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Optimal Stopping Problems for a Process with a Jump

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Abstract

In the last two decades, the market of credit derivatives has expanded rapidly, and the importance of pricing problems for credit derivatives has been recognized especially in the last decade. Among these securities, the pricing problems of credit derivatives with an early exercise feature, such as American put options, have not received enough attentions. In order to evaluate American options on defaultable bonds, it is necessarily to consider the optimal stopping problems for a process with a jump. The existence of the optimal stopping time for a process with a jump is considered in this article.

Key words: pricing American put on defaultable bonds, optimal stopping problem.
JEL classification:C61,G12

1 Introduction

The market of credit derivatives has been grown rapidly in the last deacade and the pricing problems of credit derivatives have attracted great interest both from academicd and practioners. Two main methologies have been used in the analysis of the pricing problems of credit derivatives —the structural and reduced modeling approaches. The pricing problems of contingent claims with credit risk with the reduced modeling approach has been considered in Duffie and Singleton (1999), Jarrow and Turnbull (1995) and Muroi (2002,2005), for example. Although there have been many investigations on the theory of credit derivatives, little is known about the mathematical theories relating to American put options on defaultable bonds. It is necessary to consider the optimal stopping problem for a process with a jump to evaluate the American put options on defaultable bonds within the reduced modeling approach, since the defaultable bond price process has a

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jump in the default time. See Modecki (1999) and Zhang (1995) for discussions on related issues. The existence of the optimal stopping time for a process with a jump is shown in this article. In general, the optimal stopping time is not necessarily achieved if the original process has jumps, and it is not easy to prove the existence of an optimal stopping time. This is possible, however, when the default time is represented by a totally inaccessible stopping time, as will be shown in this paper. This paper is based on the Section 5 of Muroi (2002).

2 Optima stopping problems for a process with a Jump

In the pricing problem of American put options on defaultable bonds within the reduced modeling approach, it is necessary to prove the existence of an optimal stopping time for a continuous time stochastic process which has a jump at a totally inaccessible stopping time. The assumption of the default time of the issuer of a bond as a totally inaccessible stopping time is necessary also to derive the fair value of American put options on defaultable bonds. Karatzas and Shreve (1999) (in their Appendix D) proved many results of the optimal stopping problem without a regularity condition on the left continuity of the sample path, this assumption still plays an important role in proving the existence of an optimal stopping time. In this article, however, the existence of an optimal stopping time will be proved without the left continuity of the sample paths, by assuming that the jump time is a totally inaccessible stopping time.

Two non-negative stochastic processes, $Y(t)$ and $Y'(t)$, are defined on the probability space $(\Omega, \mathcal{F}, F, P)$, with the assumptions that the two stochastic processes are $F = \{\mathcal{F}_t; t \leq T\}$ adapted and that they have continuous sample paths. The filtration, $\{\mathcal{F}_t\}_{0\leq t<T}$, satisfies the usual condition of stochastic analysis, and a random variable, τ, is defined as a totally inaccessible stopping time. A totally inaccessible stopping time means

$$P(\{\omega : \tau = \xi < \infty\}) = 0$$

for any predictable stopping time, ξ. In other words, a totally inaccessible stopping time means that, for any sequence of stopping times, the probability that the sequence approaches τ from below is 0. This permits the definition of a new random variable, $X(\cdot)$, by

$$X(t) = Y(t)1_{\{t<\tau\}} + Y'(t)1_{\{\tau\leq t\}} \,. \tag{2.1}$$

This stochastic process, $X(\cdot)$, has a possible jump, but only at a totally inaccessible stopping time (already defined as τ). A further assumption, termed the American Regularity

Condition by Duffie (2001, p.182),

$$E[\sup_{0\leq t\leq T} X(t)] < \infty \tag{2.2}$$

is required to guarantee the existence of an optimal stopping time.

Next, a new stochastic process, $Z^0(\cdot)$, termed the Snell envelope of $X(\cdot)$ is introduced, the Snell envelope being the smallest RCLL supermartingale which dominates $X(\cdot)$. Also, a new stochastic process, $Z(\cdot)$, is defined which satisfies

$$Z(t) = ess \sup_{t\leq\rho\leq T} E[X(\rho)|\mathcal{F}_t] ,$$

where $X^* = ess\sup\chi$ is the essential supremum satisfying conditions of (i) $X \leq X^*$ $a.s.$ for any $X \in \chi$ and (ii) $X^* \leq Y$ $a.s.$ for $X \leq Y$ (all $X \in \chi$), and ρ is the stopping time.

Applying Theorem D.7 of Karatzas and Shreve (1999, p354) leads to

$$Z^0(t) = Z(t)\ (a.s.).$$

To show the existence of the stopping time, ρ, such that the expected reward is maximized, is an important aim in the optimal stopping problem. Theorem 2.1, a basic result of this paper, will play an important role in the valuation problem of an American put option on a defaultable bond.

Theorem 2.1 *There exists an optimal stopping time under the assumptions of (2.2) for $X(t)$ of (2.1). Moreover, the explicit representation of the optimal stopping time is given by*

$$\eta = \inf\{t \in [v,T] | Z^0(t) = X(t)\} . \tag{2.3}$$

[*Proof*] Random variables are defined by

$$D^\lambda(v) = \inf_{t\in[v,T]}\{\lambda Z^0(t) \leq X(t)\} \wedge T \quad \lambda \in [0,1)\ (a.s.)$$

and

$$D_*(v) = \lim_{\lambda\uparrow 1} D^\lambda(v) = \lim_{n\uparrow\infty} D^{(1-\frac{1}{n})}(v)\ (a.s.)\ .$$

The stopping times, $D^\lambda(v)$, are the stopping times "just before the optimal stopping time" and $D_*(v)$ is the limit of these stopping times. In a continuous time process, it is not trivial whether $D_*(v)$ is an optimal stopping time or not unless there is the assumption of continuity of sample paths (see Karatzas and Shreve (1999, p. 358) Theorem D.12). Under the assumptions of the present paper, $\{D^{(1-\frac{1}{n})}(v)\}_{n=1}^{\infty}$ is bounded and monotone increasing in the wide sense. Hence, the random variable, $D_*(v)$, is well-defined. It is possible to derive the following relationship

$$\begin{aligned}
Z^0(v) &= E[Z^0(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&\leq \frac{n}{n-1}E[X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&= \frac{n}{n-1}\{E[1_{\{\tau-<D^{(1-\frac{1}{n})}(v)\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&+ \sum_{j=n}^{\infty} E[1_{\{D^{(1-\frac{1}{j})}(v)\leq\tau-<D^{(1-\frac{1}{j+1})}(v)\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&+ E[1_{\cap_{j=n}^{\infty}\{D^{(1-\frac{1}{j})}(v)\leq\tau-\}}1_{\{\tau\leq D_*(v)\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&+ E[1_{\{D_*(v)<\tau\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v]\}\ (a.s.)\,. \qquad (2.4)
\end{aligned}$$

The first equality of (2.4) is a direct application of Proposition D.10 of Karatzas and Shreve (1999, p. 356) which holds in the general case of the RCLL process.

Next, it is necessary to show that the third term of (2.4) equals 0. The proof of this fact is quite lengthy and this fact will be shown later as Lemma 2.1. Once, Lemma 2.1 has been proved, then, as $n \to +\infty$,

$$\begin{aligned}
Z^0(v) &= \frac{n}{n-1}\{E[1_{\{\tau-<D^{(1-\frac{1}{n})}(v)\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&+ E[1_{\{D^{(1-\frac{1}{n})}(v)\leq\tau-\leq D_*(v)\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v] \\
&+ E[1_{\{D_*(v)<\tau\}}X(D^{(1-\frac{1}{n})}(v))|\mathcal{F}_v]\} \\
&\to E[1_{\{\tau-<D_*(v)\}}X(D_*(v))|\mathcal{F}_v] + E[1_{\{D_*(v)<\tau\}}X(D_*(v))|\mathcal{F}_v] \quad (a.s.) \qquad (2.5) \\
&= E[1_{\{\tau\leq D_*(v)\}}X(D_*(v))|\mathcal{F}_v] + E[1_{\{D_*(v)<\tau\}}X(D_*(v))|\mathcal{F}_v] \\
&= E[X(D_*(v))|\mathcal{F}_v]\,.
\end{aligned}$$

Under the conditions

$$0 \leq X(D^{(1-\frac{1}{n})}(v)) \leq \sup_{0\leq t\leq T} X(t) \ \ and\ E[\sup_{0\leq t\leq T} X(t)] < \infty$$

for any integer n, the theorem may be used on taking limits under the expectation sign, as shown by Shiryaev (1989) (Theorem 2, p. 218), at (2.5).

The rest of the proof is essentially the same as in the second half of the proof for Theorem D.12 of Karatzas and Shreve (1999, p. 358). By using the fact that

$$Z^0(v) \leq E[X(D_*(v))|\mathcal{F}_v] \leq E[Z^0(D_*(v))|\mathcal{F}_v] \leq Z^0(v)\ (a.s.)\,,$$

it can be proved that $E[X(D_*(v))|\mathcal{F}_v] = Z^0(v) = ess\sup_{v\leq\rho\leq T} E[X(\rho)|\mathcal{F}_v]$.
To prove the second half of Theorem 2.1, it must be shown that

$$D_*(v) = \inf\{t \in [v,T] | Z^0(t) = X(t)\}\ (a.s.)\,.$$

Using the supermartingale property of the stochastic process, $Z^0(\cdot)$, leads to

$$E[X(D_*(v))] = E[Z^0(v)] \geq E[Z^0(D_*(v))] .$$

On the other hand, $Z^0(\cdot)$ dominates $X(\cdot)$. These two facts can be unified to conclude that $X(D_*(v)) = Z^0(D_*(v))$ $(a.s.)$, leading to the consequence that

$$D_*(v) \geq \inf\{t \in [v,T] | Z^0(t) = X(t)\} .$$

To show the reverse inequality, it is first noticed that

$$X(t) < \lambda Z^0(t) \leq Z^0(t) \ (a.s.)$$

for all $t \in (v, D^\lambda(v))$ and $\lambda \in (0,1)$, leading to

$$D_*(v) = \lim_{\lambda \uparrow 1} D^\lambda(v) \leq \inf\{t \in (v,T] | Z^0(t) = X(t)\} \ (a.s.).$$

Finally, it must be shown that $D_*(v) = v$ $(a.s.)$ on the set $\{Z^0(v) = X(v)\}$. On the set, $\{Z^0(v) = X(v) > 0\}$, the right continuity of Z^0 and X imply that $D^\lambda(v) = v$ for all $\lambda \in (0,1)$ $(a.s.)$ leading to $D_*(v) = v$ $(a.s.)$. Using the optimal stopping theorem of Elliott (1982) (Theorem 4.12, p.36) leads to

$$E[1_{\{Z^0(v)=0\}} Z^0((v+\theta) \wedge T)] \leq E[1_{\{Z^0(v)=0\}} Z^0(v)] = 0 \quad \textit{for any } \theta \geq 0 .$$

with the conclusion that $D_*(v) = v$ $(a.s.)$ for $v \leq t \leq T$ $(a.s.)$ on the set, $\{Z^0(v) = X(v) = 0\}$. On this set, $D_*(v) = v$ $(a.s.)$. (Q.E.D.)

The remaining task is to show the following Lemma 2.1.

Lemma 2.1 *The third term of (2.4) equals* 0. *i.e.*

$$E[1_{\cap_{j=n}^{\infty}\{D^{(1-\frac{1}{j})}(v) \leq \tau -\}} 1_{\{\tau \leq D_*(v)\}} X(D^{(1-\frac{1}{n})}(v)) | \mathcal{F}_v] = 0 .$$

[*Proof*] The first indicator function equals 1 on the region satisfying $D^{(1-\frac{1}{j})}(v) \leq \tau-$ for an arbitrary integer j such that $n \leq j$. Hereafter, conditioning on the events which satisfy $D^{(1-\frac{1}{j})}(v) \leq \tau-$ for an arbitrary j such that $n \leq j$, leads to

$$D^{(1-\frac{1}{j})}(v) \leq \tau- \Longleftrightarrow D^{(1-\frac{1}{j})}(v) < \tau \qquad \textit{for any } n \leq j ,$$

and hence

$$D_*(v) = \sup_n D^{(1-\frac{1}{n})}(v) \leq \tau .$$

On the other hand, the assumption that $D_*(v)$ satisfies $P[D_*(v) = \tau | \mathcal{F}_v] > 0$ leads to the contradiction as follows. By virtue of the continuity property of the stochastic process,

$Y(t)$, which is the same stochastic process as $X(t)$ until the jump time, in (2.2), τ, each $D^{(1-\frac{1}{n})}(v)$ satisfies the relation $D^{(1-\frac{1}{n})}(v) < D_*(v) \leq \tau$. Therefore, there exists a positive integer-valued, random variable, $l(\omega)$, such that

$$v = D^{(1-\frac{1}{1})}(v) = D^{(1-\frac{1}{2})}(v) = \cdots = D^{(1-\frac{1}{l(\omega)})}(v) < D^{(1-\frac{1}{l(\omega)+1})}(v) < \cdots \ (a.s.) \,. \quad (2.6)$$

To show the relation (2.6), it is evident that for each ω there is an integer values random variable, $l(\omega)$, such that

$$(1 - \frac{1}{l(\omega)+1})Z^0(v) < X(v) \leq (1 - \frac{1}{l(\omega)})Z^0(v) \ (a.s.) \,,$$

and $\alpha_n(\omega) = D^{(1-\frac{1}{n+l(\omega)})}(v)$ is a strictly monotone increasing sequence of stopping times utilizing the continuity property of the stochastic process, $Y(t)$, which must satisfy

$$\alpha_n \to D_*(v) \ (n \to \infty) \ (a.s.) \,.$$

On the other hand, there is an assumption of $P[D_*(v) = \tau | \mathcal{F}_v] > 0$, leading to

$$P[\alpha_n \uparrow \tau(n \to \infty) | \mathcal{F}_v] > 0 \,.$$

However, given v, the stopping time, τ, is a totally inaccessible stopping time, leading to a contradiction. This will lead to the relation, $D_*(v) < \tau(a.s.)$, as a consequence. In contrast, the second indicator in the third term of (2.4) equals 0 in the region satisfying $\tau > D_*(v)$. Therefore, the lemma is shown. (Q.E.D.)

If the jump time of the process is a predictable stopping time, generally it is not possible to prove (2.5) nor is it possible to show the existence of an optimal stopping time. Amin and Jarrow (1993) have provided an example where there is no optimal stopping time owing to the predictability of the jump time.

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