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The Confluence Problem for Flat TRSs

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Abstract

We prove that confluence is undecidable for flat TRSs. Here, a TRS is flat if the heights of the left and right-hand sides of each rewrite rule are at most one.

1 Introduction

A term rewriting system(TRS) is a set of directed equations (called rewrite rules). A TRS is confluent(Church-Rosser) if any two convertible terms are joinable. Confluence is an important property since it implies the unicity of normal forms [1], and has received much attention so far.

But, confluence is undecidable in general and so even if we restricts to monadic or semi-constructor TRSs [6]. On the other hand, it is known decidable for terminating TRSs [5] and right-(ground or variable) TRSs [2]. In paricular, confluence is decidable for right-linear shallow TRSs [3], hence we show here that the right-linearity condition is necessary for decidability.

Recently, Jacquemard [4] has reported that confluence is undecidable for flat TRSs. Here, a TRS is flat if the heights of the left and right-hand sides of each rewrite rule are at most one. However, we found that the proof is incorrect. In this paper, we give a correct proof of the undecidability.

2 Preliminaries

We assume that the reader is familiar with standard definitions of rewrite systems [1] and we just recall here the main notations used in this paper.

Let ε be the empty sequence. Let X be a set of variables. Let F be a finite set of operation symbols graded by an arity function $\mathsf{ar}\colon F \to \mathrm{N}(=\{0,1,2,\cdots\})$, $F_n = \{f \in F \mid \mathsf{ar}(f) = n\}$. Let T be a set of terms built from X and F We use x as a variable, f, h as function symbols, r, s, t as terms, θ as a substitution. The *height* of a term is defined as follows: $\mathsf{height}(a) = 0$ if a is a variable or a constant and $\mathsf{height}(f(t_1, \ldots, t_n)) = 1 + \max\{\mathsf{height}(t_1), \ldots, \mathsf{height}(t_n)\}$ if $n > 0$.

A position in a term is expressed by a sequence of positive integers, and positions are partially ordered by the prefix ordering $\geq$. Let $s_{|p}$ be the subterm of s at position p. Let $s \geq_{\mathsf{sub}} t$ if t is a subterm of s. For position p and term t, we use $s[t]_p$ to denote the term obtained from s by replacing a subterm $s_{|p}$ by t.

A *rewrite rule* $\alpha \to \beta$ is a directed equation over terms. A *TRS* R is a finite set of rewrite rules. A term s reduces to t at position p by a TRS R, denoted $s \xrightarrow{p}_R t$, if $s_{|p} = \alpha\theta$ and $t = s[\beta\theta]_p$ for some rewrite rule $\alpha \to \beta$ and substitution θ. For $s \xrightarrow{p}_R t$, p and R may be omitted. Let $\to^=$ be $\to \cup =$, and $\leftarrow$ the inverse of $\to$. s and t are *joinable* if $s \to^* \cdot \leftarrow^* t$, denoted $s \downarrow t$. t is *reachable* from s if $s \to^* t$. r is *confluent* on TRS R if for every $s \leftarrow_R^* r \to_R^* t$, $s \downarrow t$. A TRS R is *confluent* if every r is confluent on R. Let $\gamma\colon s_1 \xrightarrow{p_1} s_2 \cdots \xrightarrow{p_{n-1}} s_n$ be a *rewrite sequence*. This sequence is abbreviated to $\gamma\colon s_1 \to^* s_n$. γ is called p-invariant if $q > p$ for any redex position q of γ, and we write $\gamma\colon s_1 \xrightarrow{>p}{}^* s_n$.

Definition 1 A rule $\alpha \to \beta$ is *flat* if $\mathsf{height}(\alpha) \leq 1$ and $\mathsf{height}(\beta) \leq 1$. A TRS R is *flat* if every rule in R is flat.

Definition 2 A *finite automaton* is a 5-tuple $(Q, \Sigma, \delta, FS, q_0)$ where Q is a finite set of states, Σ is a finite set of iuput symbols, $\delta : Q \times \Sigma \to Q$ is a function, $FS \subseteq Q$ is a finite set of final states, and $q_0 \in Q$ is the initial state.

Let ϕ be a mapping from T to T. A mapping ϕ can be extended to TRSs as follows: $\phi(R) = \{\phi(\alpha) \to \phi(\beta) \mid \alpha \to \beta \in R\} \setminus \{t \to t \mid t \in T\}$. A mapping ϕ can be extended to substitutions as follows: let $\phi(\theta) = \{x \to \phi(t) \mid x \to t \in \theta\}$. The following lemma holds for ϕ.

Lemma 3 R is confluent iff there exists a mapping $\phi : T \to T$ that satisfies the following conditions (1)–(4).

(1) If $s \to_R t$ then $\phi(s) \to^*_{\phi(R)} \phi(t)$.

(2) $\to_{\phi(R)} \subseteq \to^*_R$.

(3) $t \to^*_R \phi(t)$.

(4) $\phi(R)$ is confluent.

Proof Only if part: Let ϕ be an identity mapping. If part: Let $s \leftarrow^*_R r \to^*_R t$. By Condition (1), $\phi(s) \leftarrow^*_{\phi(R)} \phi(r) \to^*_{\phi(R)} \phi(t)$. By Condition (4), $\phi(s) \downarrow_{\phi(R)} \phi(t)$. By Condition (2), $\phi(s) \downarrow_R \phi(t)$. By Condition (3), $s \to^*_R \phi(s)$ and $t \to^*_R \phi(t)$. Thus, $s \downarrow_R t$. □

This lemma is used in Section 4.

3 Joinability and reachability for flat TRSs

Jacquemard [4] has reported that reachability and joinability are also undecidable for flat TRSs and the confluence problem is reducible to the reachability one, so that the former is also undecidable. But, we found that the reducibility proof is incorrect and contains some errors not easy to correct. First, we give a more simplified proof of undecidability for joinability and reachability than that given in [4]. Notations used in the proof (but undefined in Section 2) are similar to those of [4].

Let $P = \{\langle u_i, v_i\rangle \in \Sigma^+ \times \Sigma^+ \mid 1 \le i \le k\}$ be an instance of the Post's Correspondence Problem (PCP). Note that the alphabet Σ is fixed. Let $l_P = \max_{1 \le i \le k}(|u_i|, |v_i|)$. Let $\_$ be a new symbol and $\Delta = \{1, \cdots, l_P\} \times (\Sigma \cup \{\_\})^2$. We shall use a product operator $\otimes$ which associates to two words of Σ^+ a word of Δ^* as follows: $a_1 \cdots a_n \otimes a'_1 \cdots a'_m = \langle 1, a_1, a'_1\rangle \cdots \langle l_P, a_{l_P}, a'_{l_P}\rangle$, where $a_1, \cdots, a_n, a'_1, \cdots, a'_m \in \Sigma$, $a_i = \_$ for all $i \in \{n+1, \cdots, l_P\}$, and $a'_j = \_$ for all $j \in \{m+1, \cdots, l_P\}$.

Example 4 Let $l_P = 4$, then $\mathsf{a} \otimes \mathsf{bab} = \langle 1, \mathsf{a}, \mathsf{b}\rangle\langle 2, \_, \mathsf{a}\rangle\langle 3, \_, \mathsf{b}\rangle\langle 4, \_, \_\rangle$.

Let $A = (Q_A, \Delta, \delta_A, F_A, q_A)$ and $B = (Q_B, \Sigma, \delta_B, F_B, q_B)$ be two finite automata recognizing the respective sets $L(A) = \{u_i \otimes v_i \mid 1 \le i \le k\}^+$ and $L(B) = \Sigma^+$. We may assume that both q_A and q_B are non final. We assume that the automata A and B are clean (i.e., any state is reachable from the initial state q_A(or q_B) by some input string).

We assume given 13 disjoint copies of the above signatures, colored with color $i \in \{0, \cdots, 12\}$: $\Sigma^{(i)} = \{a^{(i)} \mid a \in \Sigma\}$, $Q_A^{(i)} = \{q^{(i)} \mid q \in Q_A\}$, $Q_B^{(i)} = \{q^{(i)} \mid q \in Q_B\}$, $\Delta^{(i)} = \{d^{(i)} \mid d \in \Delta\}$. Let $\Theta^{012} = \Delta^{(0)} \cup \Sigma^{(1)} \cup \Sigma^{(2)}$, $\Theta^{345} = \Delta^{(3)} \cup \Delta^{(4)} \cup \Sigma^{(5)}$, and $Q = Q_A^{(6)} \cup Q_A^{(7)} \cup Q_A^{(8)} \cup Q_A^{(9)} \cup Q_B^{(10)} \cup Q_B^{(11)} \cup Q_A^{(12)}$. Let e be a constant. For a ground term t built with $\Delta \cup \Sigma \cup \{\mathsf{e}\}$, $t^{(i)}$ is defined as follows: $\mathsf{e}^{(i)} = \mathsf{e}$ and $(f(t_1))^{(i)} = f^{(i)}(t_1^{(i)})$ for $f \in \Delta \cup \Sigma$ and term t_1.

The following flat TRS R_1 is defined on an extended signature $F_0 = Q \cup \{\mathsf{e}, 0, 1\}$, $F_1 = \Theta^{012} \cup \Theta^{345}$, $F_6 = \{\mathsf{f}\}$, and $F_7 = \{\mathsf{g}\}$:

$$
\begin{array}{rcl}
T_A^{(i,j)} & = & \{q^{(i)} \to d^{(j)}(q'^{(i)}) \mid q' \in \delta_A(q,d)\} \cup \{q^{(i)} \to \mathsf{e} \mid q \in FS_A\} \\
T_B^{(i,j)} & = & \{q^{(i)} \to a^{(j)}(q'^{(i)}) \mid q' \in \delta_B(q,a)\} \cup \{q^{(i)} \to \mathsf{e} \mid q \in FS_B\} \\
S^{(i,j)} & = & \{a^{(i)}(x) \to a^{(j)}(x) \mid a \in \Sigma\} \\
P^{(i,j)} & = & \{d^{(i)}(x) \to d^{(j)}(x) \mid d \in \Delta\} \\
\Pi_1^{(i,j)} & = & \{\langle n,a,a'\rangle^{(i)}(x) \to a^{(j)}(x) \mid n \in \{1,\cdots,l_P\}, a \in \Sigma, a' \in \Sigma \cup \{\_\}\} \\
 & & \cup\{\langle n,\_,a'\rangle^{(i)}(x) \to x \mid n \in \{2,\cdots,l_P\}, a' \in \Sigma \cup \{\_\}\} \\
\Pi_2^{(i,j)} & = & \{\langle n,a,a'\rangle^{(i)}(x) \to a'^{(j)}(x) \mid n \in \{1,\cdots,l_P\}, a \in \Sigma \cup \{\_\}, a' \in \Sigma\} \\
 & & \cup\{\langle n,a,\_\rangle^{(i)}(x) \to x \mid n \in \{2,\cdots,l_P\}, a \in \Sigma \cup \{\_\}\} \\
R_0 & = & T_A^{(6,3)} \cup T_A^{(7,3)} \cup T_A^{(8,4)} \cup T_A^{(9,4)} \cup T_B^{(10,5)} \cup T_B^{(11,5)} \\
 & & \cup P^{(3.0)} \cup \Pi_1^{(3,1)} \cup \Pi_2^{(4,2)} \cup P^{(4,0)} \cup S^{(5,1)} \cup S^{(5,2)} \cup T_A^{(12,0)} \\
R_1 & = & R_0 \cup \{0 \to \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}), \\
 & & \mathsf{f}(x_3,x_3,x_4,x_4,x_5,x_5) \to \mathsf{g}(x_3,x_3,x_4,x_4,x_5,x_5,q_A^{(12)}), \\
 & & \mathsf{g}(x_0,x_1,x_2,x_0,x_1,x_2,x_0) \to 1\}
\end{array}
$$

If $0 \to_{R_1}^* 1$ then the rules of R_1 are applied as described in the following pic.

$$
\begin{array}{lccccccccl}
0 \to \mathsf{f}(& q_A^{(6)}, & q_A^{(7)}, & q_A^{(8)}, & q_A^{(9)}, & q_B^{(10)}, & q_B^{(11)} & &) & \\
 & T_A^{(6,3)} & T_A^{(7,3)} & T_A^{(8,4)} & T_A^{(9,4)} & T_B^{(10,5)} & T_B^{(11,5)} & & & \\
\mathsf{f}(& t_3, & t_3, & t_4, & t_4, & t_5, & t_5 & &) & \\
\downarrow & & & & & & & & & \\
\mathsf{g}(& t_3, & t_3, & t_4, & t_4, & t_5, & t_5, & q_A^{(12)} &) & \\
 & P^{(3,0)} & \Pi_1^{(3,1)} & \Pi_2^{(4,2)} & P^{(4,0)} & S^{(5,1)} & S^{(5,2)} & T_A^{(12,0)} & & \\
\mathsf{g}(& t_0, & t_1, & t_2, & t_0, & t_1, & t_2, & t_0 &) & \to 1
\end{array}
$$

Definition 5

(1) Let $\to_R^*(s) = \{t \mid s \to_R^* t\}$. For a set $C \subseteq \{0,\cdots,12\}$, let G^C be the set of ground terms built with e and colored function symbols in $\cup_{i\in C}(\Sigma^{(i)} \cup \Delta^{(i)} \cup Q_A^{(i)} \cup Q_B^{(i)})$.

(2) The index of an i-colored term built with $\Delta^{(i)} \cup \{\mathsf{e}\}$ is defined as follows: $\mathsf{index}(\mathsf{e}) = \varepsilon$ and $\mathsf{index}(\langle n,a,a'\rangle^{(i)}(t)) = n\ \mathsf{index}(t)$.

Lemma 6 Let $t_0 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(0)}$.

(1) If $q_A^{(6)} \to_{R_0}^* t_0$ then $q_A^{(6)} \to_{T_A^{(6,3)} \cup P^{(3,0)}}^* t_0$.

(2) If $q_A^{(9)} \to_{R_0}^* t_0$ then $q_A^{(9)} \to_{T_A^{(9,4)} \cup P^{(4,0)}}^* t_0$.

Proof

(1) By the definition of R_0, $q_A^{(6)} \to_{T_A^{(6,3)} \cup P^{(3,0)} \cup \Pi_1^{(3,1)'}}^* t_0$, where $\Pi_1^{(3,1)'} = \{\langle n,\_,a'\rangle^{(3)}(x) \to x \mid n \in \{2,\cdots,l_P\}, a' \in \Sigma \cup \{\_\}\}$. Note that $\mathsf{index}(t_0) = (1 \cdots l_P)^m$. In this rewrite sequence, if there is at least one application of some rule of $\Pi_1^{(3,1)'}$, $\mathsf{index}(t_0) = (1 \cdots l_P)^m$ does not hold, since any rule of $\Pi_1^{(3,1)'}$ can not delete any symbol of form $\langle 1,a,a'\rangle^{(3)}$. Thus, the proposition holds.

(2) Similar to (1). □

Lemma 7 $0 \to_{R_1}^* 1$ iff PCP has a solution.

Proof Only if part: By the definition of R_1, $0 \to_{R_1} \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \overset{>\varepsilon}{\to}{}_{R_0}^{*} \mathsf{f}(t_3,t_3,t_4,t_4,t_5,t_5)$ $\to_{R_1} \mathsf{g}(t_3,t_3,t_4,t_4,t_5,t_5,q_A^{(12)}) \overset{>\varepsilon}{\to}{}_{R_0}^{*} \mathsf{g}(t_0,t_1,t_2,t_0,t_1,t_2,t_0) \to_{R_1} 1$. By the definition of R_0, $\to_{R_0}^* (q_A^{(6)}) \subseteq G^{\{0,1,3,6\}}$, $\to_{R_0}^* (q_A^{(7)}) \subseteq G^{\{0,1,3,7\}}$, $\to_{R_0}^* (q_A^{(8)}) \subseteq G^{\{0,2,4,8\}}$, $\to_{R_0}^* (q_A^{(9)}) \subseteq G^{\{0,2,4,9\}}$, $\to_{R_0}^* (q_B^{(10)}) \subseteq G^{\{1,2,5,10\}}$, and $\to_{R_0}^* (q_B^{(11)}) \subseteq G^{\{1,2,5,11\}}$. We first show that the following condition (I) holds. (I) $t_i \in G^{\{i\}}$, $i \in \{0,\cdots,5\}$.

Note that $t_3 \in G^{\{0,1,3\}}$ holds, since $t_3 \in \to^*_{R_0} (q_A^{(6)}) \cap \to^*_{R_0} (q_A^{(7)})$. Similarly, $t_4 \in G^{\{0,2,4\}}$ and $t_5 \in G^{\{1,2,5\}}$ hold. Since $t_3 \to^*_{R_0} t_0$ and $t_4 \to^*_{R_0} t_0$, we have $t_0 \in G^{\{0,1,3\}} \cap G^{\{0,2,4\}} = G^{\{0\}}$. Similarly, $t_1 \in G^{\{1\}}$ and $t_2 \in G^{\{2\}}$ hold. Hence, the condition (I) holds for $i \in \{0,1,2\}$. For $t_3 \in G^{\{0,1,3\}}$, since $t_3 \to^*_{R_0} t_0 \in G^{\{0\}}$ and $t_3 \to^*_{R_0} t_1 \in G^{\{1\}}$, t_3 can not contain any symbol in $G^{\{0,1\}}$, so that $t_3 \in G^{\{3\}}$ holds. Similarly, $t_4 \in G^{\{4\}}$ and $t_5 \in G^{\{5\}}$ hold. Hence, (I) holds for $i \in \{3,4,5\}$, as claimed. By (I), we have $t_3 \to^*_{\Pi_1^{(3,1)}} t_1 \leftarrow^*_{S^{(5,1)}} t_5$ and $t_4 \to^*_{\Pi_2^{(4,2)}} t_2 \leftarrow^*_{S^{(5,2)}} t_5$. By $q_A^{(12)} \to^*_{R_0} t_0$, $t_0 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(0)}$ for some $i_1, \cdots, i_m \in \{1, \cdots, k\}$. Since an initial state q_A is not final, $m > 0$ holds. By Lemma 6, $t_3 \to^*_{P^{(3,0)}} t_0 \leftarrow^*_{P^{(4,0)}} t_4$. Thus, $t_3 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(3)}$ and $t_4 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(4)}$. By $t_3 \to^*_{\Pi_1^{(3,1)}} t_1$, $t_1 = (u_{i_1} \cdots u_{i_m}(\mathsf{e}))^{(1)}$. By $t_4 \to^*_{\Pi_2^{(4,2)}} t_2$, $t_2 = (v_{i_1} \cdots v_{i_m}(\mathsf{e}))^{(2)}$. By $t_1 \leftarrow^*_{S^{(5,1)}} t_5 \to^*_{S^{(5,2)}} t_2$, $t_5 = (u_{i_1} \cdots u_{i_m}(\mathsf{e}))^{(5)} = (v_{i_1} \cdots v_{i_m}(\mathsf{e}))^{(5)}$. Hence, PCP has a solution.

If part : Let $i_1 \cdots i_m$ be a solution of PCP, and let $s = (u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e})$ and $t = u_{i_1} \cdots u_{i_m}(\mathsf{e})$. Then, $t = v_{i_1} \cdots v_{i_m}(\mathsf{e})$ holds. By the definition of R_1, $0 \to \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \to^*$ $\mathsf{f}(s^{(3)}, s^{(3)}, s^{(4)}, s^{(4)}, t^{(5)}, t^{(5)}) \to \mathsf{g}(s^{(3)}, s^{(3)}, s^{(4)}, s^{(4)}, t^{(5)}, t^{(5)}, q_A^{(12)}) \to^* \mathsf{g}(s^{(0)}, t^{(1)}, t^{(2)}, s^{(0)}, t^{(1)}, t^{(2)}, s^{(0)}) \to 1$. Note that $\langle 1, \_, a' \rangle^{(3)}(x) \to x$ and $\langle 1, a, \_ \rangle^{(4)}(x) \to x$ need not in $\Pi_1^{(3,1)}$ and $\Pi_2^{(4,2)}$, since $P \subseteq \Sigma^+ \times \Sigma^+$. Hence, $0 \to^*_{R_1} 1$ □

Since 1 is a normal form, $0 \to^*_{R_1} 1$ iff $0 \downarrow_{R_1} 1$. Thus, the following theorem holds.

Theorem 8 Both joinability and reachability for flat TRSs are undecidable.

4 Confluence for flat TRSs

We show that confluence for flat TRSs is undecidable by reduction of the problem of the above section.

Let $\Theta_2^{012} = \{d_2 \mid d \in \Theta^{012}\}$, where d_2 has arity 2. We add TRS R_1 in the previous section the following rules:

$$R_2 = R_1 \cup \{\mathsf{e} \to 0\} \cup \{d(x) \to d_2(0,x), d_2(1,x) \to x \mid d \in \Theta^{012}\}$$

TRS R_2 is flat.

First, we show that $0 \to^*_{R_2} 1$ iff $0 \to^*_{R_1} 1$. For this purpose, we need the following definition and lemma.

Definition 9 Let ψ be the mapping over ground terms defined as follows.

$$\begin{array}{rcl} \psi(h(t_1, \cdots, t_n)) = & \mathsf{e} & (\text{if } h \in \{0, 1, \mathsf{f}, \mathsf{g}\}) \\ \psi(d_2(t_1, t_2)) = & d(\psi(t_2)) & (\text{if } d \in \Theta^{012}) \\ \psi(h(t_1, \cdots, t_n)) = & h(\psi(t_1), \cdots, \psi(t_n)) & (\text{otherwise}) \end{array}$$

Let $R_2' = R_1 \cup \{\mathsf{e} \to 0\} \cup \{d(x) \to d_2(0,x) \mid d \in \Theta^{012}\}$

Lemma 10 For any ground term s, if $s \to_{R_2'} t$ then $\psi(s) \to^{=}_{R_1} \psi(t)$.

Proof We prove this lemma by induction on the structure of s.

Basis : If $s \in Q$ then $s = \psi(s) \to_{R_1} \psi(t) = t$. If $s \in \{\mathsf{e}, 0\}$ then $\psi(s) = \psi(t) = \mathsf{e}$.

Induction step :

Case of $s \in \{\mathsf{f}(s_1, \cdots, s_6), \mathsf{g}(s_1, \cdots, s_7)\}$: In this case, $\psi(s) = \psi(t) = \mathsf{e}$.

Case of $s = d(s_1)$ where $d \in \Theta^{345}$: If $t = d(t_1)$ and $s_1 \to_{R_2'} t_1$ then $\psi(s) = d(\psi(s_1)) \to^{=}_{R_1} d(\psi(t_1)) = \psi(t)$ by the induction hypothesis. Otherwise $t = d'(s_1)$ with $d' \in \Theta^{012}$ and $\psi(s) = d(\psi(s_1)) \to_{R_1} d'(\psi(s_1)) = \psi(t)$.

Case of $s = d(s_1)$ where $d \in \Theta^{012}$: If $t = d(t_1)$ and $s_1 \to_{R_2'} t_1$ then $\psi(s) = d(\psi(s_1)) \to^{=}_{R_1} d(\psi(t_1)) = \psi(t)$ by the induction hypothesis. Otherwise $t = d_2(0, s_1)$ and $\psi(s) = \psi(t) = d(\psi(s_1))$.

Case of $s = d_2(s_1, s_2)$ where $d \in \Theta^{012}$: In this case, $t = d_2(t_1, t_2)$ holds for some t_1, t_2 and either $s_1 \to_{R_2'} t_1$ and $s_2 = t_2$ or $s_2 \to_{R_2'} t_2$ and $s_1 = t_1$, so that $\psi(s) = d(\psi(s_2)) \to^{=}_{R_1} d(\psi(t_2)) = \psi(t)$ by the induction hypothesis.
□

Lemma 11 $0 \to^*_{R_2} 1$ iff $0 \to^*_{R_1} 1$.

Proof Only if part : By the definition of R_2, if $0 \to^*_{R_2} 1$ then there exists a shortest sequence γ that satisfies $\gamma : 0 \to_{R_1} \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \overset{\geq\varepsilon *}{\to}_{R_2} \mathsf{f}(t_3, t_3, t_4, t_4, t_5, t_5) \to_{R_1} \mathsf{g}(t_3, t_3, t_4, t_4, t_5, t_5, q_A^{(12)}) \overset{\geq\varepsilon *}{\to}_{R_2}$ $\mathsf{g}(t_0, t_1, t_2, t_0, t_1, t_2, t_0) \to_{R_1} 1$. If $d_2(1,x) \to x$ is applied in γ then γ must contain a subsequence $0 \to^*_{R_2} 1$ since 1 appears only in the right-hand side of the rule $\mathsf{g}(x_0, x_1, x_2, x_0, x_1, x_2, x_0) \to 1$, and this contradicts the hypothesis that γ is the shortest sequence as above, hence $d_2(1,x) \to x$ is not applied in γ. Thus, $\mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \to^*_{R_2'} \mathsf{f}(t_3,t_3,t_4,t_4,t_5,t_5)$ and $\mathsf{g}(t_3,t_3,t_4,t_4,t_5,t_5,q_A^{(12)}) \to^*_{R_2'}$ $\mathsf{g}(t_0,t_1,t_2,t_0,t_1,t_2,t_0)$. By Lemma 10, $\mathsf{f}(\psi(q_A^{(6)}), \psi(q_A^{(7)}), \psi(q_A^{(8)}), \psi(q_A^{(9)}), \psi(q_B^{(10)}), \psi(q_B^{(11)})) \to^*_{R_1}$ $\mathsf{f}(\psi(t_3),\psi(t_3),\psi(t_4),\psi(t_4),\psi(t_5),\psi(t_5))$ and $\mathsf{g}(\psi(t_3),\psi(t_3),\psi(t_4),\psi(t_4),\psi(t_5),\psi(t_5),\psi(q_A^{(12)})) \to^*_{R_1}$ $\mathsf{g}(\psi(t_0),\psi(t_1),\psi(t_2),\psi(t_0),\psi(t_1),\psi(t_2),\psi(t_0))$. Since $\psi(q) = q$ for every $q \in Q$, $0 \to_{R_1}$ $\mathsf{f}(\psi(q_A^{(6)}), \psi(q_A^{(7)}), \psi(q_A^{(8)}), \psi(q_A^{(9)}), \psi(q_B^{(10)}), \psi(q_B^{(11)}))$. By the definition of R_1, $\mathsf{f}(\psi(t_3),\psi(t_3),\psi(t_4),\psi(t_4),\psi(t_5),\psi(t_5)) \to_{R_1} \mathsf{g}(\psi(t_3),\psi(t_3),\psi(t_4),\psi(t_4),\psi(t_5),\psi(t_5),\psi(q_A^{(12)}))$ and $\mathsf{g}(\psi(t_0),\psi(t_1),\psi(t_2),\psi(t_0),\psi(t_1),\psi(t_2),\psi(t_0)) \to_{R_1} 1$. Thus, $0 \to^*_{R_1} 1$.

If part : Obvious. □

Next, we show that R_2 is confluent iff $0 \to^*_{R_2} 1$ by using Lemma 3. Let $\phi(t)$ be the term obtained from t by replacing every maximal ground subterm (w.r.t. $\geq_{sub}$) by 1. Note that $\phi(R_0) = P^{(3.0)} \cup \Pi_1^{(3,1)} \cup \Pi_2^{(4,2)} \cup P^{(4,0)} \cup S^{(5,1)} \cup S^{(5,2)}$, $\phi(R_1) = \phi(R_0) \cup \{\mathsf{f}(x_3,x_3,x_4,x_4,x_5,x_5) \to \mathsf{g}(x_3,x_3,x_4,x_4,x_5,x_5,1), \mathsf{g}(x_0,x_1,x_2,x_0,x_1,x_2,x_0) \to 1\}$, and $\phi(R_2) = \{d(x) \to d_2(1,x), d_2(1,x) \to x \mid d \in \Theta^{012}\}$, and the rules of T_A and T_B vanish in $\phi(R_0)$.

Lemma 12 For any non-constant function symbol $h \in F \cup \Theta_2^{012}$, $h(1,\cdots,1) \to^*_{\phi(R_2)} 1$.
Proof For every $d \in \Theta^{345}$, $d(1) \to_{\phi(R_1)} d'(1)$ for some $d' \in \Theta^{012}$. For every $d' \in \Theta^{012}$, $d'(1) \to_{\phi(R_2)} d_2'(1,1)$. For every $d_2' \in \Theta_2^{012}$, $d_2'(1,1) \to_{\phi(R_2)} 1$. Moreover, $\mathsf{f}(1,\cdots,1) \to_{\phi(R_1)} \mathsf{g}(1,\cdots,1) \to_{\phi(R_1)} 1$. Thus, this lemma holds. □

We show now how the hypotheses of Lemma 3 hold for R_2 and ϕ.

Lemma 13 If $0 \to^*_{R_2} 1$ then the following propositions hold.

(1) If $s \to_{R_2} t$ then $\phi(s) \to^*_{\phi(R_2)} \phi(t)$.

(2) $\to_{\phi(R_2)} \subseteq \to^*_{R_2}$.

(3) $t \to^*_{R_2} \phi(t)$.

(4) $\phi(R_2)$ is confluent.

Proof

(1) By induction on the structure of s. If s is a ground term then $\phi(s) = \phi(t) = 1$. Thus, we assume that s is not ground. Let $s \xrightarrow{p}_{R_2} t$. If $p = \varepsilon$ then $s = \alpha\theta \to \beta\theta = t$ where $\alpha \to \beta \in R_2$. Let $s = h(s_1,\cdots,s_n)$ for some $h \in F \cup \Theta_2^{012}$ and $s_1,\cdots,s_n$, $\alpha = h(a_1,\cdots,a_n)$, and $i \in \{1,\cdots,n\}$. Since R_2 is flat, $a_i \in X \cup F_0$. Since s is not ground, $\phi(s_i) = \phi(a_i\theta)$. If a_i is a variable then $\phi(a_i\theta) = a_i\phi(\theta)$. If a_i is a constant then $\phi(s_i) = \phi(a_i) = 1$. Thus, $\phi(s) = \phi(\alpha)\phi(\theta)$. Similarly, $\phi(t) = \phi(\beta)\phi(\theta)$, so $\phi(s) \to_{\phi(R_2)} \phi(t)$ holds. If $p \neq \varepsilon$ then $s = h(s_1,\cdots,s_i,\cdots,s_n)$, $t = h(s_1,\cdots,t_i,\cdots,s_n)$, and $s_i \to_{R_2} t_i$ where $i \in \{1,\cdots,n\}$. Since s is not ground, $\phi(s) = h(\phi(s_1),\cdots,\phi(s_i),\cdots,\phi(s_n))$. By the induction hypothesis, $\phi(s_i) \to_{\phi(R_2)} \phi(t_i)$. If t is not ground then $h(\phi(s_1),\cdots,\phi(t_i),\cdots,\phi(s_n)) = \phi(t)$. If t is ground then $h(\phi(s_1),\cdots,\phi(t_i),\cdots,\phi(s_n)) = h(1,\cdots,1)$. By Lemma 12, $h(1,\cdots,1) \to^*_{\phi(R_2)} 1 = \phi(t)$. Thus, $\phi(s) \to^*_{\phi(R_2)} \phi(t)$ holds.

(2) Since $(\phi(R_2)) \setminus \{d(x) \to d_2(1,x) \mid d \in \Theta^{012}\} \subseteq R_2$, it sufficies to show that if $s = r[d(x\theta)]_p$ and $t = r[d_2(1,x\theta)]_p$ then $s \to^*_{R_2} t$. Since $d(x\theta) \to_{R_2} d_2(0,x\theta) \to^*_{R_2} d_2(1,x\theta)$, the proposition holds.

(3) It suffices to show that for any ground term t, $t \to^*_{R_2} 1$. First, we show that for any $q \in Q$, there exists s which does not have function symbols belonging to Q such that $q \to^*_{R_2} s$. Indeed, since both of the automata A and B are clean, there exists $u \in \Delta^{(3)^*} \cup \Delta^{(4)^*} \cup \Sigma^{(5)^*}$ such that $q \to^*_{R_0} u(\mathsf{e})$.
Thus, it suffices to show that for any ground term t which does not have function symbols belonging to Q, $t \to^*_{R_2} 1$ to show this lemma. We show this proposition by induction on the structure of t. Basis : By $\mathsf{e} \to_{R_2} 0 \to^*_{R_2} 1$. Induction step : Let $t = h(t_1,\cdots,t_n)$ where $n > 0$. By the induction hypothesis, $h(t_1,\cdots,t_n) \to^*_{R_2} h(1,\cdots,1)$. By Lemma 12, $h(1,\cdots,1) \to^*_{\phi(R_2)} 1$. By (2), $h(1,\cdots,1) \to^*_{R_2} 1$.

(4) We can easily show that $\phi(R_2)$ is terminating by using a lexicographic path order induced by a precedence $>$ that satisfies the following conditions: for any $d \in \Theta^{345}$, $d' \in \Theta^{012}$, $d'' \in \Theta_2^{012}$, $d > d' > d'' > 1$ and $\mathsf{f} > \mathsf{g} > 1$. Thus, it sufficies to show that every critical peak of $\phi(R_2)$ is joinable.
$\langle n,a,a'\rangle^{(0)}(x) \leftarrow \langle n,a,a'\rangle^{(3)}(x) \to a^{(1)}(x)$ is joinable by $\langle n,a,a'\rangle^{(0)}(x) \to \langle n,a,a'\rangle_2^{(0)}(1,x) \to x \leftarrow a_2^{(1)}(1,x) \leftarrow a^{(1)}(x)$.
$\langle n,\_,a'\rangle^{(0)}(x) \leftarrow \langle n,\_,a'\rangle^{(3)}(x) \to x$ is joinable by $\langle n,\_,a'\rangle^{(0)}(x) \to \langle n,\_,a'\rangle_2^{(0)}(1,x) \to x$.

$\langle n,a,a'\rangle^{(0)}(x) \leftarrow \langle n,a,a'\rangle^{(4)}(x) \rightarrow a'^{(2)}(x)$ is joinable by $\langle n,a,a'\rangle^{(0)}(x) \rightarrow \langle n,a,a'\rangle_2^{(0)}(1,x) \rightarrow x \leftarrow a_2'^{(2)}(1,x) \leftarrow a'^{(2)}(x)$.

$\langle n,a,\_\rangle^{(0)}(x) \leftarrow \langle n,a,\_\rangle^{(4)}(x) \rightarrow x$ is joinable by $\langle n,a,\_\rangle^{(0)}(x) \rightarrow \langle n,a,\_\rangle_2^{(0)}(1,x) \rightarrow x$.

$a^{(1)}(x) \leftarrow a^{(5)}(x) \rightarrow a^{(2)}(x)$ is joinable by $a^{(1)}(x) \rightarrow a_2^{(1)}(1,x) \rightarrow x \leftarrow a_2^{(2)}(1,x) \leftarrow a^{(2)}(x)$. □

Lemma 14 R_2 is confluent iff $0 \rightarrow^*_{R_2} 1$.

Proof Only if part : By $\langle n,\_,a'\rangle^{(0)}(x) \leftarrow_{P(3,0)} \langle n,\_,a'\rangle^{(3)}(x) \rightarrow_{\Pi_1^{(3,1)}} x$, confluence ensures that $\langle n,\_,a'\rangle^{(0)}(x) \downarrow_{R_2} x$. Since x is a normal form, $\langle n,\_,a'\rangle^{(0)}(x) \rightarrow^*_{R_2} x$. Thus, there exists sequence $\gamma : \langle n,\_,a'\rangle^{(0)}(x) \rightarrow_{R_2} \langle n,\_,a'\rangle_2^{(0)}(0,x) \rightarrow^*_{R_2} \langle n,\_,a'\rangle_2^{(0)}(1,x) \rightarrow_{R_2} x$. If there exists such γ, $0 \rightarrow^*_{R_2} 1$ must hold.

If part : By Lemmata 3 and 13. □

By Lemmata 7, 11, 14, the following theorem holds.

Theorem 15 Confluence for flat TRSs is undecidable.

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