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The Logarithmic Singularities of the Bergman Kernels for model domains

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1 Introduction

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and r its defining function. Let B_Ω be the Bergman kernel of the domain Ω restricted to the diagonal of $\Omega\times\Omega$. It was shown by Fefferman [F] that

$$B_\Omega=\varphi r^{-n-1}+\psi\log r$$

where $\varphi,\ \psi\in C^\infty(\bar{\Omega})$. Since singularities of the Begman kernel, $\varphi,\ \psi$ have geometric information of the domain, it is natural to use it to characterize domains. To be precise, we consider expansions of φ,ψ

$$\varphi=\sum_{k=0}^{n}\varphi_k r^k \bmod O(r^{n+1}),\qquad \psi\sim\sum_{k=0}^{\infty}\psi_k r^k$$

If we choose $r=r^F$ which satisfies certain transformation rule under biholomorphism, then φ_k,ψ_k are CR invariants, that is, polynomials in Moser's normal form coefficients satisfying certain transformation rule with weight k and $n+1+k$. By Chern-Moser theory, Moser's normal form coefficients are expressed in terms of CR curvature tensors. It implies that certain conditions on singularities φ_k,ψ_k decide the geometry of domains. (See [Hi2],[HK] for detail)

In this context Burns and Graham [G] proved :

Theorem 1. *Let $\Omega\subset\mathbb{C}^2$. The boundary of Ω is locally CR equivalent to the sphere if $\psi=O(r^2)$.*

To the direction of global characterization of domains, a well known conjecture by Ramadanov [R] is as follows:

Conjecture 1. *Let Ω be a bounded strictly pseudoconvex domain of $\mathbb{C}^n$. If its Bergman kernel does not have log term, then Ω is biholomorphic to the ball.*

Pertaining to this conjecture, Boichu and Coeuré [BC], and Nakazawa[N] proved that if $\Omega\subset\mathbb{C}^2$ is a bounded strictly pseudoconvex complete Reinhardt domain and ψ vanishes then Ω is biholomorphic to the ball. Hirachi [Hi1] proved that for general dimension, if the domains are ellipsoids close to the ball, then vanishing of log term ψ implies that the domain is the ball.

Let us state our main theorem. A domain $\Omega \in \mathcal{M}$ if and only if

$$\Omega = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^n : \Im(z_0) > F(z)\}$$

F : real analytic strictly plurisubharmonic function on $\mathbb{C}^n$ such that

1. $F(0) = \nabla F(0) = 0$
2. $F(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) = F(z_1, \cdots, z_n)$ for any $\theta_j \in \mathbb{R}$
3. There are small positive numbers c and ϵ such that $F(z) \geq c|z|^\epsilon$ for sufficiently large $|z| := (\sum_{j=1}^n |z_j|^2)^{1/2}$.

Theorem 2. *Let Ω be a domain that belongs to the class $\mathcal{M}$. Then Ω is biholomorphic to the ball if, and only if, its Bergman kernel function does not have logarithmic singularity at the boundary.*

Remark In the aspect of technique to get asymptotic expansion of Bergman kernel and compute φ, ψ in terms of defining function, Kashiwara's microlocal analysis was used in [BC], [N], [Hi1]. Graham computed expansion of ψ using higher asymptotics of Monge-Ampère equation and Moser's normal form coefficients. As a pertaining result, Hanges [Han] used Boutet de Monvel-Sjöstrand's [BS] expression of Szegö projection as Fourier integral operator to compute singularity of Szegö kernel.

2 Main ideas of Proof

First step We have an expansion formula for B_Ω on the diagonal:

Proposition 1.

$$B_\Omega(z_0, z) = \frac{1}{8\pi} \sum_{j=0}^{n+1} \varphi_j(z)(\Im z_0)^{-j-1} + \frac{1}{8\pi} \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!} \psi_p(z)(\Im z_0)^p \log(\Im z_0)$$

Our formula is based on Haslinger's formula [Has], which Kamimoto [K] used to get asymptotic expansion of the Bergman kernel for wider class of domains than ours. Haslinger's formula is as follows:

$$B_\Omega(z_0, z) = \frac{1}{2\pi} \int_0^\infty e^{-2\Im(z_0)\tau} K(z;\tau)\tau d\tau$$

where $K(\cdot;\tau)$ is Bergman kernel for

$$H_\tau(\mathbb{C}^n) = \{g \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |g|^2 e^{-2\tau F} dV < \infty\}.$$

In particular

$$K(z;\tau) = \sum_{\alpha\in\mathbf{Z}_+^n} \frac{|z|^{2\alpha}}{c_\alpha(\tau)^2}$$

where $|z|^{2\alpha} = |z_1|^{2\alpha_1}\cdots|z_n|^{2\alpha_n}$, and

$$c_\alpha(\tau)^2 = \int_{\mathbf{C}^n} |z|^{2\alpha}\, e^{-2\tau F(z)} dV(z).$$

Next we expand ψ_p. By assumption on F we have

$$F(z) = \sum_{j=1}^{n} |z_j|^2 + \sum_{k\geq 2} P_k(|z_1|^2,\ldots,|z_n|^2)$$

where

$$P_k(y_1,\ldots,y_n) = \sum_{|\beta|=k} C_\beta^{(k)} y^\beta$$

Set $S_+ = \{y \in \mathbb{R}_+^n : y_1+\cdots+y_n = 1\}$. Set $d\mu$ to be surface measure on S_+ and $d\mu_\alpha = y^\alpha d\mu$ Now expansion of ψ is given as

$$\psi_p(z) = \sum_{\alpha\in\mathbf{Z}_+^n} \psi_{p,\alpha}\, |z|^{2\alpha}$$

where

$$\begin{aligned}
\psi_{p,\alpha} &= \int_{S_+} P_{p+|\alpha|+n+3}\, d\mu_\alpha \\
&+ \int_{S_+} P_{p+|\alpha|+n+2}\, P_2\, d\mu_\alpha + \int_{S_+} P_{p+|\alpha|+n+2}\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \\
&+ \int_{S_+} P_{p+|\alpha|+n+1}\, P_3\, d\mu_\alpha + \int_{S_+} P_{p+|\alpha|+n+1}\, d\mu_\alpha \int_{S_+} P_3\, d\mu_\alpha \\
&+ \int_{S_+} P_{p+|\alpha|+n+1}\, P_2^2\, d\mu_\alpha + \int_{S_+} P_{p+|\alpha|+n+1}\, P_2\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \\
&+ \int_{S_+} P_{p+|\alpha|+n+1}\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \int_{S_+} P_2\, d\mu_\alpha \\
&+ \cdots \\
&+ \sum_{k=1}^{p+|\alpha|+n+2} \sum_{\substack{l_1+\ldots+l_k \\ =p+|\alpha|+n+2}} \int_{S_+} P_2^{l_1}\, d\mu_\alpha \cdots \int_{S_+} P_2^{l_k}\, d\mu_\alpha
\end{aligned}$$

where each term has proper constants, but we do not consider them here. We use method of stationary phase to expand $c_\alpha(\tau)^2$ in τ (See [K], section 6), which leads to the formula for $\psi_{p,\alpha}$.

Second step

Proposition 2. *If logarithmic singularity* $\Psi = \frac{1}{8\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!}\psi_p(z)(\Im z_0)^p = 0$, *that is* $\psi_{p,\alpha} = 0$, *then* $P_k = 0$ *for all* $k \geq 2$.

We consider $(C_\beta^{(k)})$ as an vector in $\mathbb{R}^{\nu_k}$, where $\nu_k =$ the number of all possible monomials in n variables of degree k. We can show that $\nu_k = (k+1)\cdots(k+n-1)/(n-1)!$. We denote it simply as $C^{(k)}$. Then we can consider $\psi_{p,\alpha}$ as a polynomial in $C^{(p+|\alpha|+n+3)},\ldots,C^{(2)}$. By algebraic operation we can show that the system $\psi_{p,\alpha} = 0$, $p \geq 0$, $\alpha \in \mathbb{Z}^n$ can be changed into a system of such form as

$$E_j^{(k)}(C^{(k)},\ldots,C^{(2)}) = 0, \quad j = 1,\ldots,N_k \quad k = n+3, n+4,\ldots$$

where $E_j^{(k)}(C^{(k)},\ldots,C^{(2)})$ is linear in $C^{(k)}$ and N_k is maximum number of all such polynomials deduced from the system $\psi_{p,\alpha} = 0$. There are two ways of getting such equations for each k. Set $k = l+n-3$. First note that for $p + |\alpha| = l$

$$\psi_{p,\alpha} = \int_{S_+} P_{l+n+3}\, y^\alpha d\mu + (P_j : j \leq l+n+2).$$

It gives desired equations for $E_j^{(k)}(C^{(k)},\ldots,C^{(2)}) = 0$ and the number of such equation is Card $\{\alpha \in \mathbb{Z}^n : |\alpha| \leq l\}$, which is $\sum_{j=1}^{l}\nu_j$. Second way of getting such equations is to use $\int_{S_+} P_{l+n+3}P_2^m y^\alpha d\mu$ which is found in

$$\psi_{p,\alpha} = \int_{S_+} P_{l+m+n+3} y^\alpha d\mu + (P_j : j \leq l+m+n+2)$$

where $p + |\alpha| = l + m$. By counting all such equations we can show that

$$N_k = \sum_{r=0}^{k-n-3} \lambda_k(r)\, \frac{(r+1)\cdots(r+n-2)}{(n-2)!}$$

where $\lambda_k(r) \in \{1,2,3\}$.

Our goal is to decide the zero set of suitable finite subsystem. We expect the zero set of such subsystem is trivial. At the same time we need some inductive relations between equations such that vanishing of $C^{(2)},\ldots,C^{(k)}$ for some k implies that $C^{(k+1)} = 0, C^{(k+2)} = 0\ldots$. First note that for small k, $N_k < \nu_k$, furthermore $N_k = 0$ for $k < n+3$. But for sufficiently large k we can show that $N_k > \nu_k$. For such k we have extra equations. By canceling $C^{(k)}$, we can rewrite such extra equations as equations in $C^{(k-1)},\ldots,C^{(2)}$. Thus by choosing k, say k_0, big enough we come to have enough equations to show that zero set of $E^{(n+3)} = 0,\ldots,E^{(k_0)} = 0$ is trivial.

Set again $k = l + n + 3$. First we show that $N_{l+n+3} - \nu_{l+n+3} > 0$ for some large l. It follows from

$$
\begin{aligned}
N_{l+n+3} - \nu_{l+n+3} &= \frac{((l+1)/2+1)\cdots((l+1)/2+n-2))}{(n-2)!} + 2\sum_{r=0}^{(l-1)/2} \frac{(r+1)\cdots(r+n-2)}{(n-2)!} \\
&\quad - \sum_{r=l+1}^{l+n+3} \frac{(r+1)\cdots(r+n-2)}{(n-2)!} \\
&= \frac{((l+1)/2+1)\cdots((l+1)/2+n-2))}{(n-2)!} + Q_{n-1}\Big(\frac{l-1}{2}\Big) - R_{n-2}(l)
\end{aligned}
$$

where Q_{n-1} is a polynomial of degree $n-1$ with positive leading coefficient and R_{n-2} is a polynomial of degree $n-2$.

Now canceling of $C^{(k)}$ in $E^{(k)} = 0$ for k such that $N_k > \nu_k$ is based on the observation that $E_j^{(k)}(C^{(k)}, \ldots, C^{(2)}) = 0, \quad j = 1, \ldots, N_k$ can be considered as

$$\sum_{|\beta|=k} C_\beta^{(k)} B(\beta, j) = \text{polynomial}(C^{(k-1)}, \ldots, C^{(2)}) \quad j = 1, \ldots, N_k$$

and $B(\beta, j)$, $|\beta| = k$, $j = 1, \ldots, \nu_k$ is nonsingular. $B(\beta, j)$ is n-dimensional beta function with some weight which increase as j increase. We finally have a system of such form as

$$E_j^{(k)}(C^{(k)}, \ldots, C^{(2)}) = 0, \quad j = 1, \ldots, \nu_k \quad k = 2, 3, \ldots, k_0$$

We can show that $E_j^{(2)}(C^{(2)}) = 0$, $j = 1, \ldots, \nu_2$ has trivial zero set, which implies that $P_2 = 0$. Inductively we can show that $P_k = 0$ for all $k > 2$.

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