数理解析研究所講究録
1488 巻 2006 年 116-122

Swaption Price by General Gram–Charlier Expansion

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November 20, 2005

Abstract

A general Gram–Charlier expansion gives an approximation of a density function by an arbitrary density function. We apply the method to an approximation of a swaption price by using a density function of a zero coupon bond with our hope to obtain the accuracy. The alternative method is constructed efficiently by combining the results of Tanaka, Yamada and Watanabe (2005), Jarrow and Rudd (1982) and Fourier inversion techniques.

1 Introduction

The valuation technique of interest rate derivatives has been receiving much attention from researchers. Tanaka, Yamada and Watanabe (2005) ("TYW" hereafter) provides an efficient method to approximate prices of several derivative products including a swaption. They use a Gram–Charlier expansion of a density function with a normal distribution. The efficiency is gained by the fact that all terms in the expansion can be obtained very accurately owing to the normal distribution. However, the approximation performance depends on the distribution of the underlying state variables that drives the interest rates.

The purpose of this paper is to describe an alternative method to approximate a swaption price by using a density function of a zero coupon bond with a general Gram–Charlier expansion. The idea comes from the fact that the main factor to affect the value of a swap is the price of zero coupon bond maturing on the final payment date of the swap. Originally such a general approximation formula is presented by Jarrow and Rudd (1982) for a stock option. We apply their approach with a swap value which may take both positive values and negative. We call it the expansion the general Gram–Charlier

*This research stemmed from my collaboration on a research activity with Takeshi Yamada and Toshiaki Watanabe.

expansion to keep the consistency in the terminology with TYW though Jarrow and Rudd (1982) call it the general Edgeworth expansion. To replace the normal distribution with an arbitrary distribution, numerical calculations are required. Fourier inversion techniques are useful for the numerical integration as discussed in Carr and Madan (1999) and Chen and Scott (1995). Hopefully our approach may contribute to improve the approximation accuracy.

The rest of this paper is organized as follows. In Section 2, we introduce the Gram–Charlier expansion along with TYW. In Section 3, we discuss the alternative method by a bond price. Section 4 concludes the paper.

2 Gram–Charlier Expansion

First we will review the results of Gram–Charlier expansion by TYW. The basic idea is to approximate a density function with one of a standard normal distribution to obtain an approximated swaption price.

The stochastic interest rates are assumed to be driven by a vector of the state variables X which is a Markov diffusion process satisfying

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where W is an n-dimensional Brownian motion on $(\Omega, \mathcal{F}, Q)$. We assume that Q is a risk-neutral probability measure. A filtration $\{\mathcal{F}_t : t \in [0, T^*]\}$ is the augmented filtration generated by W.

Consider a receiver's swaption with the expiry T_0 and the fixed rate K during a period $[T_0, T_N]$. The relevant dates are $T_0 < T_1 < \cdots < T_N$, which are set at regularly spaced time intervals, with $\delta = T_i - T_{i-1}$ for all i. The time-t price of a zero coupon bond with a maturity date of T is denoted by $P(t,T)$. By the linearity of the valuation, the value $SV(t)$ of the underlying swap at time t is written as a linear combination of the zero coupon bond prices

$$SV(t) = -P(t,T_0) + \delta K \sum_{i=1}^{N} P(t,T_i) + P(t,T_N) \equiv \sum_{i=0}^{N} a_i P(t,T_i), \tag{1}$$

where a_i is the amount of cash flow at time T_i. Then the swaption value $SOV(t)$ at time t is the discounted value of the expectation of the gain from exercising under the T_0-forward measure Q^{T_0}

$$SOV(t) = P(t,T_0)E^{T_0}\left[1_{\{SV(T_0)>0\}}SV(T_0) \mid \mathcal{F}_t\right] = P(t,T_0)\int_0^{\infty} xf(x)dx, \tag{2}$$

where f is the density function of the swap value $SV(T_0)$ at the expiry date T_0 under the T_0-forward measure conditioned on $\mathcal{F}_t$. Therefore, it is enough to obtain the density function of the value of the underlying swap under the T_0-forward measure for the calculation of the swaption price.

The first step to obtain the density function is to calculate the bond moment of the bonds involved in the valuation of the cash flow upon the exercise of the swaption. For a given set of dates $T, T_0, U_1, \dots, U_m$ ($T \le T_0 \le U_i$ for all $i = 1, \dots, m$), the bond moment is defined under the forward measure as

$$\mu^T(t, T_0, \{U_1, \cdots, U_m\}) \equiv E^{T_0}\Big[\prod_{i=1}^{m} P(T_0, U_i) \mid X_t\Big]$$

and it can be calculated as a function of X_t either analytically or numerically.

As the second step it is easy to obtain the m-th swap moment with the bond moments and the cash flows as

$$\begin{aligned} M_m(t) &= E^{T_0}\Big[SV(T_0)^m \mid X_t\Big] \\ &= E^{T_0}\Big[\Big(\sum_{i=0}^{N} a_i P(T_0, T_i)\Big)^m \mid X_t\Big] \\ &= \sum_{0 \le i_1, \dots, i_m \le N} a_{i_1} \cdots a_{i_m} \mu^{T_0}(t, T_0, \{T_{i_1}, \cdots, T_{i_m}\}). \end{aligned}$$

Then we know the n-th cumulant $c_n(t)$ from the set of the moments $\{M_m(t)\}_m$. Define the weighted cumulant $C_n = c_n(t)P(t, T_0)^n$ for $n \geq 1$, and coefficients q_n as $q_0 = 1$, $q_1 = q_2 = 0$, and for $n \geq 3$

$$q_n = \sum_{m=1}^{[n/3]} \sum_{n_1+\cdots+n_m=n, n_i \geq 3} \frac{c_{n_1} \cdots c_{n_m}}{m! n_1! \cdots n_m!} \Big(\frac{1}{\sqrt{c_2}}\Big)^n = \sum_{m=1}^{[n/3]} \sum_{n_1+\cdots+n_m=n, n_i \geq 3} \frac{C_{n_1} \cdots C_{n_m}}{m! n_1! \cdots n_m!} \Big(\frac{1}{\sqrt{C_2}}\Big)^n.$$

The definition of q_n looks complicated but the calculation is easy to do, for example,

$$q_3 = \frac{C_3}{3! C_2^{3/2}}, \quad q_4 = \frac{C_4}{4! C_2^2}, \quad q_5 = \frac{C_5}{5! C_2^{5/2}}, \quad q_6 = \frac{C_6 + 10C_3^2}{6! C_2^3}, \quad q_7 = \frac{C_7 + 35C_3C_4}{7! C_2^{7/2}}.$$

Now, let ϕ be the density function of a standard normal distribution $N(0,1)$, and H_n be the n-th Hermite polynomial defined by $H_n(x) = (-1)^n \phi(x)^{-1} \frac{d^n}{dx^n}\phi(x)$. By definition,

$$\begin{aligned} &H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \\ &H_4(x) = x^4 - 6x^2 + 3, \quad H_5(x) = x^5 - 10x^3 + 15x, \\ &H_6(x) = x^6 - 15x^4 + 45x^2 - 15, \quad H_7(x) = x^7 - 21x^5 + 105x^3 - 105x. \end{aligned}$$

The Gram–Charlier expansion is an orthogonal decomposition of a density function by $\{H_n\}_n$ with a weight of ϕ. The Gram–Charlier expansion states that the continuous density function f of a random variable Y can be expanded as a series

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\Big(\frac{x - c_1}{\sqrt{c_2}}\Big)\phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big). \tag{3}$$

The expansion is obtained by making use of the Fourier transforms of the characteristic function as shown in TYW. Since the Hermite polynomials have the orthogonal property $\int_{-\infty}^{\infty} H_k(x)H_l(x)\phi(x)dx = \delta_{kl}k!$ with respect to the Gaussian measure, q_n is also represented as $q_n = \frac{1}{n!}E[H_n(\frac{Y-c_1}{\sqrt{c_2}})]$. By the properties of the Hermite polynomials the Gram–Charlier expansion may be interpreted as the Wiener–Chaos expansion.

Applying the Gram–Charlier expansion to $Y = SV(T_0)$, the swaption value is expanded as

$$
\begin{aligned}
& SOV(t) \\
= \; & P(t,T_0)E^{T_0}\Big[1_{\{SV(T_0)>0\}}SV(T_0) \mid \mathcal{F}_t\Big] \\
= \; & P(t,T_0)\Big[c_1 N(\frac{c_1}{\sqrt{c_2}}) + \sqrt{c_2}\phi(\frac{c_1}{\sqrt{c_2}}) + \sqrt{c_2}\phi(\frac{c_1}{\sqrt{c_2}})\sum_{n=3}^{\infty}(-1)^n q_n H_{n-2}(\frac{c_1}{\sqrt{c_2}})\Big] \\
= \; & C_1 N(\frac{C_1}{\sqrt{C_2}}) + \sqrt{C_2}\phi(\frac{C_1}{\sqrt{C_2}}) + \sqrt{C_2}\phi(\frac{C_1}{\sqrt{C_2}})\sum_{n=3}^{\infty}(-1)^n q_n H_{n-2}(\frac{C_1}{\sqrt{C_2}}), \qquad (4)
\end{aligned}
$$

where N is the distribution function of a standard normal distribution $N(0,1)$. For some integer L, by truncating higher terms than $n = L$ in (4), the swaption value is approximated as

$$
SOV(t) \approx C_1 N(\frac{C_1}{\sqrt{C_2}}) + \sqrt{C_2}\phi(\frac{C_1}{\sqrt{C_2}}) + \sqrt{C_2}\phi(\frac{C_1}{\sqrt{C_2}})\sum_{n=3}^{L}(-1)^n q_n H_{n-2}(\frac{C_1}{\sqrt{C_2}}). \qquad (5)
$$

TYW suggests either $L = 3$ or $L = 7$ for a practical application.

3 Approximation of Swaption Price by Bond Price

Jarrow and Rudd (1982) shows an approximation method of an option price with an arbitrary process. It is worthwhile of regarding (3) as a decomposition by a normal distribution. Following the spirit of Jarrow and Rudd (1982), we will present an alternative approximation of the density function of the underlying swap value.

For a random variable Y we denote the characteristic functions by ϕ_Y and the n-th cumulant by $c_n(Y)$ under the T_0-forward measure. Suppose that two random variables F and G have the density function f and g, respectively, under the T_0-forward measure. By definition, the characteristic functions ϕ_F of F and ϕ_G of G are expanded as

$$
\begin{aligned}
\ln \phi_F(u) &= \sum_{n=1}^{\infty}\frac{c_n(F)}{n!}(iu)^n, \\
\ln \phi_G(u) &= \sum_{n=1}^{\infty}\frac{c_n(G)}{n!}(iu)^n.
\end{aligned}
$$

Then since $\ln\frac{\phi_F(u)}{\phi_G(u)} = \sum_{n=1}^{\infty}\frac{c_n(F)-c_n(G)}{n!}(iu)^n$, we have

$$
\phi_F(u) = \exp\Big(\sum_{n=1}^{\infty}\frac{c_n(F)-c_n(G)}{n!}(iu)^n\Big)\phi_G(u) = \Big[1 + \sum_{k=1}^{\infty}\frac{1}{k!}\Big(\sum_{n=1}^{\infty}\frac{c_n(F)-c_n(G)}{n!}(iu)^n\Big)^k\Big]\phi_G(u).
$$

By reordering the terms of $(iu)^n$, the ratio of the two functions is written as a series

$$
\phi_F(u) = \sum_{n=0}^{\infty}\frac{Q_n}{n!}(iu)^n\phi_G(u),
$$

with appropriate coefficients Q_n such as

$$Q_0 = 1, \quad Q_1 = c_1(F) - c_1(G), \quad Q_2 = c_2(F) - c_2(G) + (c_1(F) - c_1(G))^2,$$
$$Q_3 = c_3(F) - c_3(G) + 3(c_1(F) - c_1(G))(c_2(F) - c_2(G)) + (c_1(F) - c_1(G))^3.$$

Then by operating inverse Fourier transforms on the characteristic functions Jarrow and Rudd (1982) concludes that the density function f is expressed with g as

$$f(x) = \sum_{n=0}^{\infty} \frac{Q_n}{n!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} (iu)^n \phi_G(u) du = \sum_{n=0}^{\infty} \frac{(-1)^n Q_n}{n!} g^{(n)}(x). \tag{6}$$

We call the expansion (6) the general Gram–Charlier expansion to keep the consistency in the terminology with TYW though Jarrow and Rudd (1982) call it the general Edgeworth expansion. The Gram–Charlier expansion (3) is a special case of (6) with $g(x) = \phi((x - c_1)/\sqrt{c_2})$.

By assuming $\lim_{x\to\infty} g^{(n)}(x) = 0$ and using integration by parts, it is easy to observe that the expectation of the positive part of a random variable is formulated as

$$\begin{aligned} \int_0^{\infty} x f(x) dx &= \int_0^{\infty} x \sum_{n=0}^{\infty} \frac{(-1)^n Q_n}{n!} g^{(n)}(x) dx \\ &= Q_0 \int_0^{\infty} x g(x) dx + Q_1 \int_0^{\infty} g(x) dx + \frac{Q_2}{2} g(0) + \sum_{n=3}^{\infty} \frac{(-1)^n Q_n}{n!} g^{(n-2)}(0). \end{aligned} \tag{7}$$

For the application of the general Gram–Charlier expansion to a swaption valuation, the basic idea to choose the approximating random variable is that the main factor to affect the value of a swap is the price of zero coupon bond maturing on the final payment date of the swap. For simplicity of notations we assume $t = 0$. For the application the two random variables F and G are defined as follows. Let the approximated random variable F be the swap value at the expiry $SV(T_0)$

$$F = -1 + \delta K \sum_{i=1}^{N} P(T_0, T_i) + P(T_0, T_N).$$

And let the approximating random variable G be the zero coupon bond price $P(T_0, T_N)$ with maturity T_N plus a constant $-A$ which is the forward value of the coupon and the initial payment

$$G = P(T_0, T_N) - A = -1 + \delta K \sum_{i=1}^{N} P(0, T_i)/P(0, T_0) + P(T_0, T_N).$$

The difference between F and G is the terms representing the coupon payments but the expected values coincide so that $Q_1 = 0$, $Q_2 = c_2(F) - c_2(G)$, $Q_3 = c_3(F) - c_3(G)$. By truncating the higher orders than $n = 3$ in (7) we have

$$SOV(0) \approx C(A) + \frac{c_2(F) - c_2(G)}{2} P(0, T_0) g(0) - \frac{c_3(F) - c_3(G)}{6} P(0, T_0) g'(0), \tag{8}$$

where $C(x)$ is the call option price on the T_N-zero coupon bond with expiry T_0 and strike price x.

The cumulants $c_2(F), c_3(F)$ and $c_2(G), c_3(G)$ are easily calculated with the moments

$$\begin{aligned} E^{T_0}\big[F^m\big] &= E^{T_0}\Big[(-1+\delta K\sum_{i=1}^{N} P(T_0,T_i) + P(T_0,T_N))^m\Big], \\ E^{T_0}\big[G^m\big] &= E^{T_0}\Big[(P(T_0,T_N) - A)^m\Big], \end{aligned}$$

which can be easily obtained from bond moments.

The remaining issue is the derivation of $C(A)$, $g(0)$ and $g'(0)$ in (8). These numbers may be calculated either analytically or numerically within affine term structure models. Indeed it is an actually easy task if the state variables are Gaussian. Even in a non-Gaussian case it may be possible by fully utilizing the features of the affine structure. Chen and Scott (1995) examines a zero coupon bond option price within a two-factor Cox-Ingersoll-Ross (CIR) model and presents a method to numerically calculate the distribution function based on Fourier inversion techniques. For a non-negative random variable Y with the known characteristic function ϕ_Y, the distribution function is obtained as

$$Q^{T_0}(Y \le x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ux}{u}\phi_Y(u)du \tag{9}$$

by a version of the Fourier inversion formula as shown in Chen and Scott (1995) and other papers cited there. Recall that within an affine term structure model, the zero coupon bond price $P(T_0, T_N)$ is written as an exponentially affine function of X_{T_0}

$$P(T_0,T_N) = \exp\Big(\alpha(T_N - T_0) - \beta(T_N - T_0)^\top X_{T_0}\Big)$$

with some deterministic functions α and β. Let $Y = \beta(T_N - T_0)^\top X_{T_0}$. The characteristic function ϕ_Y of Y is available in some cases including the CIR model with independent state variables. If that is the case, by applying (9) to $Y = -\ln(G + A) + \alpha(T_N - T_0)$, we have

$$\begin{aligned} Q^{T_0}(G \le x) &= 1 - Q^{T_0}(Y \le -\ln(x+A) + \alpha) \\ &= 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\big(u(\ln(x+A) - \alpha(T_N - T_0))\big)}{u}\phi_Y(u)du. \end{aligned}$$

Then $g(0)$ and $g'(0)$ can be calculated with a numerical integration algorithm by

$$g^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}}Q^{T_0}(G \le x). \tag{10}$$

Similarly the call option price $C(A)$ can be obtained numerically by noting

$$C(x) = P(0,T_N)Q^{T_N}(P(T_0,T_N) > x) - xP(0,T_0)Q^{T_0}(P(T_0,T_N) > x). \tag{11}$$

At last by plugging the results by (10) and (11) into (8) we get an approximated swaption price.

4 Concluding Remarks

We demonstrate a method to approximate a swaption price by using a density function of a zero coupon bond with a general Gram–Charlier expansion. A linear combination of the state variables might be an alternative choice as the approximating random variable. Fourier inversion techniques are also useful for the numerical integration. Our approach may contribute to improve the approximation accuracy which is left for future research.

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