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The Golden Trinity
– Optimality, Inequality and Identity –

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Abstract

This paper considers a triplet of optimization, inequality and identity through the Golden ratio. When a conditional optimization problem which optimizes an objective function under a constraint function yields a pair of optimum value and optimum point, an inequality holds between the objective function and the constraint function. Conversely, if an inequality holds between two functions together with an equality condition, we can easily solve the optimization problem by applying the inequality and the equality condition. Thus an optimization is equivalent to an inequality. Further an inequality between two functions is equivalently stated as an identity form by adding a supplementary nonnegative term. Conversely, an identity between two functions involving an additional nonnegative term is stated in an inequality form without the nonnegative term. Thus an inequality is also equivalent to an identity. Therefore, the triplet is equivalent each other. In this sense, we call an equivalent triplet *trinity*.

Furthermore, we introduce the Golden ratio into trinity – Golden optimum solution, Golden inequality, and Golden identity –. Such a trinity is called *Golden*. In a class of quadratic functions, we specify six Golden trinities on the basis of three quadratic forms : $x^2+y^2,\ x^2-y^2,\ x^2+(x-y)^2$. The triplet is called the *Golden triplet of quadratic forms.*

1 Introduction

It is well known that a cetain optimization problem is easily solved through application of an inequality involving both objective function and constraint function in optimization problem. Conversely a cetain inequality is proved by finding an optimal solution – a pair of optimum value and optimum point –. In general, an inequality together with an equality condition is stated as an equivalent identity involving a nonnegative term. Thus a certain triplet of optimization problem, inequality and identity turns out to be equivalent.

Throughout the paper we call a triplet of optimization, inequality and identity *trinity* if they are equivalent each other. If an optimum solution is Golden, we call it Golden optimum solution. We transliterate the Golden optimum solution both as Golden inequality and as Golden identity. Then the trinity is called *Golden*. In a class of quadratic functions, we specify six Golden trinities both in two-variable problems and in one-variable problems.

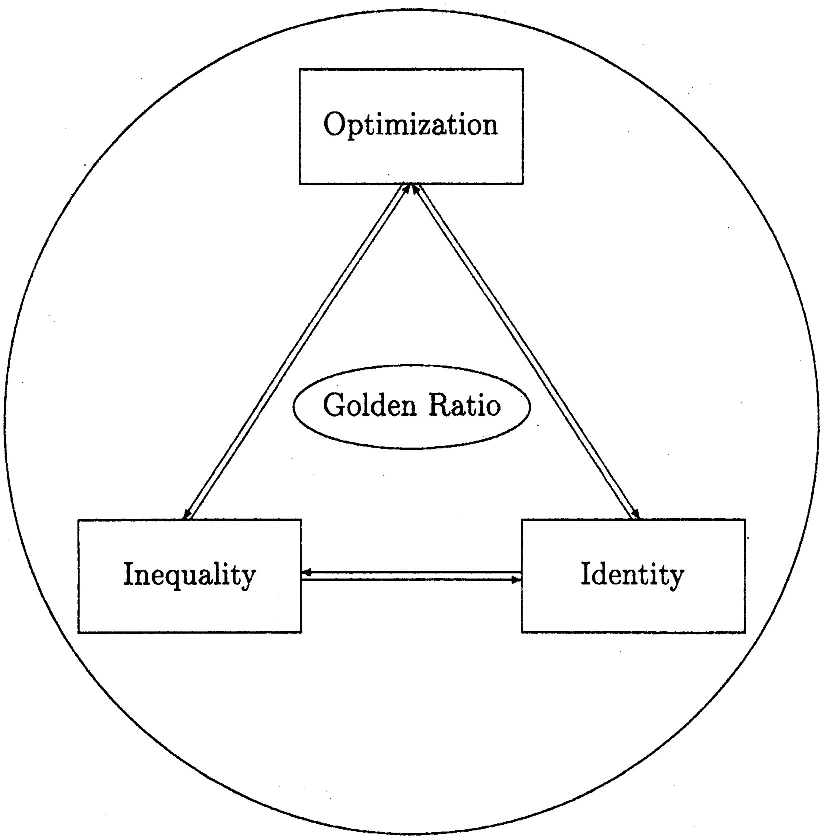

Figure 1 The Golden Trinity

2 Trinity

We take (i) optimization, (ii) inequality and (iii) identity as three expression forms. Let two two-variable functions $f, g : D \to R^1$ be given, where $D \subset R^2$. Then we consider the following triplet each of which is associated with two real constants α, β.

(i) An optimization problem

$$\mathrm{MA}_1 \qquad \begin{array}{ll} \text{Optimize} & f(x,y) \\ \text{subject to} & \text{(i)} \;\; g(x,y) = 1 \\ & \text{(ii)} \;\; (x,y) \in D \end{array}$$

has the optimum value $M = \beta$ at the point $(x^*, y^*) = \lambda(1, \alpha)$, namely, with the slope $\dfrac{y^*}{x^*} = \alpha$, where λ is a real number satisfying (i). (1)

(ii) It holds that

$$f(x,y) \ \leq (\geq)\ \beta\, g(x,y) \quad \text{on } D. \tag{2}$$

The sign of equality holds if and only if $y = \alpha x$.
(iii) It holds that

$$f(x,y) + h(\alpha x,\, y) = \beta\, g(x,y), \quad h(x,y) \geq (\leq)\ 0 \quad \text{on } D. \tag{3}$$

Lemma 2.1 *The triplet (1)-(2)-(3) is equivalent each other.*

In this paper we call an equivalent triplet *trinity.* Let us take a basic standard real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

The number ϕ is called the *Golden ratio.* ϕ is defined as the positive solution of quadratic equation

$$x^2 - x - 1 = 0.$$

We say that a pair $(\alpha,\ \beta)$ constitutes the *Golden ratio* if

$$\left|\frac{\beta}{\alpha}\right| = \phi \quad \text{or} \quad \left|\frac{\alpha}{\beta}\right| = \phi.$$

When the pair constitutes the Golden ratio, we say that the trinity (1)-(2)-(3) is *Golden.* In the case, we say as follows.

- MA_1 has a *Golden optimum solution* M, (x^*, y^*).
- The inequality (2) is *Golden.*
- The identity (3) is *Golden.*

2.1 Cauchy-Schwarz

Let us take $f(x,y) = (ax+by)^2$, $g(x,y) = x^2+y^2$, where a $(\neq 0)$, b are real constants. Then we have the following trinity (4)-(5)-(6).

(i) An optimization problem

$$\mathrm{MA}_2 \qquad \begin{array}{ll} \text{Maximize} & (ax+by)^2 \\ \text{subject to} & \text{(i)}\ \ x^2+y^2 = 1 \\ & \text{(ii)}\ \ -\infty < x,\ y < \infty \end{array}$$

has the maximum value $M = (a^2 + b^2)$ at the point $(x^*, y^*) = \pm \frac{1}{\sqrt{a^2+b^2}}(a, b)$,

namely, with the slope $\frac{y^*}{x^*} = \frac{b}{a}$. (4)

(ii) It holds that

$$(ax + by)^2 \leq (a^2 + b^2)(x^2 + y^2) \quad \text{on } R^2. \tag{5}$$

The sign of equality holds if and only if $ay = bx$.
(iii) The Lagrange identity ([1, p.3,p.60]) holds.

$$(ax + by)^2 + (bx - ay)^2 = (a^2 + b^2)(x^2 + y^2) \quad \text{on } R^2. \tag{6}$$

We note that $(bx - ay)^2 = a^2 \left(\frac{b}{a}x - y\right)^2$. Thus we have

$$\alpha = \frac{b}{a}, \quad \beta = a^2 + b^2, \quad h(x, y) = a^2(x - y)^2.$$

When α and β constitute the Golden ratio, the trinity becomes Golden.

2.2 Minkowski

Second we set $f(x, y) = (a + x)^2 + (b + y)^2$, $g(x, y) = x^2 + y^2$, where a ($\neq 0$), b are real constants. Then we have trinity (7)-(8)-(9) as follows.

(i) A maximization problem

$$\text{MA}_3 \qquad \begin{array}{ll} \text{Maximize} & (a + x)^2 + (b + y)^2 \\ \text{subject to} & \text{(i)}\ \ x^2 + y^2 = 1 \\ & \text{(ii)}\ \ -\infty < x,\ y < \infty \end{array}$$

has the maximum value $M = (\sqrt{a^2 + b^2} + 1)^2$ at the point $(x^*, y^*) = \pm \frac{1}{\sqrt{a^2+b^2}}(a, b)$,

namely, with the slope $\frac{y^*}{x^*} = \frac{b}{a}$. (7)

(ii) It holds that

$$\sqrt{(a + x)^2 + (b + y)^2} \leq \sqrt{a^2 + b^2} + \sqrt{x^2 + y^2} \quad \text{on } R^2. \tag{8}$$

The sign of equality holds if and only if $ay = bx$.
(iii) It holds that

$$\sqrt{(a + x)^2 + (b + y)^2} + H(x, y) = \sqrt{a^2 + b^2} + \sqrt{x^2 + y^2} \quad \text{on } R^2 \tag{9}$$

where

$$H(x, y) = \frac{2(bx - ay)^2}{(\sqrt{a^2 + b^2} + \sqrt{x^2 + y^2} + \sqrt{(a + x)^2 + (b + y)^2})(\sqrt{a^2 + b^2}\sqrt{x^2 + y^2} + ax + by)}.$$

2.3 Trinity for one-variable functions

Let two one-variable functions $f, g : E \to R^1$ be given, where $E \subset R^1$. We assume that $g(u) > 0$ on E.

(i) A fractional optimization problem

$$\mathrm{MA}_4 \qquad \begin{array}{ll} \text{Optimize} & \dfrac{f(u)}{g(u)} \\ \text{subject to} & \text{(i)} \ \ u \in E \end{array}$$

has the optimum value $M = \beta$ at the point $u^* = \alpha$. (10)

(ii) It holds that

$$f(u) \ \leq (\geq) \ \beta\, g(u) \qquad \text{on} \ \ E. \tag{11}$$

The sign of equality holds if and only if $u = \alpha$.
(iii) It holds that

$$f(u) + h(u) = \beta\, g(u), \quad h(u) \geq (\leq) \ 0 \quad \text{on} \ E, \quad h(\alpha) = 0. \tag{12}$$

Lemma 2.2 *The triplet (10)-(11)-(12) is equivalent each other.*

An equivalent triplet is called *trinity*. When α and β constitute the Golden ratio, we say that the trinity (10)-(11)-(12) is *Golden*. In the case, we say as follows.

- MA_4 has a *Golden optimum solution* M, u^*.
- The inequality (11) is *Golden*.
- The identity (12) is *Golden*.

For instance we have the Golden trinity (13)-(14)-(15) as follows.

(i) A fractional optimization problem

$$\begin{array}{ll} \text{Maximize} & \dfrac{1+u^2}{1+(u-1)^2} \\ \text{subject to} & \text{(i)} \ \ -\infty < u < \infty \end{array}$$

has the maximum value $M = 1 + \phi$ at the point $u^* = \phi$. (13)

(ii) It holds that

$$1 + u^2 \ \leq \ (1+\phi)\{1 + (u-1)^2\} \qquad \text{on} \ \ R^1. \tag{14}$$

The sign of equality holds if and only if $u = \phi$.
(iii) It holds that

$$1 + u^2 + \phi(\phi - u)^2 \ = \ (1+\phi)\{1 + (u-1)^2\}. \tag{15}$$

3 Non-Golden and Golden

This section illustrates a non-Golden trinity and a Golden trinity for simple two-variable quadratic forms.

3.1 Non-Golden Trinity

First of all, let us take two quadratic forms $f(x,y) = xy,\ \ g(x,y) = x^2+y^2$. Then we have trinity (16)-(17)-(18) as follows.
(i) The maximization problem

$$\begin{array}{lll} & \text{Maximize} & xy \\ \text{MA}_5 & \text{subject to} & \text{(i)}\ \ x^2+y^2=1 \\ & & \text{(ii)}\ \ -\infty < x,\ y < \infty. \end{array}$$

has the maximum value $M = \dfrac{1}{2}$ at the points $(x^*, y^*) = \pm\dfrac{1}{\sqrt{2}}(1,1)$, namely, with the slope $\dfrac{y^*}{x^*} = 1$. (16)

(ii) It holds that

$$xy \le \frac{1}{2}(x^2+y^2) \quad \text{on} \quad R^2. \tag{17}$$

The sign of equality holds if and only if $y = x$.
(iii) It holds that

$$xy + \frac{1}{2}(x-y)^2 = \frac{1}{2}(x^2+y^2) \quad \text{on} \quad R^2. \tag{18}$$

This is not Golden:

$$\alpha = 1, \quad \beta = \frac{1}{2}, \quad h(x,y) = \frac{1}{2}(x-y)^2$$

Then we call trinity (16)-(17)-(18) *non-Golden*.

3.2 Golden Trinity

Second we take another pair of quadratic forms $f(x,y) = x^2+y^2,\ \ g(x,y) = x^2+(y-x)^2$. Then we have the following trinity.

(i) The maximization problem

$$\begin{array}{lll} & \text{Maximize} & x^2+y^2 \\ \text{MA}_6 & \text{subject to} & \text{(i)}\ \ x^2+(y-x)^2=1 \\ & & \text{(ii)}\ \ -\infty < x,\ y < \infty \end{array}$$

has the maximum value $M = 1 + \phi$ at the maximum points $(x^*, y^*) = \pm\lambda(1, \phi)$, namely, with the slope $\dfrac{y^*}{x^*} = \phi$, where $\lambda = \dfrac{1}{\sqrt{3-\phi}}$. (19)

(ii) It holds that

$$x^2 + y^2 \le (1+\phi)\{x^2 + (y-x)^2\} \quad \text{on } R^2. \tag{20}$$

The sign of equality holds if and only if $y = \phi x$.
(iii) It holds that

$$x^2 + y^2 + \phi(\phi x - y)^2 = (1+\phi)\{x^2 + (y-x)^2\} \quad \text{on } R^2. \tag{21}$$

Further we see that both ϕ and $1+\phi$ constitute the Golden ratio:

$$\frac{1+\phi}{\phi} = \phi.$$

Thus we have a Golden trinity.

4 The Golden Optimum Solutions

This section shows six optimization problems whose optimum solution is Golden.

4.1 Two-variable problems

First we consider two-variable quadratic optimization problems.

(i) Now we take the maximization and minimization problem

$$\mathrm{TP}_1 \qquad \begin{array}{ll} \text{Maximize and minimize} & x^2 + y^2 \\ \text{subject to} & \text{(i)}\ \ x^2 + (y-x)^2 = 1 \\ & \text{(ii)}\ \ -\infty < x, y < \infty. \end{array}$$

Then TP_1 has the maximum value $M = 1 + \phi$ at the points $(x^*, y^*) = \pm\dfrac{1}{\sqrt{3-\phi}}(1, \phi)$ and the minimum value $m = 2 - \phi$ at the points $(\hat{x}, \hat{y}) = \pm\dfrac{1}{\sqrt{2+\phi}}(1, 1-\phi)$.

(ii) We consider the maximization and minimization problem

$$\mathrm{TP}_2 \qquad \begin{array}{ll} \text{Maximize and minimize} & -x^2 + y^2 \\ \text{subject to} & \text{(i)}\ \ x^2 + (y-x)^2 = 1 \\ & \text{(ii)}\ \ -\infty < x, y < \infty. \end{array}$$

Then TP_2 has the maximum value $M = \phi$ at the points $(x^*, y^*) = \pm\frac{1}{\sqrt{2+\phi}}(1,\, 1+\phi)$ and the minimum value $m = 1 - \phi$ at the points $(\hat{x}, \hat{y}) = \pm\frac{1}{\sqrt{3-\phi}}(1,\, 2-\phi)$.

(iii) Let us now consider the maximization problem

$$
\mathrm{TP}_3 \qquad \begin{array}{ll} \text{Maximize} & -y^2 - (y-x)^2 \\ \text{subject to} & \text{(i)} \quad -x^2 + y^2 = 1 \\ & \text{(ii)} \quad -\infty < x, y < \infty. \end{array}
$$

Then TP_3 has the maximum value $M = -\phi$ at the points $(x^*, y^*) = \pm\frac{1}{\sqrt{1+3\phi}}(1,\, 1+\phi)$.

(iv) Let us consider the related minimization problem

$$
\mathrm{TP}_4 \qquad \begin{array}{ll} \text{minimize} & y^2 + (y-x)^2 \\ \text{subject to} & \text{(i)} \quad x^2 - y^2 = 1 \\ & \text{(ii)} \quad -\infty < x, y < \infty. \end{array}
$$

Then TP_4 has the minimum value $m = -1+\phi$ at the points $(\hat{x}, \hat{y}) = \pm\frac{1}{\sqrt{-4+3\phi}}(1,\, 2-\phi)$.

4.2 One-variable problems

Second we consider one-variable quadratic-fractional optimization problems.

(i) Now we take the maximization and minimization problem

$$
\mathrm{OP}_1 \qquad \begin{array}{ll} \text{Maximize and minimize} & \dfrac{1+u^2}{1+(u-1)^2} \\ \text{subject to} \quad \text{(i)} & -\infty < u < \infty. \end{array}
$$

Then OP_1 has the maximum value $M = 1 + \phi$ at the point $u^* = \phi$ and the minimum value $m = 2 - \phi$ at the point $\hat{u} = 1 - \phi$.

(ii) We consider the maximization and minimization problem

$$
\mathrm{OP}_2 \qquad \begin{array}{ll} \text{Maximize and minimize} & \dfrac{-1+v^2}{1+(v-1)^2} \\ \text{subject to} \quad \text{(i)} & -\infty < v < \infty. \end{array}
$$

Then OP_2 has the maximum value $M = \phi$ at the point $v^* = 1 + \phi$ and the minimum value $m = 1 - \phi$ at the point $\hat{v} = 2 - \phi$.

(iii) Let us now consider the maximization problem

$$\mathrm{OP}_3 \qquad \begin{array}{ll} \text{Maximize} & \dfrac{u^2+(1-u)^2}{1-u^2} \\ \text{subject to} & \text{(i)} \ \ |u|>1. \end{array}$$

Then OP_3 has the maximum value $M = -\phi$ at the point $u^* = 1+\phi$.

(iv) Let us consider the corresponding minimization problem

$$\mathrm{OP}_4 \qquad \begin{array}{ll} \text{minimize} & \dfrac{u^2+(1-u)^2}{1-u^2} \\ \text{subject to} & \text{(i)} \ \ -1<u<1. \end{array}$$

Then OP_4 has the minimum value $m = -1+\phi$ at the point $\hat{u} = 2-\phi$.

5 The Golden Inequalities

This section shows six Golden inequalities both in two-variable functions and in one-variable functions.

5.1 Two-variable functions

First we consider Golden inequalities between two-variable quadratic functions $f,\ g : R^2 \to R^1$. Each of (22) and (23) yields a pair of Golden inequalities. Both (24) and (25) constitute a pair of Golden inequalities. Thus we have six Golden inequalities in the following.

Theorem 5.1 (i) *It holds that*

$$(2-\phi)\{x^2+(y-x)^2\} \le x^2+y^2 \le (1+\phi)\{x^2+(y-x)^2\} \quad \text{on} \ \ R^2. \tag{22}$$

The sign of gleft equality holds if and only if $y=(1-\phi)x$ and the sign of right equality holds if and only if $y=\phi x$.
(ii) *It holds that*

$$(1-\phi)\{x^2+(y-x)^2\} \le -x^2+y^2 \le \phi\{x^2+(y-x)^2\} \quad \text{on} \ \ R^2. \tag{23}$$

The sign of left equality holds if and only if $y=(2-\phi)x$ and the sign of right equality holds if and only if $y=(1+\phi)x$.
(iii) *The middle-right inequality (resp. left-middle) in (22) is equivalent to the left-middle (resp. middle-right) inequality in (23).*

Theorem 5.2 (i) *It holds that*

$$(-1+\phi)(x^2-y^2) \le y^2+(y-x)^2 \quad \text{on} \ \ R^2. \tag{24}$$

The sign of equality holds if and only if $y = (2-\phi)x$. It holds that

$$-y^2 - (y-x)^2 \geq \phi(x^2 - y^2) \quad \text{on} \quad R^2. \tag{25}$$

The sign of equality holds if and only if $y = (1+\phi)x$.
(ii) *The inequality (24) is equivalent to the middle-right inequality in (23). The inequality (25) is equivalent to the left-middle inequality in (23).*

5.2 One-variable functions

Second we consider six Golden inequalities between two one-variable functions $f,\ g : R^1 \to R^1$. The inequalities (26) and (27) are pairs of Golden inequalities. The inequalities (28) and (29) are Golden. Thus we have also six Golden inequalities in the following.

Lemma 5.1 (i) *It holds that*

$$(2-\phi)\{1+(u-1)^2\} \leq 1+u^2 \leq (1+\phi)\{1+(u-1)^2\} \quad \text{on} \quad R^1. \tag{26}$$

The sign of left equality holds if and only if $u = 1-\phi$ and the sign of right equality holds if and only if $u = \phi$ (Figure 2).
(ii) *It holds that*

$$(1-\phi)\{1+(v-1)^2\} \leq -1+v^2 \leq \phi\{1+(v-1)^2\} \quad \text{on} \quad R^1. \tag{27}$$

The sign of left equality holds if and only if $v = 2-\phi$ and the sign of right equality holds if and only if $v = 1+\phi$.
(iii) *The middle-right inequality (resp. left-middle) in (26) is equivalent to the left-middle (resp. middle-right) inequality in (27).*

Lemma 5.2 (i) *It holds that*

$$(-1+\phi)(1-u^2) \leq u^2 + (u-1)^2 \quad \text{on} \quad R^1 \tag{28}$$

The sign of equality holds if and only if $u = 2-\phi$. It holds that

$$-u^2 - (u-1)^2 \leq \phi(1-u^2) \quad \text{on} \quad R^1 \tag{29}$$

The sign of equality holds if and only if $u = 1+\phi$.
(ii) *The inequality (28) is equivalent to the middle-right inequality in (27). The inequality (29) is equivalent to the left-middle inequality in (27).*

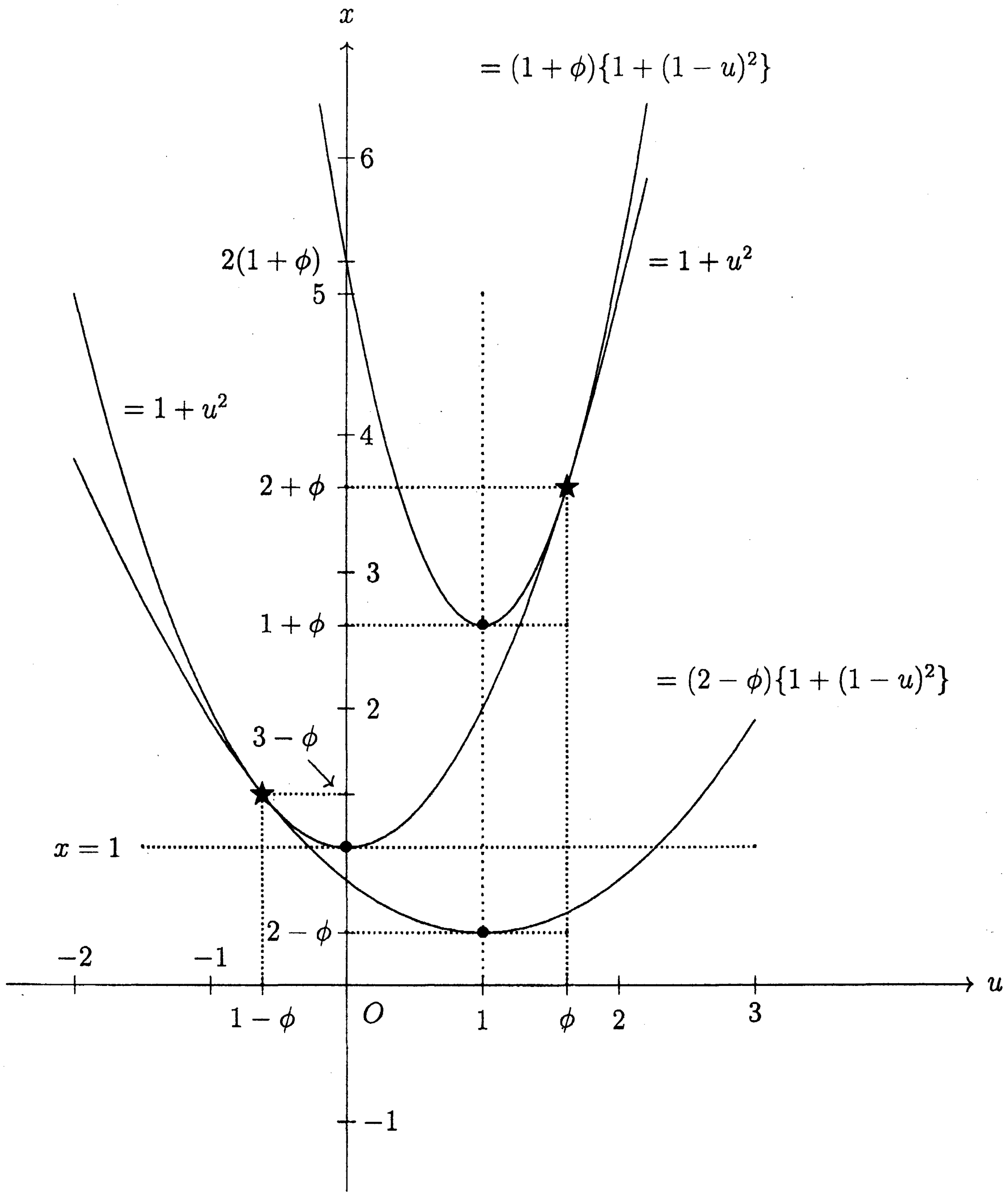

Figure 2 A pair of Golden Inequalities

$(2-\phi)\{1+(1-u)^2\} \le 1+u^2 \le (1+\phi)\{1+(1-u)^2\}$.

The left and right equalities attain at $\hat{u} = 1-\phi$ and $u^* = \phi$, respectively.

6 The Golden Identities

This section shows six Golden identities both in two-variable and in one-variable.

6.1 Two-variable quadratic functions

We have six Golden identities between three quadratic forms.

Theorem 6.1 (i) *It holds that*

$$x^2 + y^2 + \phi(\phi x - y)^2 = (1+\phi)\{x^2 + (y-x)^2\} \tag{30}$$

$$(2-\phi)\{x^2 + (y-x)^2\} + (\phi - 1)\{(1-\phi)x - y\}^2 = x^2 + y^2. \tag{31}$$

(ii) *It holds that*

$$-x^2 + y^2 + (\phi - 1)\{(1+\phi)x - y\}^2 = \phi\{x^2 + (y-x)^2\} \tag{32}$$

$$(1-\phi)\{x^2 + (y-x)^2\} + \phi\{(2-\phi)x - y\}^2 = -x^2 + y^2. \tag{33}$$

(iii) *The identity (30) (resp. (31)) is equivalent to the identity (33) (resp. (32)).*

Theorem 6.2 (i) *It holds that*

$$(-1+\phi)(x^2 - y^2) + (1+\phi)\{(2-\phi)x - y\}^2 = y^2 + (y-x)^2 \tag{34}$$

$$-\{y^2 + (y-x)^2\} + (2-\phi)\{(1+\phi)x - y\}^2 = \phi(x^2 - y^2). \tag{35}$$

(ii) *The identity (34) (resp. (35)) is equivalent to the identity (33) (resp. (32)).*

6.2 One-variable quadratic functions

We have the corresponding Golden identities between one-variable quadratic functions.

Theorem 6.3 (i) *It holds that*

$$1 + u^2 + \phi(\phi - u)^2 = (1+\phi)\{1 + (u-1)^2\} \tag{36}$$

$$(2-\phi)\{1 + (u-1)^2\} + (\phi - 1)(1 - \phi - u)^2 = 1 + u^2. \tag{37}$$

(ii) *It holds that*

$$-1 + v^2 + (\phi - 1)(1 + \phi - v)^2 = \phi\{1 + (v-1)^2\} \tag{38}$$

$$(1-\phi)\{1 + (v-1)^2\} + \phi(2 - \phi - v)^2 = -1 + v^2. \tag{39}$$

(iii) *The identity (36) (resp. (37)) is equivalent to the identity (39) (resp. (38)).*

Theorem 6.4 (i) *It holds that*

$$(-1+\phi)(1 - u^2) + (1+\phi)(2 - \phi - u)^2 = u^2 + (u-1)^2 \tag{40}$$

$$-\{u^2 + (u-1)^2\} + (2-\phi)(1 + \phi - u)^2 = \phi(1 - u^2). \tag{41}$$

(ii) *The identity (40) (resp. (41)) is equivalent to the identity (39) (resp. (38)).*

7 The Golden Trinities

As a summary, we enumerate six Golden trinities both in two-variable and in one-variable.

7.1 Two-variable functions

1. Maximum solution in TP_1 – Right-middle inequality (22) – Identity (30)

This Golden trinity is stated in (19)-(20)-(21).

2. Minimum solution in TP_1 – Left-middle inequality (22) – Identity (31)

3. Maximum solution in TP_2 – Right-middle inequality (23) – Identity (32)

4. Minimum solution in TP_2 – Left-middle inequality (23) – Identity (33)

5. Maximum solution in TP_3 – Inequality (24) – Identity (34)

6. Minimum solution in TP_4 – Inequality (25) – Identity (35)

7.2 One-variable functions

1. Maximum solution in OP_1 – Right-middle inequality (26) – Identity (36)

This Golden trinity is stated in (13)-(14)-(15).

2. Minimum solution in OP_1 – Left-middle inequality (26) – Identity (37)

3. Maximum solution in OP_2 – Right-middle inequality (27) – Identity (38)

4. Minimum solution in OP_2 – Left-middle inequality (27) – Identity (39)

5. Maximum solution in OP_3 – Inequality (28) – Identity (40)

6. Minimum solution in OP_4 – Inequality (29) – Identity (41)

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