{R}} V$. The *gamma function* Γ_Ω of the symmetric cone Ω is defined by the following absolutely convergent integral: if $\mathrm{Re}\, s_j > d(j - 1)/2$ for $j = 1, \ldots, r$,

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\mathrm{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-n/r}\, dx.$$

By [2, Theorem VII.1.1] one knows that

$$(1.6) \qquad \Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n-r)/2} \prod_{j=1}^{r} \Gamma\!\left(s_j - \tfrac{d}{2}(j - 1)\right).$$

If $\mathbf{s} = (\lambda, \ldots, \lambda)$, we shall write $\Delta_\lambda(x)$ and $\Gamma_\Omega(\lambda)$ for $\Delta_{\mathbf{s}}(x)$ and $\Gamma_\Omega(\mathbf{s})$ respectively.

Let us set $\mathbf{s}^* := (s_r, \ldots, s_1)$ for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. Let $\Delta_1^*, \ldots, \Delta_r^*$ stand for the principal minors corresponding to the reverse Jordan algebra frame $\{e_r, \ldots, e_1\}$. By Propositions VII.1.2 and VII.1.5 in [2], we have for $\mathrm{Re}\, s_j > d(r - j)/2$ $(j = 1, \ldots, r)$,

$$(1.7) \qquad \Gamma_\Omega(\mathbf{s}^*)\Delta_{-\mathbf{s}}(y) = \int_\Omega e^{-\langle x | y \rangle} \Delta^*_{\mathbf{s}^* - n/r}(x)\, dx \qquad (y \in \Omega).$$

This shows in particular that the function $\Delta_{-\mathbf{s}}$ originally defined on Ω is analytically continued to a holomorphic function on the tube domain $\Omega + iV$.

1.3. Berezin kernels. We now introduce the Bergman spaces $H_\lambda^2(D)$ on D following [9, p. 583]. First we extend the inner product $\langle \cdot | \cdot \rangle$ of V to a complex bilinear form on $V_{\mathbb{C}} = W$, which we still denote by $\langle \cdot | \cdot \rangle$. On W, we have the Hermitian inner product defined by $(w_1 | w_2) := \langle w_1 | w_2^* \rangle$, whereas on U we take the Hermitian inner product $(u_1 | u_2) := \langle \Phi(u_1, u_2) | e \rangle$. Let $dm(u)$ and $dm(w)$ be the euclidean measures on U and W respectively corresponding to each of the euclidean structures $\mathrm{Re}(\cdot | \cdot)$. We put

$$(1.8) \qquad m := \dim_{\mathbb{C}} U, \qquad p := \frac{2n+m}{r}, \qquad N := n + m.$$

Let G denote the identity component of the group of holomorphic automorphisms of D. We know that G is a semisimple Lie group with trivial center. Moreover, the G-invariant measure $d\mu$ on D is given by

$$(1.9) \qquad d\mu(u, w) = \Delta(w + w^* - \Phi(u, u))^{-p} \, dm(u) dm(w).$$

For every λ satisfying $\lambda > p - 1$, set

$$(1.10) \qquad d\mu_\lambda(u, w) := c_\lambda \cdot \Delta(w + w^* - \Phi(u, u))^{\lambda-p} \, dm(u) dm(w),$$

$$(1.11) \qquad c_\lambda := \frac{1}{\pi^N} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - N/r)},$$

and let $H_\lambda^2(D)$ be the Hilbert space of holomorphic functions f on D such that

$$\int_D |f(z)|^2 \, d\mu_\lambda(z) < \infty.$$

The reproducing kernel $\kappa_\lambda(z_1, z_2)$ $(z_1, z_2 \in D)$ of $H_\lambda^2(D)$ is given by

$$(1.12) \qquad \kappa_\lambda(z_1, z_2) = \Delta(w_1 + w_2^* - \Phi(u_1, u_2))^{-\lambda} \quad (z_j = (u_j, w_j), \; j = 1, 2).$$

Observe here that $\mathrm{Re}(w_1 + w_2^* - \Phi(u_1, u_2)) \in \Omega$. Thus $\kappa_\lambda(z_1, z_2)$ is indeed holomorphic in $z_1 \in D$ and anti-holomorphic in $z_2 \in D$ by (1.7).

The *Berezin kernel* on D associated to $H_\lambda^2(D)$ is the function A_λ defined by

$$(1.13) \qquad A_\lambda(z_1, z_2) := \frac{|\kappa_\lambda(z_1, z_2)|^2}{\kappa_\lambda(z_1, z_1) \, \kappa_\lambda(z_2, z_2)} \qquad (z_1, z_2 \in D).$$

We know that A_λ is G-invariant:

$$(1.14) \qquad A_\lambda(g \cdot z_1, \; g \cdot z_2) = A_\lambda(z_1, z_2) \qquad (g \in G).$$

Put $\mathbf{e} := (0, e) \in D$ for simplicity, and set

$$(1.15) \qquad a_\lambda(g) := A_\lambda(g \cdot \mathbf{e}, \; \mathbf{e}) \qquad (g \in G).$$

Let K be the stabilizer at $\mathbf{e}$ in G. Then K is a maximal compact subgroup of G. We shall normalize the Haar measure dg on G in such a way that

$$(1.16) \qquad \int_G f(g \cdot \mathbf{e}) \, dg = \int_D f(z) \, d\mu(z).$$

By (1.14) it is clear that a_λ is K-biinvariant.

Lemma 1.1. *Suppose* $\lambda > p - 1$. *One has* $a_\lambda \in L^1(K \backslash G / K)$.

Proof. Note that we have, in view of (1.9), (1.10) and (1.12), $\kappa_\lambda(z,z)\,d\mu_\lambda(z) = c_\lambda\,d\mu(z)$ with c_λ as in (1.11). Then by (1.13) and (1.16) we get

$$\int_G a_\lambda(g)\,dg = \int_G A_\lambda(g \cdot \mathbf{e},\ \mathbf{e})\,dg = \int_D A_\lambda(z, \mathbf{e})\,d\mu(z)$$

$$= \frac{c_\lambda^{-1}}{\kappa_\lambda(\mathbf{e}, \mathbf{e})} \int_D |\kappa_\lambda(z, \mathbf{e})|^2\,d\mu_\lambda(z) = c_\lambda^{-1} < \infty.$$

Since $a_\lambda \geq 0$, this implies that a_λ is integrable over G. $\qquad\square$

2. ZONAL SPHERICAL FUNCTIONS

In order to give an expression of zonal spherical functions on G in the present context, we need to specify some of the subgroups of G and of the Lie subalgebras of $\mathfrak{g} := \mathrm{Lie}(G)$. We first observe ([7] and [8]) that elements of $\mathfrak{g}$ are holomorphic polynomial vector fields $p(z)\,\partial/\partial z$ on D, so that the bracket operation in $\mathfrak{g}$ is the Poisson bracket

$$\left[p(z)\frac{\partial}{\partial z},\ q(z)\frac{\partial}{\partial z}\right] := \Big(p'(z)\big(q(z)\big) - q'(z)\big(p(z)\big)\Big)\frac{\partial}{\partial z}.$$

We will drop the symbol $\partial/\partial z$ in what follows for simplicity. Thus we simply think of elements of $\mathfrak{g}$ as holomorphic polynomial maps $Z \to Z$. With the JTS frame $\{e_1,\ldots,e_r\}$ fixed in §1 let $\mathfrak{a} := \sum_{1\leq j\leq r}^{\oplus} \mathbb{R}\,(e_j \,\square\, e_j)$. Then $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that $\mathrm{ad}\,\mathfrak{a}$ consists of semisimple operators on $\mathfrak{g}$. Recalling (1.3) and (1.4), we set

$$\mathfrak{g}_{ij}^0 := \{x \,\square\, e_i\ ;\ x \in V_{ij}\} \qquad (1 \leq i < j \leq r),$$

$$\mathfrak{g}_j^{1/2} := \{u + 2e\,\square\,u\ ;\ u \in U_j\} \qquad (j = 1,\ldots,r),$$

$$\mathfrak{g}_{jk}^1 := \{ia\ ;\ a \in V_{jk}\} \qquad (1 \leq j \leq k \leq r),$$

and $\mathfrak{n} := \left(\sum_{i<j}^{\oplus} \mathfrak{g}_{ij}^0\right) \oplus \left(\sum_{1\leq j\leq r}^{\oplus} \mathfrak{g}_j^{1/2}\right) \oplus \left(\sum_{j\leq k}^{\oplus} \mathfrak{g}_{jk}^1\right)$. Then, with $\mathfrak{k} := \mathrm{Lie}(K)$, we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$. The corresponding Iwasawa decomposition of G is $G = NAK$ with $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. For $g \in G$, we write $g \in N\big(\exp l(g)\big)K$ with $l(g) \in \mathfrak{a}$.

Let $\phi_\nu(g)$ $(\nu \in \mathfrak{a}_{\mathbb{C}}^*)$ be the zonal spherical function on G. We have

$$(2.1) \qquad\qquad \phi_\nu(g) = \int_K e^{(i\nu+\rho)(l(kg))}\,dk,$$

where $\rho(H) := \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}(H)|$ $(H \in \mathfrak{a})$ (see [4]). To rewrite the general formula (2.1) into the present context, we put

$$\mathfrak{n}^0 := \sum_{i<j}^{\oplus} \mathfrak{g}_{ij}^0, \qquad \mathfrak{n}_D := \left(\sum_{1\leq j\leq r}^{\oplus} \mathfrak{g}_j^{1/2}\right) \bigoplus \left(\sum_{j\leq k}^{\oplus} \mathfrak{g}_{jk}^1\right),$$

and $\mathfrak{s}^0 := \mathfrak{a} \oplus \mathfrak{n}^0$, $\mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{s}^0 \oplus \mathfrak{n}_D$. The subgroup $S^0 := \exp \mathfrak{s}^0$ acts on Ω simply transitively, and similarly $S := \exp \mathfrak{s}$ acts on D simply transitively. For each $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$, let ξ_α be the complex character of A defined by $\xi_\alpha(\exp H) = e^{\alpha(H)}$ $(H \in \mathfrak{a})$. The character ξ_α extends to S by setting $\xi_\alpha(na) = \xi_\alpha(a)$ $(na \in NA)$. We define a function ξ_α^D on D by $\xi_\alpha^D(s \cdot \mathbf{e}) := \xi_\alpha(s)$ $(s \in S)$. Similarly, for the zonal spherical

function ϕ_ν, we set $\phi_\nu^D(s \cdot \mathbf{e}) := \phi_\nu(s)$ $(s \in S)$. Since we have $ks \cdot \mathbf{e} = n' \exp l(ks) \cdot \mathbf{e}$ for some $n' \in N$, it holds that if $z := s \cdot \mathbf{e} \in D$, then $\xi_\alpha^D(k \cdot z) = e^{\alpha(l(ks))}$. Hence (2.1) is rewritten as

$$(2.2) \qquad \phi_\nu^D(z) = \int_K \xi_{i\nu+\rho}^D(k \cdot z) \, dk \qquad (z \in D).$$

Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to the basis $e_1 \square e_1, \ldots, e_r \square e_r$ of $\mathfrak{a}$. From now on we identify $\mathfrak{a}^*$ (resp. $\mathfrak{a}_{\mathbb{C}}^*$) with $\mathbb{R}^r$ (resp. $\mathbb{C}^r$) through the basis $\alpha_1, \ldots, \alpha_r$: $\sum a_j \alpha_j \leftrightarrow (a_1, \ldots, a_r)$. Let $\Delta_\mathbf{s}$ be as in (1.5). Then, if $s = n s_0$ with $n \in N_D := \exp \mathfrak{n}_D$ and $s_0 \in S^0$, we have

$$(2.3) \qquad \xi_\mathbf{s}^D(s \cdot \mathbf{e}) = \xi_\mathbf{s}(s) = \xi_\mathbf{s}(s_0) = \Delta_\mathbf{s}(s_0 e),$$

the third equality being a consequence of [2, Proposition VI.3.10]. To make the function $\xi_\mathbf{s}^D$ more explicit, we observe that N_D acts on D by affine maps. In fact, putting

$$n(a, b) = \exp(ia + b + 2e \square b) \in N_D \qquad (a \in V, \; b \in U),$$

we have by the proof of [7, Lemma 10.7 (3)] together with (1.2),

$$(2.4) \quad n(a, b) \cdot (u, w) = \left(u + b, \; w + ia + \tfrac{1}{2}\Phi(b, b) + \Phi(u, b)\right) \qquad (u \in U, \; w \in W).$$

Now if $(u, w) \in D$, then $\operatorname{Re} w - \tfrac{1}{2}\Phi(u, u) \in \Omega$ by definition, so that there exists a unique element $s_0 \in S^0$ such that $s_0 e = \operatorname{Re} w - \tfrac{1}{2}\Phi(u, u)$. By (2.4), the element $n := n(\operatorname{Im} w, u) \in N_D$ yields $n \cdot (0, \operatorname{Re} w - \tfrac{1}{2}\Phi(u, u)) = (u, w)$. Therefore we have $n s_0 \cdot \mathbf{e} = (u, w)$. These observations together with (2.3) lead us to

$$(2.5) \qquad \xi_\mathbf{s}^D(u, w) = \Delta_\mathbf{s}(\operatorname{Re} w - \tfrac{1}{2}\Phi(u, u)).$$

The formulas (2.2) and (2.5) will enable us to calculate explicitly the spherical Fourier transform $(a_\lambda)^\frown$ of $a_\lambda \in L^1(K \backslash G / K)$ defined by

$$(2.6) \qquad (a_\lambda)^\frown(\nu) := \int_G a_\lambda(g) \phi_{-\nu}(g) \, dg \qquad (\nu \in \mathfrak{a}^*).$$

3. Integral formulas related to Ω

In this section we prepare two integral formulas that are necessary to calculate $(a_\lambda)^\frown$. Recall the decomposition (1.1) of Z, where U (resp. W) is the 1/2-Peirce (resp. 1-Peirce) space for the maximal tripotent e. To every $a \in W$ we associate an operator $\pi(a)$ on U by $\pi(a)u := 2\{a, e, u\}$ $(u \in U)$. By [8, Proposition V.6.2], the map $\pi : a \mapsto \pi(a) \in \operatorname{End}(U)$ is a unital Jordan algebra representation of $(W, \circ)$. In other words, we have

$$\pi(a \circ b) = \frac{1}{2}\big(\pi(a)\pi(b) + \pi(b)\pi(a)\big), \qquad \pi(e) = I.$$

Taking the inner product $\operatorname{tr}_Z(u_1 \square u_2)$ in U, we see by [7, Lemma 10.2 (3)] that $\pi(a^*) = \pi(a)^*$ $(a \in W)$, the right hand side being the adjoint operator of $\pi(a)$. In particular, π gives rise to a selfadjoint representation of the simple euclidean Jordan algebra $(V, \circ)$. Thus, recalling (1.8), we get the following lemma by virtue of [2, Proposition IV.4.2].

Lemma 3.1. *Let* $\det_\mathbb{R} \pi(a)$ *denote the determinant of the real linear operator* $\pi(a)$ *on the underlying real vector space* $U_\mathbb{R}$ *of* U. *Then*

$$\det_\mathbb{R} \pi(a) = \Delta(a)^{2m/r} \qquad (a \in V).$$

Let P be the quadratic representation of the Jordan algebra $(W, \circ)$ (cf. [2, II.3]). Recalling our sesqui-linear map Φ defined by (1.2), we have the following formula by [7, Proposition 10.11 (2)].

Lemma 3.2. *Let* $a \in W$ *and* $u_1, u_2 \in U$. *Then*

$$P(a)\big(\Phi(u_1, u_2)\big) = \Phi(\pi(a)u_1, \pi(a^*)u_2).$$

With these preparations, we now evaluate some integrals related to the symmetric cone Ω. Recall that the euclidean measure dm on U comes from the real inner product $\mathrm{Re}(\Phi(u_1, u_2) \,|\, e)$. The first one is the following integral.

Proposition 3.3. *Suppose* $\mathrm{Re}\, s_j > \frac{d}{2}(r - j) + \frac{m}{r}$ *for* $j = 1, \ldots, r$. *If* $x \in \Omega$, *then*

$$(3.1) \qquad \int_U \Delta_{-s}\big(x + \tfrac{1}{2}\Phi(u, u)\big)\, dm(u) = (2\pi)^m \frac{\Gamma_\Omega(s^* - m/r)}{\Gamma_\Omega(s^*)} \Delta_{-s+m/r}(x).$$

Proof. Since $y = P(y^{1/2})e$ for $y \in \Omega$, and since $P(y^{1/2})$ restricted to V is a symmetric operator relative to the inner product $\langle \cdot \,|\, \cdot \rangle$ in V, Lemma 3.2 gives

$$\langle\, \Phi(u, u) \,|\, y \,\rangle = \langle\, P(y^{1/2})\Phi(u, u) \,|\, e \,\rangle$$
$$= \langle\, \Phi\big(\pi(y^{1/2})u, \pi(y^{1/2})u\big) \,|\, e \,\rangle = \|\pi(y^{1/2})u\|^2.$$

This together with (1.7) gives

$$\Delta_{-s}\big(x + \tfrac{1}{2}\Phi(u, u)\big) = \frac{1}{\Gamma_\Omega(s^*)} \int_\Omega e^{-\langle x \,|\, y \rangle} e^{-\|\pi(y^{1/2})u\|^2/2} \Delta^*_{s^*-n/r}(y)\, dy.$$

Hence, by interchanging the order of integrations, we see that the left hand side I of (3.1) is rewritten as

$$I = \frac{1}{\Gamma_\Omega(s^*)} \int_\Omega e^{-\langle x \,|\, y \rangle} \Delta^*_{s^*-n/r}(y)\, dy \int_U e^{-\|\pi(y^{1/2})u\|^2/2}\, dm(u).$$

Now Lemma 3.1 says $\det_\mathbb{R} \pi(y^{-1/2}) = \Delta(y)^{-m/r}$. Thus we obtain

$$I = \frac{(2\pi)^m}{\Gamma_\Omega(s^*)} \int_\Omega e^{-\langle x \,|\, y \rangle} \Delta^*_{s^*-m/r-n/r}(y)\, dy,$$

from which the proposition follows in view of (1.7). $\qquad\square$

The second one is an analogue of the beta integral

$$\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt = B(p, q) \qquad (\mathrm{Re}\, p > 0,\ \mathrm{Re}\, q > 0),$$

and is a generalization of the formula given by [2, Exercise VII.4]. The formula in the following theorem was given incorrectly by [3, Proposition 2.6]. Although our shorter proof works also for non-symmetric convex cones, we restrict ourselves here to the symmetric case for simplicity.

Theorem 3.4. *If* $\operatorname{Re} p_j > \frac{d}{2}(j-1)$, $\operatorname{Re} q_j > \frac{d}{2}(r-j)$ $(j = 1, \ldots, r)$, *then*

$$(3.2) \qquad \int_\Omega \Delta_{\mathbf{p}-n/r}(x) \Delta_{-\mathbf{p}-\mathbf{q}}(e + x) \, dx = \frac{\Gamma_\Omega(\mathbf{p})\Gamma_\Omega(\mathbf{q}^*)}{\Gamma_\Omega(\mathbf{p}^* + \mathbf{q}^*)}.$$

Proof. By (1.7) we have

$$\Delta_{-\mathbf{p}-\mathbf{q}}(e + x) = \frac{1}{\Gamma_\Omega(\mathbf{p}^* + \mathbf{q}^*)} \int_\Omega e^{-\langle e + x \mid y\rangle} \Delta^*_{\mathbf{p}^* + \mathbf{q}^* - n/r}(y) \, dy.$$

Hence interchanging the order of the integrations, we see that the left hand side of (3.2) is rewritten as

$$(3.3) \qquad \frac{1}{\Gamma_\Omega(\mathbf{p}^* + \mathbf{q}^*)} \int_\Omega \left\{ \int_\Omega e^{-\langle x \mid y\rangle} \Delta_{\mathbf{p}-n/r}(x) \, dx \right\} e^{-\operatorname{tr}(y)} \Delta^*_{\mathbf{p}^* + \mathbf{q}^* - n/r}(y) \, dy.$$

By Propositions VII.1.2 and VII.1.5 in [2], we see that the inner integral in (3.3) equals $\Gamma_\Omega(\mathbf{p})\Delta^*_{-\mathbf{p}^*}(y)$. Thus (3.3) reduces to the right hand side of (3.2) by virtue of (1.7) again. $\qquad \square$

4. Spherical Fourier transform of a_λ

We are now able to compute the spherical Fourier transform $(a_\lambda)^\wedge$.

Theorem 4.1. *If* $\lambda > p - 1$, *then one has*

$$(a_\lambda)^\wedge(\nu) = \frac{\pi^N}{\Gamma_\Omega(\lambda)^2} \Gamma_\Omega(-i\nu + \rho + \lambda - N/r) \Gamma_\Omega(i\nu^* - \rho^* + \lambda) \qquad (\nu \in \mathfrak{a}^*).$$

Proof. Let $\nu \in \mathfrak{a}^* \equiv \mathbb{R}^r$ and we suppose that the real number λ is so large that the integrals in (4.1), (4.2) and (4.3) below are absolutely convergent. We have by (1.14), (1.15), (1.16), (2.2) and (2.6)

$$(a_\lambda)^\wedge(\nu) = \int_D A_\lambda(z, \mathbf{e}) \phi^D_{-\nu}(z) \, d\mu(z) = \int_D A_\lambda(z, \mathbf{e}) \xi^D_{-i\nu + \rho}(z) \, d\mu(z).$$

Since $d\mu(z) = c_\lambda^{-1} \kappa_\lambda(z, z) \, d\mu_\lambda(z)$ and since $\kappa_\lambda(\mathbf{e}, \mathbf{e}) = \Delta(2e)^{-\lambda} = 2^{-r\lambda}$, we get by using (1.10), (1.12), (1.13) and (2.5),

$$(a_\lambda)^\wedge(\nu) = \frac{2^{r\lambda}}{c_\lambda} \int_D |\kappa_\lambda(z, \mathbf{e})|^2 \, \xi^D_{-i\nu + \rho}(z) \, d\mu_\lambda(z)$$

$$= 2^{r(2\lambda - p)} \iiint_D |\Delta(x + e + iy)|^{-2\lambda} \Delta_{-i\nu + \rho + \lambda - p}\left(x - \tfrac{1}{2}\Phi(u, u)\right) \, dm(u) dx dy.$$

We first perform the integration with respect to x and change the variable $x \mapsto t = x - \frac{1}{2}\Phi(u, u)$. Then we see that $(a_\lambda)^\wedge(\nu)$ equals

$$2^{r(2\lambda - p)} \int_\Omega \Delta_{-i\nu + \rho + \lambda - p}(t) \, dt \int_V dy \int_U \left|\Delta\left(t + \tfrac{1}{2}\Phi(u, u) + e + iy\right)\right|^{-2\lambda} dm(u).$$

Let us set $s := t + \frac{1}{2}\Phi(u, u) + e \in \Omega$ for simplicity. Then by [2, Proposition III.4.2],

$$\Delta(s + iy) = \Delta\big(P(s^{1/2})(e + iP(s^{-1/2})y)\big) = \Delta(s)\Delta\big(e + iP(s^{-1/2})y\big).$$

Since $\det P(s^{1/2}) = \Delta(s)^{n/r}$ (loc. cit.), the change of the variable $y \mapsto P(s^{1/2})y$ with s fixed yields

$$(a_\lambda)^\wedge(\nu) = 2^{r(2\lambda-p)} \int_\Omega \Delta_{-i\nu+\rho+\lambda-p}(t)\, dt \times$$

$$\times \int_U \Delta\big(t + \tfrac{1}{2}\Phi(u,u) + e\big)^{-2\lambda+n/r} dm(u) \int_V \Delta(e + y \circ y)^{-\lambda}\, dy.$$

Now the formula [2, Exercise VII.5] shows

(4.1) $$\int_V \Delta(e + y \circ y)^{-\lambda}\, dy = 4^{n-r\lambda}\pi^n \frac{\Gamma_\Omega(2\lambda - n/r)}{\Gamma_\Omega(\lambda)^2},$$

whereas Proposition 3.3 together with (1.8) gives

(4.2)
$$\int_U \Delta\big(t + \tfrac{1}{2}\Phi(u,u) + e\big)^{-2\lambda+n/r} dm(u) =$$
$$= (2\pi)^m \frac{\Gamma_\Omega(2\lambda - N/r)}{\Gamma_\Omega(2\lambda - n/r)} \Delta_{-2\lambda+N/r}(t + e).$$

Recalling (1.8) again, we therefore obtain

(4.3)
$$(a_\lambda)^\wedge(\nu) = \pi^N \frac{\Gamma_\Omega(2\lambda - N/r)}{\Gamma_\Omega(\lambda)^2} \int_\Omega \Delta_{-i\nu+\rho+\lambda-p}(t)\Delta_{-2\lambda+N/r}(t + e)\, dt$$
$$= \pi^N \frac{\Gamma_\Omega(2\lambda - N/r)}{\Gamma_\Omega(\lambda)^2} \cdot \frac{\Gamma_\Omega(-i\nu + \rho + \lambda - N/r)\Gamma_\Omega(i\nu^* - \rho^* + \lambda)}{\Gamma_\Omega(2\lambda - N/r)}$$

in view of Theorem 3.4. Hence we arrive at the formula in the theorem in case λ is sufficiently large.

Now both sides of the formula are analytic in λ, so that Theorem 4.1 follows by analytic continuation. $\square$

We conclude this paper by touching on the result of [9] concerning the Berezin transforms on D. Put $d\mu_0(z) := \kappa_\lambda(z,z)\,d\mu_\lambda(z,z)$. The *Berezin transform* B_λ associated to $H^2_\lambda(D)$ is the integral operator on $L^2(D, d\mu_0)$ given by

$$B_\lambda f(z) = \int_D A_\lambda(z, z')f(z')\, d\mu_0(z') \qquad (f \in L^2(D, d\mu_0)).$$

Since $d\mu_0 = c_\lambda d\mu$, the map I_λ defined by $I_\lambda f(gK) := c_\lambda^{1/2} f(g \cdot e)$ $(g \in G)$ gives rise to a unitary isomorphism of $L^2(D, d\mu_0)$ onto $L^2(G/K)$ in view of (1.16). Using I_λ, we transfer B_λ to the operator $B_\lambda^{G/K}$ on $L^2(G/K)$: $B_\lambda^{G/K} = I_\lambda B_\lambda I_\lambda^{-1}$. An easy computation making use of (1.14) and (1.15) shows

(4.4) $$B_\lambda^{G/K} f(gK) = c_\lambda \int_G a_\lambda(h^{-1}g)f(hK)\, dh \qquad (f \in L^2(G/K)).$$

On the other hand, let M be the centralizer of a in K. We put $A(gK, kM) := l(k^{-1}g)$ $(g \in G,\ k \in K)$. Then the Helgason-Fourier transform $\tilde{f}$ of a function

$f \in C_c^\infty(G/K)$ is defined as (see [5, p. 223])

$$\tilde{f}(\nu, b) := \int_{G/K} f(x) e^{(-i\nu+\rho)(A(x,b))} \, dx \qquad (\nu \in \mathfrak{a}^*, \ b \in K/M).$$

Let $\mathfrak{a}^+$ be the Weyl chamber in $\mathfrak{a}$ corresponding to $\mathfrak{n}$, and $\mathfrak{a}_+^*$ the dual Weyl chamber. Let $c(\nu)$ be the c-function of Harish-Chandra (cf. [4]). We denote by $d\nu$ and db the euclidean measure on $\mathfrak{a}^*$ and the normalized K-invariant measure on K/M respectively. Then by [5, Theorem III.1.5], the Helgason-Fourier transform $f \mapsto \tilde{f}$ extends to a unitary isomorphism of $L^2(G/K)$ onto $L^2(\mathfrak{a}_+^* \times K/M)$ with the measure $|c(\nu)|^{-2} \, d\nu db$. By (4.4) we have $B_\lambda^{G/K} f = c_\lambda f \times a_\lambda$ in the notation of [5, p. 225 (9)], so that Lemma III.1.4 in [5] shows

$$\left(B_\lambda^{G/K} f\right)^\sim (\nu, b) = c_\lambda \cdot (a_\lambda)^\wedge (\nu) \tilde{f}(\nu, b).$$

This together with Theorem 4.1 and (1.11) yields the spectral expression of B_λ given by Unterberger-Upmeier [9].

<div align="center">

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