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正則グラフの1頂点を削除した部分グラフにおける2-因子について
2-factor in $2r$-regular graph

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Abstract

Let r be a positive integer such that $r \geq 2$, G be a $2r$-regular graph of odd order and G be connected. Then, there is some $x \in V(G)$ such that $G - x$ has a 2-factor.

1 Introduction

We consider finite undirected graphs which may have *loops* and *multiple edges*. Let G be a graph. For $x \in V(G)$, we denote by $\deg_G(x)$ the *degree* of x in G. The set of *neighbours* of $x \in V(G)$ is denoted by $N_G(x)$. If $\deg_G(x) = r$ for any $x \in V(G)$, we call the graph *r-regular*. For subsets S and T of $V(G)$, we denote by $e_G(S,T)$ the number of the edges joining S and T. If $S \cap T \neq \emptyset$, the edges of $S \cap T$ are counted twice. If S is a singleton $\{x\}$, we write $S = x$ instead of $S = \{x\}$. For example, we write $e_G(x,T)$ instead of $e_G(\{x\},T)$. Let k be a constant. A spanning subgraph F of G such that $\deg_F(x) = k$ for each $x \in V(G)$ is called a *k-factor* of G. When no fear of confusion arises, we often identify a k-factor with its edge set.

Petersen proved the next theorem in 1891.

Theorem A (Petersen [1]) *Every $2r$-regular graph can be decomposed into r disjoint 2-factors.*

This theorem implies that if G is a $2r$-reguler graph, then G has a k-factor for every even integer k, $2 \leq k \leq 2r$.

Katerinis showed the next theorem in 1985.

Theorem B (Katerinis [2]) *Let G be a connected graph of even order, and let a, b, and c be odd integers such that $1 \leq a < b < c$. If G has both a-factor and c-factor, then G has a b-factor.*

If a $2r$-regular graph G has a 1-factor, we can obtain a $(2r-1)$-factor by excluding the 1-factor from G. By the 1-factor and the $(2r-1)$-factor of G and by Theorem B, G has a k-factor for any odd integer k, $1 \leq k \leq 2r-1$. Thus, by the above two theorems, if a $2r$-regular graph G has a 1-factor, then G has a k-factor for every integer k, $1 \le k \le 2r-1$. Note that the order of G is even. For the case that the order of G is odd, Katerinis proved the next theorem in 1994.

Theorem C (Katerinis [3]) *Let G be a $2r$-regular, $2r$-edge-connected graph of odd order, and k be an integer such that $1 \le k \le r$. Then for every $x \in V(G)$, the graph $G-x$ has a k-factor.*

Let us focus our attention that the condition "$2r$-edge-connected" of Theorem C is replaced by "connected". What resulet can be obtained under the weakered condition? Now we will present our main theorem.

Theorem 1 *Let r be a positive integer such that $r \ge 2$, G be a $2r$-regular graph of odd order and G be connected. Then, there is some $x \in V(G)$ such that $G-x$ has a 2-factor.*

We believe that following conjecture.

Conjecture 1 *Let r be a positive integer such that $r \ge 2$, G be a $2r$-regular graph of odd order. and G be connected. Then for any even k, $2 \le k \le r$, there is some $x \in V(G)$ such that $G-x$ has a k-factor.*

In order to prove Theorem 1, we use the following Tutte's Theorem. Let G be a graph. For disjoint subsets S and T of $V(G)$, we define $\delta_G(S,T;k)$ by

$$\delta_G(S,T;k) = k|S| + \sum_{y\in T} \deg_{G-S}(y) - k|T| - h_G(S,T;k),$$

where $h_G(S,T;k)$ is the number of components C of $G-(S\cup T)$ such that $k|V(C)| + e_G(V(C),T)$ is odd. These components are called *odd* components. We denote by $\mathcal{H}_G(S,T;k)$ the set of the odd components. That is $|\mathcal{H}_G(S,T;k)| = h_G(S,T)$. If $\delta_G(S,T;k) = \delta_G(T,S;k)$, then we say that S and T are *symmetric*.

Theorem D (Tutte [4]) *Let G be a graph, and let k be a positive integer. Then*

(1) $\delta_G(S,T;k) \equiv k|V(G)| \pmod 2$ *for each disjoint subsets S and T of $V(G)$, and*

(2) *G has a k-factor if and only if $\delta_G(S,T;k) \ge 0$ for each pair of disjoint subsets S and T of $V(G)$.*

2 Proof of Theorem 1

We apply induction on $|V(G)|$. For $|V(G)| = 1$ the assertion is true. Now let G be given with $|V(G)| \ge 3$, and assume that the theorem holds for graphs with fewer vertices. Assume on the contrary that $G-x$ has no 2-factor for any $x \in V(G)$. Then, there is some pair of disjoint subsets $S', T' \subseteq V(G)-x$ for every $x \in V(G)$ such that $\delta_{G-x}(S',T';2) \le -2$ by Theorem D. Let $S = S' \cup \{x\}$, $T = T'$, and $U = G-(S\cup T)$. Then,

$$\delta_{G-x}(S-x,T;2) \le -2. \qquad (1)$$

Since G is $2r$-regular,

$$\delta_G(S,T;2r) \geq 0 \tag{2}$$

for each disjoint subsets S and T of $V(G)$. By the definition of odd component, $h_{G-x}(S-x,T;2) = h_G(S,T;2)$ holds. Let $h_G(S,T) = h_{G-x}(S-x,T;2) = h_G(S,T;2)$. Subtracting (2) from (1), we have

$$\begin{aligned}(2-2r)|S| - 2 - (2-2r)|T| &\leq -2 \\ -(2-2r)|T| &\leq -(2-2r)|S| \\ |T| &\leq |S|. \end{aligned} \tag{3}$$

By (1) and (3),

$$\sum_{y \in T} \deg_{G-S}(y) \leq h_G(S,T). \tag{4}$$

On the other hand, by the definition of odd component,

$$\sum_{y \in T} \deg_{G-S}(y) \geq e_G(T,U) \geq h_G(S,T). \tag{5}$$

By (4) and (5),

$$\sum_{y \in T} \deg_{G-S}(y) = h_G(S,T). \tag{6}$$

By (1) and (6),

$$\begin{aligned} 2|S| - 2 - 2|T| &\leq -2 \\ 2|S| &\leq 2|T| \\ |S| &\leq |T|. \end{aligned} \tag{7}$$

By (3) and (7),

$$|S| = |T|. \tag{8}$$

Since $\delta_G(S,T;2) = \delta_G(T,S;2)$ by (8), S and T are symmetric. Moreover, $|U|$ is odd. By (6),

$$e_G(T,T) + e_G(T,U) = h_G(S,T). \tag{9}$$

By (5) and (9),

$$e_G(T,T) = 0 \quad \text{and} \quad e_G(T,U) = h_G(S,T) \tag{10}$$

If there is no odd component of U, $e_G(T,S) = 2r|T|$ holds by (9). Then, since $e_G(S \cup T, U) = 0$ holds, G is disconnected. This is a contradiction. Thus, there is some odd component of U. Note that $e_G(V(C),T) = 1$ for each odd component $C \in \mathcal{H}_G(S,T)$. Let $\mathcal{H}_G(S,T) = \{C_1, \ldots, C_z\}$. Let $a_i, b_i \in V(C_i)$, $s_i \in S$, $t_i \in T$ for every odd component $C_i \in \mathcal{H}_G(S,T)$, $1 \leq i \leq z$, such that $N_G(a_i) \cap \{t_i\} \neq \emptyset$ and $N_G(b_i) \cap \{s_i\} \neq \emptyset$. We show that there is subgraph H_i of G such that $\deg_{H_i}(s_i) = \deg_{H_i}(t_i) = 1$ and $\deg_{H_i}(x_i) = 2$ for any $x \in V(C_i)$ for any odd component $C_i \in \mathcal{H}_G(S,T)$. Now, for every odd component $C_i \in \mathcal{H}_G(S,T)$ $\deg_{C_i}(x) = 2r$ for every $x \in V(C_i) - \{a_i, b_i\}$ and $\deg_{C_i}(a_i) = \deg_{C_i}(b_i) = 2r-1$. Therefore, $C_i \cup \{a_i b_i\}$ is $2r$-regular for any odd component $C_i \in \mathcal{H}_G(S,T)$. $C_i \cup \{a_i b_i\}$ has r disjoint 2-factors by Theorem A in $C_i \cup \{a_i b_i\}$. Let

F_{C_i} be a 2-factor including new edge $\{a_ib_i\}$ for each odd component $C_i \in \mathcal{H}_G(S,T)$ in $C_i \cup \{a_ib_i\}$. Then, $(F_{C_i} - \{a_ib_i\}) \cup \{a_it_i\} \cup \{b_is_i\}$ is the desired subgraph H_i of G for each odd component $C_i \in \mathcal{H}_G(S,T)$. On the other hand, there is also 2-factor $F_{C_i'}$ not to include new edge $\{a_ib_i\}$ for each odd component $C_i \in \mathcal{H}_G(S,T)$ in $C_i \cup \{a_ib_i\}$, that is, C_i has a 2-factor for each odd component $C_i \in \mathcal{H}_G(S,T)$ in C_i.

Next, we show that there is some $x \in V(C_i)$ for some odd component $C_i \in \mathcal{H}_G(S,T)$ such that $C_i - x$ has a 2-factor, or there is a subgraph H of G including every vertices of $C_i - x$, $s_i \in S$ and $t_i \in T$ as above. Let C be this odd component C_i, $s = s_i$, $t = t_i$, $a = a_i$ and $b = b_i$. By the induction hypothesis, for this odd component $C \in \mathcal{H}_G(S,T)$ there is some x such that $(C \cup \{ab\}) - x$ has a 2-factor F_C since $C \cup \{ab\}$ is $2r$-regular and $|V(C)| < |V(G)|$.

If $F_C \cap \{ab\} \neq \emptyset$ for this odd component $C \in \mathcal{H}_G(S,T)$, $(F_C - \{ab\}) \cup \{at\} \cup \{bs\}$ is the desired subgraph H. Then, there is a path P from s to t such that $C \cap P \neq \emptyset$ for this odd component $C \in \mathcal{H}_G(S,T)$. As well as this odd component $C \in \mathcal{H}_G(S,T)$, we can obtain a path P_i for every odd component $C_i \in \mathcal{H}_G(S,T)$. Let G' be a graph obtained from G by contracting the path P_i into a new edge p_i, and excluding $C_i - P_i$ in $G - x$ for every odd component $C_i \in \mathcal{H}_G(S,T)$. Let $p = p_i$ for $p_i \in C$ for some odd component $C \in \mathcal{H}_G(S,T)$. Then, graph G' becomes $2r$-regular graph. By Theorem A, G' has a 2-factor F' avoiding p. If $F' \cap \{p_i\} \neq \emptyset$, we can use the subgraph H_i of G. If $F' \cap \{p_i\} = \emptyset$, we can use the 2-factor $F_{C_i'}$ in C_i excluding new edge a_ib_i for any odd component $C_i \in \mathcal{H}_G(S,T) - C$. Thus, G has a 2-factor.

If $F_C \cap \{ab\} = \emptyset$, $C - x$ has a 2-factor. There is a path P_i from s to t such that $C_i \cap P_i = \emptyset$ for each odd component $C_i \in \mathcal{H}_G(S,T) - C$. Let G' be a graph obtained from G by contracting the path P_i into a new edge p_i, and excluding $C_i - P_i$ in $G - x$. Then, graph G' becomes $2r^-$-regular graph. Note that $2r^-$-regular graph is graph obtained from $2r$-regular graph by excluding an edge. Since $G' \cup \{st\}$ is $2r$-regular, $G' \cup \{st\}$ has a 2-factor avoiding st by Theorem A, that is, G' has a 2-factor F'. If $F' \cap \{p_i\} \neq \emptyset$, we can use the subgraph H_i of G. If $F' \cap \{p_i\} = \emptyset$, we can use the 2-factor $F_{C_i'}$ in C_i excluding new edge a_ib_i for any odd component $C_i \in \mathcal{H}_G(S,T) - C$. Thus, G has a 2-factor.

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