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ADIABATIC LIMITS OF η-INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS

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CONTENTS

1. Introduction

Let Σ_g be a closed oriented suface of genus g and let $\mathcal{M}_g$ be the mapping class group of genus g, namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g. Meyer introduced a cocycle $\tau_g : \mathcal{M}_g\times\mathcal{M}_g\to\mathbb{Z}$, called the signature cocycle or the Meyer cocycle, and he gave a signature fomula for the signature of surface bundles over surfaces ([21]). Let $[\tau_g]\in H^2(\mathcal{M}_g,\mathbb{Z})$ denotes the cohomology class of τ_g. When $g = 1$, since $\mathcal{M}_1 = SL_2(\mathbb{Z})$, $H^1(SL_2(\mathbb{Z}),\mathbb{Z}) = 0$ and $3[\tau_1] = 0$, there exists a unique 1-cocycle $\phi_1 : SL_2\mathbb{Z}\to\frac{1}{3}\mathbb{Z}$ such that cobounds τ_1. The function ϕ_1 is called the Meyer function of genus one, which has the following property: Let $\pi : Z\to X$ be a Σ_1-bindle over a compact oriented surface with boundary $\partial Z = c_1 \amalg\cdots\amalg c_k$. Let $A_1,\cdots,A_k$ be the monodromies around each component of the boundary. Since the Picard-Lefschetz transformation along c_i is an automorphism of $H^1(\Sigma_1,\mathbb{Z})$ preserving the intersection form, one has $A_i \in SL_2(\mathbb{Z})$ by fixing a symplectic basis of $H^1(\Sigma_1,\mathbb{Z})$. Then the signature of Z, which is defined as the signature of the cup-product pairing on $H^2(Z,\partial Z,\mathbb{R})$, satisfies

$$\mathrm{Sign}(Z) = \sum_{i=1}^{k}\phi_1(A_i). \tag{1}$$

The explicite formula of ϕ_1 was obtained by Meyer ([21]).

When $g = 2$, since $5[\tau_2] = 0 \in H^2(\mathcal{M}_2,\mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ and $H^1(\mathcal{M}_2,\mathbb{Z}) = 0$, there exists a unique 1-cocycle $\phi_2 : \mathcal{M}_2\to\frac{1}{5}\mathbb{Z}$ satisfying (1), for every Σ_2-bundles over compact orienred surfaces. The function ϕ_2 is called the Meyer function of genus two.

In [1], Atiyah investigated the Meyer function ϕ_1 from the several view points. For an odd dimensional closed oriented Riemannian manifold M, let $\eta(M)$ be the η-invariant of M with respect to the signature operator of M [2]. For $\sigma \in SL_2\mathbb{Z}$, let $\pi : M_\sigma\to S^1$ be the mapping torus associated with σ, i.e., Σ_1-bundle over S^1 with monodromy σ. Then Atiyah showed the following identity, when M_σ is equipped with a certain metric:

$$\phi_1(\sigma) = \eta(M_\sigma)$$

Moreover, he gave several interpretation of ϕ_1 interms of the following quantities: (1)Hirzebruch's signature defect; (2)the transformation lows of the logarithm of the Dedekind η-function; (3)the logarithm of the monodromy of Quillen's line bundle; (4)the special value of the Shimizu L-function at the origin.

In this note, we study an extension of the result of Atiyah to the case $g = 2$ and higher dimansional manifold. We shall construct a higher dimensional analogue of the Meyer function for smooth theta divasors of odd dimension.

Notation : For a complex manofold M, $T^{1,0}M$ (resp. $T^{0,1}M$) denotes the holomorphic (resp. anti-holomorphic) tangent bendle and TM denotes the real tangent bundle. We set $d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial})$. Hence $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.

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2. Preliminaries from Riemannian geometry

In this section, we recall some results of Riemannian geometry which will be used in the proof of the main theorem. Following [10], we define connections of fiber bundles and the connection of relative tangent bundles. Let M be a manifold and let $\pi : Z \to B$ be a fiber bundle with typical fiber M.

The ***relative tangent bundle*** $T(Z/B)$ is the subbundle of TZ defined by

$$T(Z/B) := \mathrm{Ker}\{\pi_* : TZ \to \pi^*TB\}.$$

A vector of $T(Z/B)$ is said to be *vertical.*

Definition 2.1. A subbundle $T_HZ \subset TZ$ with $TZ = T(Z/B) \oplus T_HZ$ is called a *connection* of the fiber bundle $\pi : Z \to B$.

For a connection, one has $T_HZ \cong \pi^*TB$ via the projection $\pi_* : TZ \to \pi^*TB$. A vector of T_HZ is said to be *horizontal.*

When Z is trivial, i.e., $Z = M \times B$, TZ is naturally isomorphic to the direct sum $(\mathrm{pr}_1)^*TM \oplus (\mathrm{pr}_2)^*TB$. This connection is called the *trivial connection* of the trivial fiber bundle.

Given a connection, one can define the projection $P_Z : TZ \to T(Z/B)$ with kernel T_HZ. We often identify P_Z with the corresponding connection $T_HZ := \mathrm{Ker}(P_Z)$. In the rest of Section 2, we fix a connection T_HZ, or equivalently P_Z. One can define the pull-back of a connection, as follows: Let B' be a manifold and let $h : B' \to B$ be a C^∞ map. The fiber product $Z' := Z \times_B B' = \{(x,b) \in Z \times B' \mid \pi(x) = h(b)\}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{\tilde{h}} & Z \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[h]{} & B \end{array} \qquad \tilde{h} = \mathrm{pr}_1,\ \pi' = \mathrm{pr}_2.$$

Since the map $P_Z \circ \tilde{h}_* : TZ' \to h^*T(Z/B)$ is surjective, $\mathrm{Ker}(P_Z \circ \tilde{h}_*)$ is a subbundle of TZ'. Since $T(Z'/B')$ is canonically isomorphic to $h^*T(Z/B)$, the map $P_Z \circ \tilde{h}_*$ is identified with a projection from TZ' to $T(Z'/B')$.

Definition 2.2. The connection of $\pi' : Z' \to B'$ induced from $T_H Z$ by h is defined by

$$T_H Z' := \mathrm{Ker}\big(P_Z \circ \tilde{h}_* : TZ' \to T(Z/B)\big),$$

under the identification between $T(Z'/B')$ and $h^* T(Z/B)$. The projection corresponding to $T_H Z'$ is denoted by $h^* P_Z$.

We fix a metric $g^{Z/B}$ on the relative tangent bundle, a Riemannian metric g^B on B, and the connection $T_H Z$ and the corresponding projection P_Z. We define the Riemannian metric g^Z on the total space Z by

$$g^Z := g^{Z/B} \oplus \pi^* g^B,$$

under the isomorphism $TZ \cong T(Z/B) \oplus T_H Z \cong T(Z/B) \oplus \pi^* TB$. Let ∇^Z be the Levi-Civita connection of (Z, g^Z). We define the connection $\nabla^{Z/B}$ on $T(Z/B)$ by

$$\nabla^{Z/B} := P_Z \circ \nabla^Z.$$

Then $\nabla^{Z/B}$ preserves the metric $g^{Z/B}$.

Lemma 2.3. *The connection $\nabla^{Z/B}$ is independent of a choice of g^B*

Proof. See [10, Proposition 10.2] □

Lemma 2.4. *Let B' be a manifold and let $h : B' \to B$ be a C^∞-map, and set $Z' := Z \times_B B'$. Let $g^{Z'/B'} = h^* g^{Z/B}$ be the metric on $T(Z'/B')$ induced from $g^{Z/B}$, and let $P_{Z'} = h^* P_Z$ be the connection of Z' induced from P_Z. Then $\nabla^{Z'/B'} = h^* \nabla^{Z/B}$.*

Proof. See [15] □

With respect to the decomposition $TZ = T(Z/B) \oplus T_H Z$, We put for $\varepsilon \in \mathbb{R}^+$

$$g^{Z,\varepsilon} := g^{Z/B} \oplus \varepsilon^{-1} \pi^* g^B.$$

The Levi-Civita connections of $(Z, g^{Z,\varepsilon})$ and (B, g^B) are denoted by $\nabla^{Z,\varepsilon}$ and ∇^B, respectively. Let $R^{Z,\varepsilon}$ and R^B be the curvature of $\nabla^{Z,\varepsilon}$ and ∇^B, respectively. Then $g^Z := g^{Z,1}$ and $\nabla^Z := \nabla^{Z,1}$. We define another connection ∇ on Z by

$$\nabla := \nabla^{Z/B} \oplus \pi^* \nabla^B,$$

and we put

$$S^{(\varepsilon)} := \nabla^{Z,\varepsilon} - \nabla \in \mathcal{A}^1(\mathrm{End}(TZ)), \qquad S := S^{(1)}.$$

Then ∇ preseves the Riemannian meteric $g^{Z,\varepsilon}$, and P_Z is paralel with respect to ∇, i.e. $\nabla \circ P_Z - P_Z \circ \nabla = 0$.

Let $\{e_1, \cdots, e_k\}$ be a local orthogonal framing for $(T(Z/B), g^{Z/B})$, and let $\{f_1, \cdots, f_l\}$ be a local orthgonal framing for $(T_H Z, \pi^* g^B)$.

Proposition 2.5. *With respect to the splitting $TZ = T(Z/B) \oplus T_H B$, the following identity holds:*

$$\lim_{\varepsilon \to 0} R^{Z,\varepsilon} = \begin{pmatrix} R^{Z/B} & P_Z(\nabla S) \\ 0 & \pi^* R^B \end{pmatrix}.$$

Proof. See [7] (3.195). □

3. η-invariants

In this section, we recall the definition and some properties of η-invariants. Let (M, g^M) be a coled oriented Riemannian manifold of dimension $(2l-1)$. Denote the space of C^∞ k-forms on M by $\mathcal{A}^k(M)$. Let $*: \mathcal{A}^k(M) \to \mathcal{A}^{2l-k-1}(M)$ be the Hodge star operation with respect to g^M. The signature operator $D: \oplus_{p\geq 0}\mathcal{A}^{2p}(M) \to \oplus_{p\geq 0}\mathcal{A}^{2p}(M)$ of M is defined by

$$D: \omega \longmapsto (\sqrt{-1})^l(-1)^{p+1}(*d - d*)\omega, \quad \omega \in \mathcal{A}^{2p}(M).$$

Then D is an elliptic self-adjoint differential operator of first order acting on $\oplus_{p\geq 0}\mathcal{A}^{2p}(M)$. Let $\sigma(D)$ be the spectrum of D. The η-function of M is defined by

$$\eta(s) := \sum_{\lambda\in\sigma(D)\setminus\{0\}} \frac{\text{sign}\lambda}{\lambda^s},$$

for $s\in\mathbb{C}$ with $\text{Re}(s) \gg 0$. Then $\eta(s)$ extends meromorphically to $\mathbb{C}$ and is holomorphic at $s=0$ by [2], [7].

Definition 3.1. The real number $\eta(0)$ is called the *η-invariant* of (M, g^M) and is denoted by $\eta(M, g^M)$.

Let (X, g^X) be a $4k$-dimensional, oriented, compact, Riemannian manifold with boundary Y. Put $g^Y := g^X|_Y$ and fix a color neighborhood $U \supset Y$ such that $U \cong Y \times [0,1)$. Assume that $g^X|_U = g^Y \oplus dt^2$ under the above isomorphism. Let ∇^L be the Levi-Civita connction of (X, g^X).

Theorem 3.2 (Atiyah-Patodi-Singer [2]). *The following equation holds:*

$$\text{Sign}(X) = \int_X L(TX, \nabla^L) - \eta(Y, g^Y)$$

Here L denotes the Hirzebruch L-polynomial, which is a multiplicative genus associated with the power series: $L(x) := x/\tanh(x)$.

Let X, B and M be closed oriented manifolds. Let $\pi: X \to B$ be a C^∞-submersion, whose fibers are isomorphic to M. Assume that $\dim X = 4k$. Let $g^{X/B}$ be a metric on $T(X/B)$ and let g^B be a metric on TB. Let $T_H X \subset TX$ be a connection . We identify $T_H X$ with $\pi^* TB$ via π. With respect to the decomposition $TX = T(X/B) \oplus \pi^* TB$, we define the metric on X by $g^X := g^{X/B} \oplus \pi^* g^B$ and we consider the one parameter family of metrics on X defined by

$$g^{X,\varepsilon} := g^{X/B} \oplus \varepsilon^{-1}\pi^* g^B, \quad \varepsilon \in \mathbb{R}^+.$$

Theorem 3.3 (Bismut-Cheeger, [6]). *The limit $\lim_{\varepsilon\to 0} \eta(X, g^{X,\varepsilon})$ exists.*

The limit $\lim_{\varepsilon\to 0} \eta(X, g^{X,\varepsilon})$ is called the *adiabatic limit of the η-invariants* and is denoted by $\eta^0(X)$. By definition, $\eta^0(X, g^X)$ depends on the three data: $g^{X/B}$, g^B and $T_H X$.

4. Family of smooth theta divisors

We fix the following notation. Let $\mathfrak{S}_g$ be the Siegel upper-half space of degree g and let Γ_g be the integral symplectic group, i.e.,

$$\begin{aligned} \mathfrak{S}_g &:= \{\tau \in M(g, \mathbb{C}) \mid {}^t\tau = \tau, \ \text{Im}\tau > 0\} \\ \Gamma_g &:= \{\gamma \in GL(2g, \mathbb{Z}) \mid \gamma\, J_g\, {}^t\gamma = J_g\}, \end{aligned}$$

where $J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$ and 1_g denotes the $g \times g$ identity matrix. Γ_g acts on $\mathfrak{S}_g$ by

$$\gamma\cdot\tau := (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad \tau \in \mathfrak{S}_g.$$

For $\tau \in \mathfrak{S}_g$, write $\tau = {}^t(\tau_1, \cdots, \tau_g)$ and set

$$\Lambda_\tau := \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_g \oplus \mathbb{Z}\tau_1 \oplus \cdots \oplus \mathbb{Z}\tau_g \subset \mathbb{C}^g$$

where $1_g = {}^t(e_1, \cdots, e_g)$ and $\tau = {}^t(\tau_1, \cdots, \tau_g) \in \mathfrak{S}_g$. Define the $\mathbb{Z}^{2g}$-action on $\mathbb{C}^g \times \mathfrak{S}_g$ by

$$(m,n)\cdot(z,\tau) := (z + m\tau + n, \tau), \qquad (z,\tau) \in \mathbb{C}^g \times \mathfrak{S}_g, \quad m,n \in \mathbb{Z}^{2g}.$$

Then

$$f : \mathbb{A}_g := (\mathbb{C}^g \times \mathfrak{S}_g)/\mathbb{Z}^{2g} \to \mathfrak{S}_g$$

is the universal family of principally polarized Abelian varieties over $\mathfrak{S}_g$, whose fiber at τ is $A_\tau := \mathbb{C}^g/\Lambda_\tau$. For $(a,b)\in\mathbb{R}^{2g}$, $z\in\mathbb{C}^g$ and $\tau\in\mathfrak{S}_g$ we define the theta function with characteristic by

$$\vartheta_{a,b}(z,\tau) := \sum_{n\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}(n+a)\tau{}^t(n+a) + (n+a){}^t(z+b)\right),$$

where $\mathbf{e}(t) = \exp(2\pi\sqrt{-1}t)$. Let

$$f : \Theta_{a,b} := \{(z,\tau)\in\mathbb{A}_g \mid \vartheta_{a,b}(z,\tau) = 0\} \to \mathfrak{S}_g.$$

be the *universal family of theta divisors.* For simplicity we write ϑ for $\vartheta_{0,0}$ and set $\Theta = \Theta_{0,0}$. On $\mathbb{A}_g$, Γ_g acts by

$$\gamma\cdot(z,\tau) := (z(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}), \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \ z \in \mathbb{C}^g, \ \tau \in \mathfrak{S}_g.$$

For any $(m,n)\in\mathbb{R}^{2g}$, we define an automorphism $t_{m,n} : \mathbb{A}_g \to \mathbb{A}_g$ by

$$(z,\tau) := (z + m\tau + n, \tau).$$

Then $t_{(m,n)}$ has no fixed points when $(m,n)\in\mathbb{R}^{2g}\backslash\mathbb{Z}^{2g}$ and the subgroup $\mathbb{Z}^{2g}\subset\mathbb{R}^{2g}$ acts trivially on $\mathbb{A}_g$. For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, we define

$$\tilde{\gamma} := t_{(m,n)}\circ\gamma \in \mathrm{Aut}(\mathbb{A}_g), \qquad (m,n) := \frac{1}{2}((C{}^tD)_0, (A{}^tB)_0).$$

Then $\tilde{\gamma}$ preserves the family $f : \Theta \to \mathfrak{S}_g$.

Proposition 4.1. *For any* $\gamma_1, \gamma_2 \in \Gamma_g$,

$$\tilde{\gamma}_1 \circ \tilde{\gamma}_2 = \widetilde{\gamma_1\gamma_2}$$

Proof. See [15] □

We set

$$g^{\mathbb{A}_g/\mathfrak{S}_g} := dz\cdot(\mathrm{Im}\tau)^{-1}\cdot{}^t d\bar{z}.$$

Then $g^{\mathbb{A}_g/\mathfrak{S}_g}$ is a Γ_g-invariant Hermitian metric on the relative tangent bundle $T(\mathbb{A}_g/\mathfrak{S}_g)$. The next purpose of this section is to construct a Γ_g-invariant Kähler metric on $T\mathbb{A}_g$ such that $g^{\mathbb{A}_g}|_{A_\tau} = dz\cdot(\mathrm{Im}\tau)^{-1}\cdot{}^t d\bar{z}$ for all $\tau \in \mathfrak{S}_g$.

Put $T^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Define a $\mathbb{Z}^{2g}$-action on $\mathbb{R}^{2g} \times \mathfrak{S}_g$ by $(m,n)\cdot(x,y,\tau) := (x+m, y+n, \tau)$ for $(m,n) \in \mathbb{Z}^{2g}$, $(x,y) \in \mathbb{R}^{2g}$, $\tau \in \mathfrak{S}_g$. Then $(\mathbb{R}^{2g} \times \mathfrak{S}_g)/\mathbb{Z}^{2g}$ is the trivial T^{2g}-bundle $T^{2g} \times \mathfrak{S}_g$. We define a C^∞-map $\tilde{\rho} : \mathbb{R}^{2g} \times \mathfrak{S}_g \to \mathbb{C}^g \times \mathfrak{S}_g$ by

$$\tilde{\rho}((x,y),\tau) := (x\tau + y, \tau), \qquad x,y \in\mathbb{R}^g, \ \tau\in\mathfrak{S}_g.$$

Since $\tilde{\rho}$ is a $\mathbb{Z}^{2g}$-equivariant map, $\tilde{\rho}$ induces a C^∞-isomorphism $\rho : T^{2g} \times \mathfrak{S}_g \to \mathbb{A}_g$ as T^2-bundles over $\mathfrak{S}_g$. Define a Γ_g-action on $T^{2g}\times\mathfrak{S}_g$ by

$$\gamma\cdot((x,y),\tau) := ((x,y)\gamma^{-1}, \gamma\cdot\tau), \quad \gamma\in\Gamma_g.$$

Then for any $\gamma\in\Gamma_g$, the following diagram is commutative.

$$\begin{array}{ccc} T^{2g}\times\mathfrak{S}_g & \xrightarrow{\rho} & \mathbb{A}_g \\ \gamma\downarrow & & \downarrow\gamma \\ T^{2g}\times\mathfrak{S}_g & \xrightarrow[\rho]{} & \mathbb{A}_g \end{array}$$

Since the trivial connection on $T^{2g}\times\mathfrak{S}_g$ is Γ_g-invariant, $\mathbb{A}_g$ has the induced Γ_g-invariant connection $T_H\mathbb{A}_g \subset T\mathbb{A}_g$ via the Γ_g-equivariant isomorphism ρ. We denote the Γ_g-equivariant projection corresponding to $T_H\mathbb{A}_g$ by P_ρ. Let $P_\rho^{\mathbb{C}} : T\mathbb{A}_g \otimes \mathbb{C} \to T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ be the complexification of P_ρ. Then $P_\rho^{\mathbb{C}}$ is also Γ_g-equivariant.

Under the projection, the horizontal lift of a $(1,0)$ (resp. $(1,0)$) tangent vector is a $(1,0)$ (resp. $(1,0)$) tangent vector. Therefore the extension $P_\rho^{\mathbb{C}} : T\mathbb{A}_g\otimes\mathbb{C}\to T(\mathbb{A}_g/\mathfrak{S}_g)\otimes\mathbb{C}$ decomposes

(2) $$P_\rho^{\mathbb{C}} = P_\rho^{1,0}\oplus P_\rho^{0,1},$$

under the isomorphism $T\mathbb{A}_g\otimes\mathbb{C} = T^{1,0}\mathbb{A}_g\oplus T^{0,1}\mathbb{A}_g$ and $T(\mathbb{A}_g/\mathfrak{S}_g)\otimes\mathbb{C} = T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g)\oplus T^{0,1}(\mathbb{A}_g/\mathfrak{S}_g)$. Hence P_ρ induces a Γ_g-equivariant C^∞-isomorphism

(3) $$T^{1,0}\mathbb{A}_g\cong T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g)\oplus f^*T^{1,0}\mathfrak{S}_g.$$

Let $g^{\mathfrak{S}_g}$ be the Bergman metric on $\mathfrak{S}_g$ with Kähler form

(4) $$\omega_{\mathfrak{S}_g} = -2\sqrt{-1}\partial\bar{\partial}\log\det\mathrm{Im}\tau.$$

Then $g^{\mathfrak{S}_g}$ is Γ_g-invariant. Using the Γ_g-equivariant isomorphism (3), we define the Γ_g-invariant Hermitian metric $g^{\mathbb{A}_g}$ on $T\mathbb{A}_g$ by

$$g^{\mathbb{A}_g} := g^{\mathbb{A}_g/\mathfrak{S}_g}\oplus f^*g^{\mathfrak{S}_g}.$$

Theorem 4.2. *The Hermitian metric $g^{\mathbb{A}_g}$ is Kähler.*

Proof. See [15] □

We put

$$A_k(\Gamma_g,\chi) = \{f\in\mathcal{O}(\mathfrak{S}_g) \mid f(\gamma\cdot\tau) = j(\tau,\gamma)^k\chi(\gamma)f(\tau),\ \gamma\in\Gamma_g\}$$

where χ is a character of Γ_g and $j(\tau,\gamma) = \det(C\tau + D)$ for $\gamma\in\begin{pmatrix} A & B \\ C & D\end{pmatrix}$. An element of $A_k(\Gamma_g,\chi)$ is called a *Siegel modular form* of weight k with character χ . In particular, an element of $A_k(\Gamma_g,1)$ is called a Siegel modular form. Let $\mathcal{F}_g^k := \mathfrak{S}_g\times\mathbb{C}^g$ be the trivial holomorphic line bundle over $\mathfrak{S}_g$ with the Γ_g-action

$$\gamma\cdot(\tau,\xi) := (\gamma\cdot\tau, j(\gamma,\tau)^k\xi).$$

A Siegel modular form of weight k is regarded as a Γ_g-invariant holomorphic setion of $\mathcal{F}_g^k$. Define the *Peterson metric* on $\mathcal{F}_g^k$ by

$$\|\xi\|^2_{\mathcal{F}_g^k} := (\det\mathrm{Im}\tau)^k|\xi|^2,\quad (\tau,\xi)\in\mathcal{F}_g^k.$$

By the automorphic property of $\det\mathrm{Im}(\gamma\cdot\tau) = |j(\tau,\gamma)|^{-2}\det\mathrm{Im}\tau$, we see that $\|\cdot\|_{\mathcal{F}_g^k}$ is Γ_g-invariant.

Let $\mathcal{N}_g := \{\tau \in \mathfrak{S}_g \mid \mathrm{Sing}\Theta_\tau\neq\emptyset\}$ be the Andreotti-Mayer locus, which is the locus of Abelian varieties whose theta divisors is singular. The followings are known for the locus $\mathcal{N}_g$.

Theorem 4.3 ([12]). *$\mathcal{N}_g$ is a divisor of $\mathfrak{S}_g$, consisting of two irreducible components as a divisor of the modular variety $\Gamma_g\backslash\mathfrak{S}_g$:*

$$\mathcal{N}_g = \theta_{null,g} + 2\mathcal{N}_g^{'}.$$

Here $\theta_{null,g}$ is the zero divisor of Igusa's modular form $\chi_g(\tau)$ which is the Siegel modular form of weight $2^{g-2}(2^g+1)$ defined as the product of all even theta constants and $\mathcal{N}_g^{'} = \emptyset$ for $g = 2, 3$. For a generic point $\tau\in\theta_{null,g}$, $\mathrm{Sing}(\Theta_\tau)$ consists of one ordinary double point.

Theorem 4.4 ([25]). *There is a Siegel cusp form $\Delta_g(\tau)$ of weight $\frac{(g+3)\cdot g!}{2}$ with zero divisor $\mathcal{N}_g$. By the Proposition 4.3, this implies that there exists $J_g(\tau)$ which is a Siegel modular form of weight $\frac{(g+3)\cdot g!}{4} - 2^{g-3}(2^g+1)$ with zero divisor $\mathcal{N}_g^{'}$ such that*

$$\Delta_g := \chi_g(\tau)J_g(\tau)^2.$$

We put $\mathfrak{S}_g^{'} := \mathfrak{S}_g - \mathcal{N}_g$, $\Theta_g^{'} := \Theta|_{\mathfrak{S}_g^{'}}$. Then $f : \Theta^{'} \to \mathfrak{S}_g^{'}$ is a family of smooth theta divisors. Endow $T^{1,0}(\Theta^{'}/\mathfrak{S}_g^{'})$ the Hermitian metric $g^{\Theta^{'}/\mathfrak{S}_g^{'}} := g^{\mathbb{A}_g/\mathfrak{S}_g}|_{\Theta^{'}}$. Let $g^{\Theta^{'}} := g^{\mathbb{A}_g}|_{\Theta^{'}}$ be the restriction of the Kähler metric $g^{\mathbb{A}_g}$. Consider $g^{\Theta^{'}/\mathfrak{S}_g^{'}}$ and $g^{\Theta^{'}}$ as Riemannian metric on $T(\Theta/\mathfrak{S}_g^{'})$ and $T\Theta^{'}$. Let

$$T_H\Theta' := (T(\Theta'/\mathfrak{S}_g'))^{\perp}$$

be the orthgonal complement of $T(\Theta'/\mathfrak{S}_g')$ in $T\Theta'$, which induces a connection $P_{\Theta'} : T\Theta^{'} \to T\Theta^{'}/\mathfrak{S}_g^{'}$. Hence we obtain the connection $\nabla^{\Theta'/\mathfrak{S}_g'}$ on $T(\Theta'/\mathfrak{S}_g')$ by using $g^{\Theta'/\mathfrak{S}_g'}$ and $P_{\Theta'}$ as in Section 2.2. Let ∇^h be the holomorphic Hermitian connection on $T^{1,0}(\Theta'/\mathfrak{S}_g')$ with respect to the Hemitian metric $g^{\Theta'/\mathfrak{S}_g'}$.

Lemma 4.5. *Under the C^∞-isomorphism $T(\Theta'/\mathfrak{S}_g')\otimes\mathbb{C}\cong T^{1,0}(\Theta'/\mathfrak{S}_g')\oplus T^{0,1}(\Theta'/\mathfrak{S}_g')$, the following equality of connections holds.*

$$\nabla^{\Theta'/\mathfrak{S}_g'}\otimes\mathbb{C} = \nabla^h\oplus\bar{\nabla}^h$$

Proof. Let ∇^L be the Levi-Civita connection on $T\mathbb{A}_g$ and let ∇^H be the holomorphic Hermitian connection on $T^{1,0}\mathbb{A}_g$. Since $g^{\mathbb{A}_g}$ is Kähler, the following equality holds ([18])

$$\nabla^L\otimes\mathbb{C} = \nabla^H\oplus\bar{\nabla}^H$$

under the isomorphism $T\mathbb{A}_g\otimes\mathbb{C} = T^{1,0}\mathbb{A}_g\oplus T^{0,1}\mathbb{A}_g$. By (2), we get

$$\begin{aligned}\nabla^{\Theta'/\mathfrak{S}_g'}\otimes\mathbb{C} &= (P_\rho\nabla^L P_\rho)\otimes\mathbb{C}\\ &= P_\rho^{\mathbb{C}}(\nabla^L\otimes\mathbb{C})P_\rho^{\mathbb{C}}\\ &= P_\rho^{1,0}\nabla^H P_\rho^{1,0}\oplus P_\rho^{0,1}\bar{\nabla}^H P_\rho^{0,1}.\end{aligned}$$

Since $P_\rho^{1,0}\nabla^H P_\rho^{1,0} = \nabla^h$ (see [18] Capter I, Section 6), we get the result. □

Let g_{1_g} be the restriction of the Hermitian metric $|dz|^2$ on $T\mathbb{A}_g/\mathfrak{S}_g$ to the relative tangent bundle $T\Theta^{'}/\mathfrak{S}_g^{'}$. Let $F(T\Theta^{'}/\mathfrak{S}_g^{'}, g_{1_g})$ be the corresponding Chern-Weil form for $F(x)$ and the holomorphic Hermitian connection of $(T\Theta^{'}/\mathfrak{S}_g^{'}, g_{1_g})$.

Proposition 4.6 ([24], Proposition 2.1). *The following equality holds:*

$$[F(T\Theta^{'}/\mathfrak{S}_g^{'}, g_{1_g})]^{(g,g)} \equiv 0.$$

In particular one has

$$[f_*F(T\Theta^{'}/\mathfrak{S}_g^{'}, g_{1_g})]^{(1,1)} \equiv 0.$$

Let $\|\Delta_{2g}(\tau)\|^2 := (\mathrm{detIm}\tau)^{\frac{(2g+3)\cdot(2g)!}{2}}|\Delta_{2g}(\tau)|^2$ denote the Peterson norm of the Siegel modular form $\Delta_{2g}(\tau)$ and let B_k be the k-th Bernoulli number, i.e.,

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{x^{2k}}{(2k)!}.$$

Theorem 4.7. *The followig equality holds:*

$$\begin{aligned}\left[f_*L(T(\Theta'/\mathfrak{S}'_{2g}), \nabla^{\Theta'/\mathfrak{S}'_{2g}})\right]^{(2)} &= \frac{(-1)^g2^{2g+1}(2^{2g+2}-1)}{(2g+1)(g+1)}B_{g+1}dd^c\log\mathrm{detIm}\tau \\ &= \frac{(-1)^g2^{2g+3}(2^{2g+2}-1)}{(2g+3)!}B_{g+1}dd^c\log\|\Delta_{2g}(\tau)\|^2.\end{aligned}$$

By Lemma 4.5 and the fact that $(\nabla^h)^2$ is a $(1,1)$-form, we see that the left-hand side is equal to $\left[f_*L(T^{1,0}(\Theta'/\mathfrak{S}'_{2g}), \nabla^h)\right]^{(1,1)}$. By Proposition 4.6 we obtain

$$[L(T^{1,0}(\Theta'/\mathfrak{S}'_{2g}), \nabla^h)]^{(1,1)} = -dd^cf_*[\tilde{L}(T^{1,0}(\Theta'/\mathfrak{S}'_{2g}), g_{1_g}, g^{\Theta'/\mathfrak{S}'_g})]^{(2g-1,2g-1)}.$$

Hence we deduced the proof to the computation of the Bott-Chern form and we can compute it by using the same idea in [25]. Since this is rather complicated, we omit the proof.

Remark 4.8. In Section 5, it will be cruicial that $d^c\log\|\Delta_g(\tau)\|^2$ is Γ_g-invariant and that $dd^c\log\|\Delta_g(\tau)\|^2$ is an eaxct form as a 2-form on $\Gamma_g\backslash\mathfrak{S}'_g$.

5. The signature cocycle for smooth theta divisors

Since Γ_g acts on $\mathfrak{S}'_g$ properly discontinuously the space $\Gamma_g\backslash\mathfrak{S}'_g$ has naturally orbifold structure and can be regarded as the moduli space of smooth theta divisors. We shall consider the orbifold fundamental group of $\Gamma_g\backslash\mathfrak{S}'_g$ and construct a 2-cocycle of this group.

In the rest of this section we fix a generic base point $*\in\mathfrak{S}'_g$, i.e., $*$ satisfies $\{\gamma\in\Gamma_g \mid \gamma\cdot * = *\} = \{\pm 1_{2g}\}$. Let (B, b) be a topological space with a base point and let $\pi : \tilde{B}\to B$ be the universal covering. Then the fundamental group $\pi_1(B, b)$ acts on $\tilde{B}$ as the deck transformation. Fix a lift $\tilde{b}\in\tilde{B}$ of $b\in B$. We set

$$[B, \Gamma_g\backslash\mathfrak{S}'_g]^{orb} := \{(p, \beta) \mid p : \tilde{B}\to\mathfrak{S}'_g, \beta : \pi_1(B, b)\to\Gamma_g, \text{ s.t. } p(\tilde{b}) = *,\ p(\gamma\cdot x) = \beta(\gamma)\cdot p(x)\}/\sim.$$

Here the relation $(p_0, \beta_0)\sim(p_1, \beta_1)$ holds if and only if $\beta_0 = \beta_1$ and there is a map $\tilde{p} : \tilde{B}\times[0,1]\to\mathfrak{S}'_g$ such that $\tilde{p}(x, 0) = p_0$, $\tilde{p}(x, 1) = p_1$ and $\tilde{p}(\gamma\cdot x, t) = \beta(\gamma)\cdot\tilde{p}(x, t)$ for any $\gamma\in\Gamma_g$, $x\in\tilde{B}$, $t\in[0,1]$.

Definition 5.1. We define the *orbifold fundamental group* of $\Gamma_g\backslash\mathfrak{S}'_g$ by

$$\begin{aligned} S_g &:= [S^1, \Gamma_g\backslash\mathfrak{S}'_g]^{orb} \\ &= \{(\alpha, \gamma) \mid \gamma\in\Gamma_g,\ \alpha : \mathbb{R}\to\mathfrak{S}'_g, \text{s.t. } \alpha(0) = *,\ \alpha(t) = \gamma\cdot\alpha(t+1),\ t\in\mathbb{R}\}/\sim.\end{aligned}$$

Then

$$S_g = \{(\alpha, \gamma) | \gamma\in\Gamma_g,\ \alpha : [0,1]\to\mathfrak{S}'_g, \text{s.t. } \alpha(0) = \gamma\cdot\alpha(1) = *\}/\sim.$$

Here $(\alpha_0, \gamma_0)\sim(\alpha_1, \gamma_1)$ if and only if $\gamma_0 = \gamma_1$ and there exists a homotopy $\alpha(s,t) : [0,1]\times[0,1]\to\mathfrak{S}'_g$ connecting α_0 and α_1, such that $\alpha(s, 0) = \gamma_0\cdot\alpha(s, 1) = *$ for $s\in[0,1]$.

The group law of S_g is defined as follows. Let $[(\alpha_1, \gamma_1)]$, $[(\alpha_2, \gamma_2)]\in S_g$. Then $\gamma_2^{-1}\cdot\alpha_1$ is a path path from $\gamma_2^{-1}\cdot *$ to $(\gamma_1\gamma_2)^{-1}\cdot *$. We define the new path $\alpha : [0,1]\to\mathfrak{S}'_g$ by $\alpha(t) := \alpha_2(2t)$ for $0\le t\le\frac{1}{2}$, $\alpha(t) := \gamma_2^{-1}\cdot\alpha_1(2t-1)$ for $\frac{1}{2}\le t\le 1$. Then we define $[(\alpha_1, \gamma_1)]\cdot[(\alpha_2, \gamma_2)] := [(\alpha, \gamma_1\gamma_2)]\in S_g$.

Let $p : S_g \to \Gamma_g$ be the projection to the second factor. Since the kernel of p is isomorphic to $\pi_1(\mathfrak{S}_g', *)$, we have an exact sequence

(5) $$1 \to \pi_1(\mathfrak{S}_g', *) \to S_g \to \Gamma_g \to 1.$$

Remark 5.2. When $g = 1$, $\Gamma_1 \backslash \mathfrak{S}_1' = SL_2\mathbb{Z} \backslash \mathfrak{S}_1$ is the moduli space of curves of genus 1 and $S_1 = \mathcal{M}_1$. When $g = 2$, $\Gamma_2 \backslash \mathfrak{S}_2'$ is the moduli space of curves of genus 2 by the Torelli theorem and $S_2 = \mathcal{M}_2$.

Recall that a $\pi_1(B, b)$-equivariant map $f : (\tilde{B}, \tilde{b}) \to (\mathfrak{S}_g^\circ, *)$ induces the homomorphism of groups $f_* : \pi_1(B, b) \to S_g$ such that $f_*([c]) = [f \circ c]$ for $[c] \in \pi_1(B, b)$.

Proposition 5.3. *Let (B, b) be a compact orienrted surface with base point and with non empty boundary. Then the map*

$$[B, \Gamma_g \backslash \mathfrak{S}_g']^{orb} \ni [f] \mapsto f_* \in \mathrm{Hom}(\pi_1(B, b), S_g).$$

is a bijection.

Proof. It is known that B is homotopy equivalent to an n-bouquet $\vee_{k=1}^n S_k^1$ for some n and the fundamental group $\pi_1(B, b) \cong \pi_1(\vee_{k=1}^n S_k^1, o)$ is isomorphic to the free group of rank n. Hence we get

$$[B, \Gamma_g \backslash \mathfrak{S}_g']^{orb} \simeq [\vee_{k=1}^n S_k^1, \Gamma_g \backslash \mathfrak{S}_g']^{orb} \simeq \mathrm{Hom}(\pi_1(\vee_{k=1}^n S_k^1, o), S_g) \simeq \mathrm{Hom}(\pi_1(B, b), S_g).$$

which completes the proof. □

In the rest of this section we assume that $B = S^2 - \amalg_{k=1}^3 D_k$, where D_1, D_2, D_3 are mutually disjoint open discs. Since B is homotopy equivalent to a 2-bouquet $\pi_1(B, b)$ is the free group of rank 2. Let g_1, g_2 be generators of $\pi_1(B, b)$ represented by the loops which are mutually homotopy equivalent to ∂D_1, ∂D_2. By Proposition 5.3 we have a bijection

(6) $$[B, \Gamma_g \backslash \mathfrak{S}_g']^{orb} \simeq S_g \times S_g,$$

which is given by $[f] \mapsto (f_*(g_1), f_*(g_2)) \in S_g \times S_g$ for $[f] \in [B, \Gamma_g \backslash \mathfrak{S}_g']^{orb}$.

For $[f] \in [B, \Gamma_g \backslash \mathfrak{S}_g']^{orb}$ the fiber product $\pi : \widetilde{B} \times_f \Theta \to \widetilde{B}$ is a $\pi_1(B, b)$-equivariant fiber bundle because $f : \widetilde{B} \to \mathfrak{S}_g'$ is a $\pi_1(B, b)$-equivariant map. We get the fiber bundle $\pi : (\widetilde{B} \times_f \Theta)/\pi_1(B, b) \to B$, which is uniquely determined by $[f] \in [B, \Gamma_g \backslash \mathfrak{S}_g']^{orb}$ up to an isomorphism and whichi is $2g$-dimensional compact oriented manifold with boundary. For $(\sigma_1, \sigma_2) \in S_g \times S_g$, Let $\pi : X(\sigma_1, \sigma_2) \to B$ denote the corresponding fiber bundle under the isomorphism (6).

Definition 5.4. Define the map $c_{2g} : S_{2g} \times S_{2g} \to \mathbb{Z}$ by

$$c_{2g}(\sigma_1, \sigma_2) := \mathrm{Sign}(X(\sigma_1, \sigma_2)).$$

We call c_{2g} the *signature cocycle for smooth theta divisors.*

Remark 5.5. We only consider the case of an even genus because in the case of an odd genus $\mathrm{Sign}(X(\sigma_1, \sigma_2))$ always vanishes.

Lemma 5.6. *The followig relation holds:*

$$c_{2g}(\sigma_1, \sigma_2) + c_{2g}(\sigma_1\sigma_2, \sigma_3) = c_{2g}(\sigma_2, \sigma_3) + c_{2g}(\sigma_2\sigma_3, \sigma_1),$$

for any $\sigma_1, \sigma_2, C \in S_{2g}$. In particular, c_{2g} is a 2-cocycle of the group S_{2g} ([11]).

Proof. By the same argument in [1], we obtain the assertion. □

Let $[c_{2g}] \in H^2(S_{2g}, \mathbb{Z})$ be the cohomology class of c_{2g}. When $g = 1$, c_2 is the Meyer cocycle.

6. Construction of the Meyer function

Let $\sigma = [(\alpha,\gamma)]$ be an element of S_{2g}, where $\alpha : \mathbb{R}\to\mathfrak{S}'_{2g}$ and $\gamma\in\Gamma_{2g}$. Let $\mathbb{R}\times_\alpha\Theta'$ be the fiber product, which has a natural $\pi_1(S^1)$-action. We define the *mapping torus M_σ for σ* by

$$\pi : M_\sigma := (\mathbb{R}\times_\alpha\Theta')/\pi_1(S^1)\to S^1.$$

Since the metric $g^{\Theta'/\mathfrak{S}'_{2g}}$ on $T(\Theta'/\mathfrak{S}'_g)$ and the connection $P_{\Theta'}$ on Θ' are Γ_{2g}-invariant and the map $p : \widetilde{S^1} = \mathbb{R}\to\mathfrak{S}'_{2g}$ is $\pi_1(S^1)$-equivariant, the mtric g^{M_σ/S^1} on $T(M_\sigma/S^1)$ and the conection on P_σ on M_σ are naturally induced via the map p. Using the connection P_σ we define the 1-parameter family of Riemannian metrics $\{g^{M_\sigma,\varepsilon}\}_{\varepsilon>0}$ on M_σ by

$$g^{M_\sigma,\varepsilon} := g^{M_\sigma/S^1}\oplus\varepsilon^{-1}\pi^*dt^2, \quad \varepsilon\in\mathbb{R}_{>0}.$$

Here we regard S^1 as $\mathbb{R}/\mathbb{Z}$ and $t\in\mathbb{R}$ as a coordinate of S^1. By the theorem 3.3, the adiabtic limit

$$\eta^0(M_\sigma, g^{M_\sigma,\varepsilon}) := \lim_{\varepsilon\to 0}\eta(M_\sigma, g^{M_\sigma,\varepsilon})$$

exists. Recall that the Siegel modular form $\Delta_{2g}(\tau)$ with zero divisors $\mathcal{N}_{2g}$ (see Section 3.3.). Since the 1-form $d^c\log\|\Delta_{2g}(\tau)\|^2$ is Γ_{2g}-invariant the pull-back $p^*d^c\log\|\Delta_{2g}(\tau)\|^2$ can be regarded as a 1-form on S^1.

Definition 6.1. For $\sigma\in S_{2g}$ we fix (p,γ) which represents $\sigma = [(p,\gamma)]$, where $\gamma\in\Gamma_{2g}$ and $p : \mathbb{R}\to\mathfrak{S}'_g$. we set

$$\Phi_{2g}(p,\gamma) := \eta^0(M_\sigma, g^{M_\sigma}) + \frac{(-1)^g 2^{2g+3}(2^{2g+2}-1)B_{g+1}}{(2g+3)!}\int_{S^1} p^*d^c\log\|\Delta_{2g}(\tau)\|^2.$$

The following theorem is the main result of this paper.

Theorem 6.2. *(a) The value $\Phi_{2g}(p,\gamma)$ is independent of a choice of (p,γ) which represents $\sigma\in S_{2g}$. In particular Φ_{2g} is a function on S_{2g}.*
(b) The cocycle $-c_{2g}$ is the coboundary of the function Φ_{2g}. In particular $[c_{2g}]\otimes\mathbb{Q} = 0 \in H^2(S_{2g},\mathbb{Z})$.

As a corollary of the Theorem 6.2, it follows that $\phi_2 = \Phi_2$ by the uniqueness of Meyer's function of genus 2. On the other hand, $\Delta_2(\tau)$ coincides with the Igusa's modular form $\chi_2(\tau)$ ([25]), which is the product of all even theta constans.Then we can derive the following formula:

Corollary 6.3 ([15]). *Let $\sigma = [(p,\gamma)]$ be an element of $S_2 = \mathcal{M}_2$ as before. Then we have*

$$\phi_2(\sigma) = \eta^0(M_\sigma, g^{M_\sigma,\varepsilon}) - \frac{2}{15}\int_{S^1} p^*d^c\log\|\chi_2(\tau)\|^2.$$

Proof of Theorem 6.2. (a) Assume that (p_0,γ) and (p_1,γ) represents the same element $\sigma\in S_{2g}$. Put $I := [0,1]$. There is a map

$$\tilde{p} : I\times\mathbb{R}\to\mathfrak{S}'_{2g}$$

which satisfies $\tilde{p}(s,0) = *$ for $s\in I$ and $\tilde{p}(s,t) = \gamma\cdot\tilde{p}(s,t+1)$ for $(s,t)\in I\times\mathbb{R}$ and the following condition

$$\tilde{p}(s,t) = p_0(t), \quad s\in[0,\frac{1}{3}) \quad \text{and} \quad \tilde{p}(s,t) = p_1(t), \quad s\in(\frac{2}{3},1]. \tag{7}$$

Since $\tilde{p}$ is $\pi_1(I\times\mathbb{R})$-equivariant, the fiber product $(I\times\mathbb{R})\times_{\tilde{p}}\Theta'$ has the $\pi_1(I\times S^1)$-action and the quotien space

$$\bar{\pi} : \bar{M}_\sigma := (I\times\mathbb{R})\times_{\tilde{p}}\Theta'/\pi_1(I\times S^1)\to I\times S^1$$

has the induced metric $g^{\bar{M}_\sigma/I\times S^1}$ on $T(\bar{M}_\sigma/I\times S^1)$ from the metric $g^{\Theta'/\mathfrak{S}'_g}$ and the connection $\bar{P}_\sigma$ on $\bar{M}_\sigma$ from the connection $P_{\Theta'}$ mutually via the map p. Using the connection $\bar{P}_\sigma$ we set

$$g^{\bar{M}_\sigma,\varepsilon} := g^{\bar{M}_\sigma/I\times S^1}\oplus\varepsilon^{-1}\pi^*(ds^2\oplus dt^2), \quad \varepsilon\in\mathbb{R}_{>0}.$$

Let $g_i^{M_\sigma,\varepsilon}$ be the metrics on M_{σ_i} induced from the map p_i for $i=0,1$ as above. The condition (7) implies that

$$g^{\bar{M}_\sigma,\varepsilon}|_{[0,\frac{1}{3})\times S^1} = g_0^{M_\sigma,\varepsilon}\oplus\varepsilon^{-1}dt^2, \quad g^{\bar{M}_\sigma,\varepsilon}|_{(\frac{2}{3},1]\times S^1} = g_1^{M_\sigma,\varepsilon}\oplus\varepsilon^{-1}dt^2.$$

Then we can apply the Atiyah-Patodi Singer's index theorem to $(\bar{M}_\sigma, g^{\bar{M}_\sigma,\varepsilon})$:

$$\text{(8)} \qquad \mathrm{Sign}(\bar{M}_\sigma) = \int_{I\times S^1} \bar{\pi}_* L(T\bar{M}_\sigma, g^{\bar{M}_\sigma,\varepsilon}) - (\eta(M_\sigma, g_0^{M_\sigma,\varepsilon}) - \eta(M_\sigma, g_1^{M_\sigma,\varepsilon})).$$

Since $\bar{M}_\sigma$ is isomorphic to the product $M_\sigma\times I$, we have (see [3]),

$$\text{(9)} \qquad \mathrm{Sign}(\bar{M}_\sigma) = \mathrm{Sign}(M_\sigma)\times\mathrm{Sign}(I) = 0.$$

By Proposition 2.4 and the Proposition 2.5, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{I\times S^1} \bar{\pi}_* L(T\bar{M}_\sigma, g^{\bar{M}_\sigma,\varepsilon}) &= \int_{I\times S^1} \bar{\pi}_*\Big(L(T(\bar{M}_\sigma/(I\times S^1)))\cdot\bar{\pi}^* L(T(I\times S^1))\Big) \\
&= \int_{I\times S^1} \Big(\bar{\pi}_* L(T(\bar{M}_\sigma/(I\times S^1)), \nabla^{\bar{M}_\sigma/(I\times S^1)})\Big)^{(2)} \\
&= \int_{I\times S^1} \Big[\bar{\pi}_*\bar{p}^* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2})\Big]^{(2)} \\
\text{(10)} \qquad &= \int_{I\times S^1} \bar{p}^*\Big[\bar{\pi}_* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2})\Big]^{(2)}
\end{aligned}$$

where $\nabla^{\bar{M}_\sigma/(S^1\times I)}$ is the connection on the relative tangent bundle $T(\bar{M}_\sigma/(S^1\times I))$ associated with $g^{\bar{M}_\sigma/(S^1\times I)}$ and $\bar{P}_\sigma$ and we used the commutativity of fiber integrals and base changes in the last equality. By the Proposition 4.7, we have

$$\begin{aligned}
\text{(11)} \qquad & \int_{I\times S^1} \bar{p}^*\Big[\bar{\pi}_* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2})\Big]^{(2)} \\
= & -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!}\int_{I\times S^1} \bar{p}^* dd^c\log\|\Delta_{2g}(\tau)\|^2 \\
= & -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!}\Big(\int_{\{1\}\times S^1} p_1^* d^c\log\|\Delta_{2g}(\tau)\|^2 - \int_{\{0\}\times S^1} p_0^* d^c\log\|\Delta_{2g}(\tau)\|^2\Big),
\end{aligned}$$

where we used the Γ_{2g}-invariance of the 1-form $d^c\log\|\Delta_{2g}(\tau)\|^2$) in the last equality. By (25) $\sim$ (12) and the Definition 6.1, we obtain

$$0 = \Phi_{2g}(p_1,\gamma) - \Phi_{2g}(p_0,\gamma),$$

which completes the proof of (a).

(b) Let $\sigma_1 = [(p_1,\gamma_1)]$, $\sigma_2 = [(p_2,\gamma_2)]$, $\sigma_3 := (\sigma_1\sigma_2)^{-1} = (p_3, (\gamma_1\gamma_2)^{-1})\in S_{2g}$. Set $B := S^2 - \amalg_{k=1}^3 D_k$. Recall that the fiber bundle $\pi: X(\sigma_1,\sigma_2)\to B$ for σ_1,σ_2 defined at the Section 3.2. By the definition of Φ_{2g}, we have $\Phi_{2g}(\sigma^{-1}) = -\Phi_{2g}(\sigma)$ for any $\sigma\in S_{2g}$. Therefore to show that $-c_{2g}$ is the coboundary of Φ, we have to show that

$$\text{(12)} \qquad \mathrm{Sign}(X(\sigma_1,\sigma_2)) = -\sum_{i=1}^{3}\Phi_{2g}(\sigma_i)$$

Let U_i be the neighborhood of ∂D_i in B such that $U_i \cong [0,1) \times \partial D_i$. Let $\beta_i : \widetilde{U}_i \cong [0,1) \times \mathbb{R} \to \widetilde{B}$ be the lift of the map $U_i \hookrightarrow B$. Let $g_1, g_2 \in \pi_1(B,b)$ be the generators represented by the loops $\partial D_1, \partial D_2$. Let $[(p,\alpha)] \in [B, \Gamma_{2g} \backslash \mathfrak{S}'_{2g}]^{orb}$ be the corresponding element for $(\sigma_1,\sigma_2) \in S_{2g} \times S_{2g}$ under the isomorphism (6) where $\alpha : \pi_1(B,b) \to \Gamma_{2g}$ is a group homomorphism and $p : \tilde{B} \to \mathfrak{S}'_{2g}$ is a $\pi_1(B,b)$-equivariant homomorphism preserving the basepoint. Since ∂D_1, ∂D_2 and ∂D_3 are homotopy equivalent to the loops which represent g_1, g_2 and $(g_1 g_2)^{-1} \in \pi_1(B,b)$ we can assume that

$$p \circ \beta_i |_{\widetilde{U}_i}(s_i,t) = p_i(t), \quad (s_i,t) \in \widetilde{U}_i \cong [0,1) \times \mathbb{R}, \ i = 1 \sim 3. \tag{13}$$

Let $g^{X(\sigma_1,\sigma_2)/B}$ and $P_{X(\sigma_1,\sigma_2)}$ be the metric on $TX(\sigma_1,\sigma_2)$ and the connection on $X(\sigma_1,\sigma_2)$ induced from the metric $g^{\Theta'/\mathfrak{S}'_{2g}}$ and the connection $P_{\Theta'}$ via the map p. Let g^B be the metric on TB such that $g^B|_{U_i} = ds_i^2 \oplus dt^2$. Using the conection $P_{X(\sigma_1,\sigma_2)}$ we define the metric on $TX(\sigma_1,\sigma_2)$ by

$$g^{X(\sigma_1,\sigma_2),\varepsilon} := g^{X(\sigma_1,\sigma_2)/B} \oplus \varepsilon^{-1} \pi^* g^B, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Let $g^{M_{\sigma_i},\varepsilon}$ be the metric on M_{σ_i} induced from p_i for $i = 1 \sim 3$ as above. Let $\nabla^{X(\sigma_1,\sigma_2)/B}$ be the connection on $T(X(\sigma_1,\sigma_2))$ defined by the metric $g^{X(\sigma_1,\sigma_2)/B}$ and the connction $P_{X(\sigma_1,\sigma_2)}$. Since the condition (13) implies that the metric $g^{X(\sigma_1,\sigma_2),\varepsilon}$ is a product metric near the boundary of $X(\sigma_1,\sigma_2)$ we can apply the Atiyah-Patodi-Singer's index theorem to $(X(\sigma_1,\sigma_2), g^{X(\sigma_1,\sigma_2),\varepsilon})$:

$$\begin{aligned}
\operatorname{Sign}(X(\sigma_1,\sigma_2)) &= \int_{X(\sigma_1,\sigma_2)} L(TX(\sigma_1,\sigma_2), g^{X(\sigma_1,\sigma_2),\varepsilon}) - \sum_{i=1}^{3} \eta(M_{\sigma_i}, g^{M_{\sigma_i},\varepsilon}) \\
&= \int_B \pi_* L(T(X(\sigma_1,\sigma_2)/B), \nabla^{X(\sigma_1,\sigma_2)/B}) - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i},\varepsilon}) \\
&= \int_B p^* \left[f_* L(T(\Theta'/\mathfrak{S}'_{2g}), \nabla^{\Theta'/\mathfrak{S}'_{2g}}) \right]^{(2)} - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i},\varepsilon}) \\
&= -\sum_{i=1}^{3} \int_{\partial D_i} -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!} d^c \log \|\Delta_{2g}(\tau)\|^2 \\
&\quad - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}, g^{M_{\sigma_i},\varepsilon}) \\
&= -\sum_{i=1}^{3} \Phi_{2g}(\sigma_i)
\end{aligned}$$

which completes the proof of (b). □

7. The first cohomology of S_g

The uniqueness of a 1-cocycle that cobounds the 2-cocycle c_{2g} is equivalent to the vanishing of $H^1(S_{2g}, \mathbb{Z})$. In deed, if there is another 1-cocycle $\Phi'_{2g} : S_{2g} \to \mathbb{R}$ that cobounds c_{2g}, the difference $\Phi_{2g} - \Phi'_{2g}$ is an element of $\operatorname{Hom}(S_{2g}, \mathbb{R}) \cong H^1(S_{2g}, \mathbb{R})$. While $H^1(S_1, \mathbb{Z}) = H^1(S_2, \mathbb{Z}) = 0$, the uniqueness no longe valid for higher genus.

Theorem 7.1. *The following holds:*

$$H^1(S_g, \mathbb{Z}) = \begin{cases} 0 & 1 \leq g \leq 3, \\ \mathbb{Z} & g \geq 4. \end{cases}$$

In particular, the cochain cobounding the signature cocycle c_{2g} is not unique when $g \geq 2$.

By (5) and [11], we have the 5-term exact sequence

$$(14) \qquad 1 \to H^1(\Gamma_g, \mathbb{Z}) \to H^1(S_g, \mathbb{Z}) \to H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \xrightarrow{\delta} H^2(\Gamma_g, \mathbb{Z}) \to H^2(S_g, \mathbb{Z}).$$

We have $H^1(\Gamma_g, \mathbb{Z}) = 0$ for $g \geq 1$ and $H^2(\Gamma_g, \mathbb{Z}) = \mathbb{Z}$ for $g \geq 3$. By the Hurwitz theorem we see that

$$(15) \qquad H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \cong H^1(\mathfrak{S}'_g, \mathbb{Z}).$$

Lemma 7.2. *Let X be a connected complex manifold of $\dim_{\mathbb{C}} X \geq 2$. Assume that*

$$(16) \qquad H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0.$$

Let $D = \sum_{\lambda \in \Lambda} n_\lambda D_\lambda$ be a divisor on X such that $n_\lambda \neq 0$ and D_λ is irreducible for all $\lambda \in \Lambda$. Then

$$H^1(X - D, \mathbb{Z}) \cong \mathbb{Z}^\Lambda.$$

The generator of the cohomology $H^1(X - D, \mathbb{Z})$ corresponding to $\lambda \in \Lambda$ is represented by the map $l_\lambda \mapsto 1$ and $l_\mu \mapsto 0$ for $\mu \neq \lambda \in \Lambda$, where l_μ denotes the loop around a small disk and intersecting D_μ transversally.

Proof. Since the real codimension of $\mathrm{Sing}D$ in X is greater than or equal to 4, we have $\pi_k(X, X - \mathrm{Sing}D, *) = 0$ for $1 \leq k \leq 3$. The relative Hurwitz theorem asserts that $H_k(X, X - \mathrm{Sing}D, \mathbb{Z}) = 0$ for $k \leq 3$. Hence $H^k(X, X - \mathrm{Sing}D, \mathbb{Z}) = 0$ for $k \leq 3$, which together with the cohomology exact sequence for the triple $(X, X - \mathrm{Sing}D, X - D)$, yields that

$$(17) \qquad H^2(X, X - D, \mathbb{Z}) \cong H^2(X - \mathrm{Sing}D, X - D, \mathbb{Z}).$$

By the cohomology exact equence for the pair $(X, X - D)$ and (16), we obtaion

$$(18) \qquad H^1(X - D, \mathbb{Z}) \cong H^2(X, X - D, \mathbb{Z}).$$

Since $D - \mathrm{Sing}D$ is a closed submanifold in $X - \mathrm{Sing}D$ and $X - D = (X - \mathrm{Sing}D) - (D - \mathrm{Sing}D)$, the Thom isomorphism asserts that

$$(19) \qquad H^2(X - \mathrm{Sing}D, X - D, \mathbb{Z}) \cong H^0(D - \mathrm{Sing}D, \mathbb{Z}).$$

By the irreducibility of D_λ, $D_\lambda - \mathrm{Sing}D_\lambda$ is path connected so that

$$(20) \qquad H^0(D - \mathrm{Sing}D, \mathbb{Z}) \cong \mathbb{Z}^\Lambda.$$

The result follows from (17)$\sim$(20). □

Lemma 7.3. *The following holds:*

$$H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} = \begin{cases} \mathbb{Z} & 1 \leq g \leq 3 \\ \mathbb{Z}^{\oplus 2} & g \geq 4. \end{cases}$$

By regarding $H^1(\mathfrak{S}'_g, \mathbb{C})$ as the de Rham cohomology group, the image of the generators under the natural map $H^1(\mathfrak{S}'_g, \mathbb{Z}) \to H^1(\mathfrak{S}'_g, \mathbb{C})$ are represented by the 1-forms $\frac{1}{2\pi\sqrt{-1}} d\log \chi_g(\tau)$ and $\frac{1}{2\pi\sqrt{-1}} d\log J_g(\tau)$. Here $J_g(\tau) \equiv 1$ and hence $d\log J_g(\tau) = 0$ for $1 \leq g \leq 3$.

Proof. By Proposition 4.3, Proposition 4.4, the isomorphism (15) and Lemma 7.2, we get the assertion. □

Recall that the automorphic factor $j(\tau,\gamma)$ is a nowhere vanishing holomorphic function on $\mathfrak{S}_g$. Since $\mathfrak{S}_g$ is simmply connected, the logarithm of $j(\tau,\gamma)$ makes sence. Choose a branch of the logarithm of $j(\tau,\gamma)$ and denote it by $\log_\sigma j(\tau,\gamma)$ for $\gamma\in\Gamma_g$. Define the function $\lambda_\sigma:\Gamma_g\times\Gamma_g\to\mathbb{Z}$ by

$$(21)\quad \lambda_\sigma(A,B):=\frac{1}{2\pi\sqrt{-1}}\{\log_\sigma j(\tau,AB)-\log_\sigma j(B\cdot\tau,A)-\log_\sigma j(\tau,B)\},\qquad (A,B)\in\Gamma_g\times\Gamma_g.$$

Lemma 7.4. *The function λ_σ is a 2-cocycle of Γ_g, whose cohomology class generates $H^2(\Gamma_g,\mathbb{Z})$.*

Proof. For $g=1$ see [4]. When $g\geq 1$, we follow [4]. Let $G:=Sp(2g,\mathbb{R})$ be the symplectic group and let G^δ be the same group endowed with the discrete topology. Let $u\in H^2(G^\delta,\mathbb{Z})$ be the cohomology class corresponding to the universal covering

$$0\to\mathbb{Z}\to\widetilde{G}\to G\to 1.$$

We choose the branch $\log_\sigma j(\tau,\gamma)$ satisfying

$$(22)\qquad \operatorname{Im}\log_\sigma j(\sqrt{-1}\cdot 1_{2g},\gamma)\in[0,2\pi).$$

Since the function $\bar\lambda_\sigma$ is measurable, the cohomology class $[\bar\lambda_\sigma]$ is a constant multiple of u by [20]. Therefore it suffices to determine the restriction of the cohomology class $[\bar\lambda_\sigma]$ to the maximal compact subgroup of G. We shall identify the unitary group $U(g)$ with the maximal compact subgroup of G by the inclusion map defined as

$$\iota:U(g)\ni Z\longmapsto\begin{pmatrix}\operatorname{Re}Z & \operatorname{Im}Z\\ -\operatorname{Im}Z & \operatorname{Re}Z\end{pmatrix}\in G,\qquad Z\in U(g).$$

Since $j(\sqrt{-1}\cdot 1_{2g},\iota(Z))=\det(Z)^{-1}$ for $Z\in U(g)$ and the isotropy subgroup at $\sqrt{-1}\cdot 1_{2g}\in\mathfrak{S}_g$ is just $U(g)$, we have

$$(23)\qquad 2\pi\sqrt{-1}\bar\lambda_\sigma(Z_1,Z_2)=-\log_\sigma\det(Z_1Z_2)+\log_\sigma\det(Z_1)+\log_\sigma\det(Z_2)$$

for $(Z_1,Z_2)\in U(g)\times U(g)$. By (23), the restriction of the cohomology class $[\bar\lambda_\sigma]$ to $U(g)$ is the pull-back of the cohomology class corresponding to the universal covering $0\to\mathbb{Z}\to\widetilde{U(1)}\cong\mathbb{R}\to U(1)\to 1$, via the map $\det:U(g)\to U(1)$. Since the induced map $(\det)_*:\pi_1(U(g))\to\pi_1(U(1))$ is an isomorphism, we obtain $[\bar\lambda_\sigma]=u$. Since the cohomology class $[\bar\lambda_\sigma]$ is independent of the choice of the branch of $\log_\sigma j(\tau,\gamma)$ and since the restriction of u to Γ_g is the generator of the cohomology $H^2(\Gamma_g,\mathbb{Z})$ we obtain the assertion. □

Lemma 7.5. *Let $g\geq 2$. The map $\delta:H^1(\pi_1(\mathfrak{S}_g',*),\mathbb{Z})\to H^2(\Gamma_g,\mathbb{Z})$ is given by*

$$(m,n)\longmapsto(k_1(g)m+k_2(g)n)\in H^2(\Gamma_g,\mathbb{Z})\cong\mathbb{Z}$$

for $(m,n)\in H^1(\pi_1(\mathfrak{S}_g',*),\mathbb{Z})\cong\mathbb{Z}^{\oplus 2}$. Here,

$$k_1(g)=2^{g-2}(2^g+1),\quad k_2(g)=\frac{(g+3)\cdot g!}{4}-2^{g-3}(2^g+1)$$

are the weights of Siegel modular forms $\chi_g(\tau),J_g(\tau)$, respectively.

Proof. Let $\sigma:\Gamma_g\to S_g$ be a section, and write $\sigma(\gamma)=[(l_\gamma,\gamma)]\in S_g$ for $\gamma\in\Gamma_g$. We can assume that $l_{\gamma^{-1}}=-\gamma\cdot l_\gamma$, where $-l(t):=l(1-t)$, $t\in[0,1]$ for a path $l(t)$. Hence $\sigma(\gamma^{-1})=\sigma(\gamma)^{-1}$. Let α be an element of $H^1(\pi_1(\mathfrak{S}_g',*),\mathbb{Z})^{\Gamma_g}\cong\operatorname{Hom}(\pi_1(\mathfrak{S}_g',*),\mathbb{Z})^{\Gamma_g}$. Then $\delta(\alpha):\Gamma_g\times\Gamma_g\to\mathbb{Z}$ is given by

$$(A,B)\longmapsto\alpha(\sigma(A)\sigma(B)\sigma(AB)^{-1})\in\mathbb{Z},\qquad (A,B)\in\Gamma_g\times\Gamma_g,$$

where we identufy $\sigma(A)\sigma(B)\sigma(AB)^{-1}\in\operatorname{Im}\{\pi_1(\mathfrak{S}_g',*)\to S_g\}$ with the corresponding preimage of $\pi_1(\mathfrak{S}_g',*)$ under the inclusion $\pi_1(\mathfrak{S}_g',*)\hookrightarrow S_g$. Write $\sigma(A)\sigma(B)\sigma(AB)^{-1}=[(l_{(A,B)},1)]\in\pi_1(\mathfrak{S}_g',*)$.

Here $l_{(A,B)}$ is a loop on $\mathfrak{S}_g'$, which is the composition of the paths l_B, $B^{-1}{\cdot}l_A$ and $-l_{AB}$. Under the identification $H^1(\pi_1(\mathfrak{S}_g',*),\mathbb{Z})^{\Gamma_g}\cong\mathbb{Z}^{\oplus 2}$ given in Lemma 7.3, the cochain $\delta(m,n)$ is given by

$$\delta(m,n)(A,B)=\frac{1}{2\pi\sqrt{-1}}\int_{l_{(A,B)}}d\log\chi_g(\tau)^m J_g(\tau)^n\in\mathbb{Z},\qquad (A,B)\in\Gamma_g\times\Gamma_g,$$

for $(m,n)\in H^1(\pi_1(\mathfrak{S}_g',*),\mathbb{Z})^{\Gamma_g}\cong\mathbb{Z}^{\oplus 2}$. Using σ, we choose the branch $\log_\sigma j(\tau,\gamma)$ for $\gamma\in\Gamma_g$ such that

$$\log_\sigma j(*,\gamma):=\frac{1}{k_1(g)}\int_{l_{\gamma^{-1}}}d\log\chi_g(\tau).$$

Then we get

$$\begin{aligned}
2\pi\sqrt{-1}\delta(1,0)(A,B) &= \int_{l_{(A,B)}}d\log\chi_g(\tau)\\
&= \int_{AB\cdot l_{(A,B)}}d\log\chi_g(AB\cdot\tau)\\
&= \int_{AB\cdot l_{(A,B)}}[k_1(g)d\log j(\tau,AB)+d\log\chi_g(\tau)]\\
&= \int_{AB\cdot l_B}d\log\chi_g(\tau)+\int_{A\cdot l_A}d\log\chi_g(\tau)-\int_{AB\cdot l_{AB}}d\log\chi_g(\tau)\\
&= -\int_{l_{B^{-1}}}d\log\chi_g(A\cdot\tau)-\int_{l_{A^{-1}}}d\log\chi_g(\tau)+\int_{l_{(AB)^{-1}}}d\log\chi_g(\tau)\\
&= -\int_{l_{B^{-1}}}[k_1(g)d\log j(\tau,A)+d\log\chi_g(\tau)]\\
&\quad -k_1(g)\log_\sigma j(*,A)+k_1(g)\log_\sigma j(*,AB)\\
&= k_1(g)\bigl[-\log_\sigma j(B\cdot *,A)+\log_\sigma j(*,A)-\log_\sigma j(*,B)\\
&\quad -\log_\sigma j(*,A)+\log_\sigma j(*,AB)\bigr]\\
&= k_1(g)\bigl[\log_\sigma j(*,AB)-\log_\sigma j(B\cdot *,A)-\log_\sigma j(*,B)\bigr].
\end{aligned}$$

By Lemma 7.4 we get $\delta(1,0)=k_1(g)\in H^2(\Gamma_g,\mathbb{Z})\cong\mathbb{Z}$. Similarly, $\delta(0,1)=k_2(g)\in H^2(\Gamma_g,\mathbb{Z})\cong\mathbb{Z}$. This completes the proof. □

Proof of Theorem 7.1. Since $H^1(\Gamma_g,\mathbb{Z})$ in the exact sequence (5), we get $H^1(S_g,\mathbb{Z})=\ker\delta$. By Lemma 7.5, we get $\ker\delta=0$ for $1\le g\le 3$ and $\ker\delta\cong\mathbb{Z}$ for $g\ge 4$. This completes the proof of Theorem 7.1. □

8. The value for the Dehn twist

In this section, we shall compute the value of Φ_{2g} for the *Dehn twist*, which is defined as follows (cf. [16]). Let $\Delta\subset\mathbb{C}$ be the unit disk. Recall that the Andreotti-Mayer locus $\mathcal{N}_{2g}$ has two irreducible components $\theta_{null,2g}$ and $\mathcal{N}_{2g}'$ by Theorem 4.3. Let $\rho:\Delta\to\mathfrak{S}_{2g}$ be a C^∞-map such that $\rho(0)\in\theta_{null,2g}$ is a generic point, $\rho(z)\notin\mathcal{N}_{2g}$ for $z\in\Delta\backslash\{0\}$ and $\rho(\Delta)$ intersects with $\theta_{null,2g}$ at $\rho(0)$ transversally. For simplicity we assume that the base point $*$ lies in $\rho(\partial\Delta)$ and we denote the monodromy corresponding to the loop $\rho|_{\partial\Delta}:\partial\Delta\to\mathfrak{S}_{2g}'$ by $\sigma_{2g}\in S_{2g}$. The element σ_{2g} is called the Dehn twist. We put

$$\varpi: X_{2g}:=\Delta\times_\rho\Theta\longrightarrow\Delta,$$

which is smooth family of theta divisors over Δ induced from the universal family $\pi:\Theta\to\mathfrak{S}_{2g}$ by ρ. Let $\tilde\rho: X_{2g}\to\Theta$ be the lift of the map ρ defined as the projection to the second factor. By the assumption of ρ and the Theorem 4.3, $\mathrm{Sing}(\varpi^{-1}(0))$ consists of one ordinary double point and $\varpi^{-1}(z)$ is a smooth theta divisor for $z\in\Delta\backslash\{0\}$. Notice that ∂X_{2g} endowed with the the orientation induced from X_{2g} is diffeomorphic to the mapping torus $M_{\sigma_{2g}^{-1}}$ endowed with the natural orientation, i.e., $\partial X_{2g} = -M_{\sigma_{2g}}$.

Theorem 8.1. *The following equality holds:*

$$\Phi_{2g}(\sigma_{2g}) = \begin{cases} -\frac{4}{5} & \text{if } \ g=1, \\ (-1)^{g+1}\frac{(2g+1)2^{2g+2}(2^{2g+2}-1)}{(2g+3)!}B_{g+1} & \text{if } \ g>1. \end{cases}$$

Proof. Put $\Delta_r := \{z\in\Delta \mid |z| < r\}\subset\Delta$ for $0 < r < 1$. We choose ρ such that the restriction $\rho|_{\Delta_{1/3}} : \Delta_{1/3}\to\rho(\Delta_{1/3})\subset\mathfrak{S}_{2g}$ is a holomorphic embedding that

$$\rho(re^{\sqrt{-1}\theta}) = \rho\left(\frac{2}{3}e^{\sqrt{-1}\theta}\right), \quad \frac{2}{3} < r\leq 1, \quad 0\leq\theta < 2\pi. \tag{24}$$

Let g^{Δ} be the metric on $T\Delta$ which is a product metric near the bondary $\partial\Delta$ and coincides with the metric $\rho^{*}g^{\mathfrak{S}_g}$ on $\Delta_{1/3}$. Let $p\in X_{2g}$ be the unique singular point on the singular fiber X_0. Let $g^{X_{2g}/\Delta}$ be the metric on $T(X_{2g}/\Delta)|_{X_{2g}-\{p\}}$ induced from the metric $g^{\Theta/\mathfrak{S}_g}$ via the map ρ. Let $g^{X_{2g}}$ be the metric on TX_{2g} which coincides with $g^{X_{2g}/\Delta}\oplus\varpi^{*}g^{\Delta}$, where we used the connection induced from the connection $P_{\Theta'}$ on Θ' via the map ρ, on $X_{2g}-\{p\}$ and coincides with the metric induced from the metric $g^{\Theta'}$ via the map $\tilde{\rho}$ on a neighbourhood of p. Set

$$g^{X_{2g},\varepsilon} := g^{X_{2g}}\oplus\varepsilon^{-1}\varpi^{*}g^{\Delta}, \quad \varepsilon\in\mathbb{R}_{>0}.$$

By the assumption of g^{Δ} and the condition (24), $g^{X_{2g},\varepsilon}$ is the product metric near the boundary ∂X_{2g} for $\varepsilon\in\mathbb{R}_{>0}$. By the Atiyah-Patodi-Singer index theorem,

$$\mathrm{Sign}(X_{2g}) = \int_{X_{2g}} L(TX_{2g}, g^{X_{2g},\varepsilon}) + \eta(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}). \tag{25}$$

Here ∂X_{2g} is identified with $-M_{\sigma_{2g}}$, and $g^{M_{\sigma_{2g}},\varepsilon}$ is the restriction of $g^{X_{2g},\varepsilon}$ to the boundary $\partial X_{2g}\cong -M_{\sigma_{2g}}$. By the formula in [26], the first term of the right-hand side of (25):

$$\lim_{\varepsilon\to 0} L(TX_{2g}, g^{X_{2g},\varepsilon}) = L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta}) + P(-t,\cdots,(-t)^{2g})|_{t^{2g}}\cdot\mu(p)\delta_p \tag{26}$$

Here $L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta})$ is only defined on $X_{2g}-\{p\}$ but has the natural smooth extension on whole X_{2g}. The constant $\mu(p)$ is the Milnor number of the singular point p, δ_p is the Dirac delta current supported at p and $P(x_1,\cdots,x_{2g})\in\mathbb{C}[[x_1,\cdots,x_{2g}]]$ is defined by

$$\prod_{k=1}^{2g} L(x_k) = P(\sigma_1,\cdots,\sigma_{2g}),$$

where $L(x) = x/\tanh(x)$ and $\sigma_1 = \sum_k x_k, \sigma_2 = \sum_{i>j} x_i x_j, \cdots, \sigma_{2g} = \prod_k x_k$ are the fundamental symmetric polynomials. Notice that

$$P(-t,\cdots,(-t)^{2g})|_{t^{2g}} = L^{-1}(t)|_{t^{2g}}.$$

Since p is a non-degenerate critical point of $\pi : X\to\Delta$, we get $\mu(p) = 1$, which together with (25), (26) and Theorem 4.7, yields that

$$\begin{aligned} \mathrm{Sign}(X_{2g}) = & \frac{(-1)^{g}2^{2g+1}(2^{2g+2}-1)}{(g+1)(2g+1)}B_{g+1}\int_{\Delta}\rho^{*}dd^{c}\mathrm{logdetIm}\tau \\ & + \frac{(-1)^{g}2^{2g+2}(2^{2g+2}-1)}{(2g+2)!}B_{g+1} + \eta^{0}(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}). \end{aligned} \tag{27}$$

By (27) and Definition 6.1, we get

$$\begin{aligned}\Phi_{2g}(\sigma_{2g}) &= \eta^0(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}))\\ &+ \frac{(-1)^g 2^{2g+3}(2^{2g+2}-1)}{(2g+3)!}B_{g+1}\int_{\partial\Delta} p^* d^c\left(\log|\Delta_{2g}(\tau)|^2(\det\mathrm{Im}\tau)^{\frac{(2g+3)\cdot(2g)!}{2}}\right)\\ &= -\frac{(-1)^g 2^{2g+2}(2^{2g+2}-1)}{(2g+2)!}B_{g+1} + \mathrm{Sign}(X_{2g})\\ &+ \frac{(-1)^g 2^{2g+3}(2^{2g+2}-1)}{(2g+3)!}B_{g+1}\int_{\Delta} p^* dd^c\log|\Delta_{2g}(\tau)|^2\\ &= \frac{(-1)^{g+1}(2g+1)2^{2g+2}(2^{2g+2}-1)}{(2g+3)!}B_{g+1} + \mathrm{Sign}(X_{2g}),\end{aligned}$$

where we used the Poincaré-Lelong formula and Theorem 4.4 to get the last equality. When $g=1$, since the singular fiber has two irreducible components and $\mathrm{Sign}(X_2) = -1$, we obtain the proof for the case $g=1$. We complete the proof by the following Lemma in the case $g>1$. □

Lemma 8.2. *Let $\pi : \mathfrak{X} \to \Delta$ be a Lefschetz degeneration of relative dimension $2n-1$, i.e., π is a proper holomorphic surjective map from a $2n$-dimensional complex manifold $\mathfrak{X}$ to the unit disk Δ and there is a point $p \in \mathfrak{X}_0$ and an open neighbourhood $p \in U \cong \{(z_1,\cdots,z_{2n}) \in \mathbb{C}^{2n} \mid \sum_{k=1}^{2n}|z_k|^2 < 1\}$ such that*

$$\pi(z_1,\cdots,z_{2n}) = \sum_{k=1}^{2n} z_k^2, \qquad (z_1,\cdots,z_{2n}) \in U$$

and π_* has maximal rank on $\mathfrak{X}\setminus p$. Assume that $n>1$. Then $\mathrm{Sign}(\mathfrak{X}) = 0$.

Proof. For $\in \Delta$, we set $U_t := \mathfrak{X}_t \cap U$. Then a sequence of inclusions

$$\mathfrak{X}_0 \setminus U_0 \subset \mathfrak{X}_0 \setminus \{p\} \subset \mathfrak{X}_0 \subset \mathfrak{X}$$

induces a sequence of isomorphisms:

$$H_{2n}(\mathfrak{X}_0\setminus U_0,\mathbb{Z}) \cong H_{2n}(\mathfrak{X}_0\setminus\{p\},\mathbb{Z}) \cong H_{2n}(\mathfrak{X}_0,\mathbb{Z}) \cong H_{2n}(\mathfrak{X},\mathbb{Z}). \tag{28}$$

Here the first isomorphism follows from the homotopy equivalence of $\mathfrak{X}_0\setminus U_0$ and $\mathfrak{X}_0\setminus\{p\}$, the second isomorphism follows from the fact $\mathrm{codim}_{\mathbb{R}}\{p\}/\mathfrak{X}_0 = 4n-2 > 2n+1$, and the third isomorphism follows from the fact that the inclusion $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$ is a deformation retraction. By Ehresman's Theorem, $\mathfrak{X}\setminus U$ is diffeomorphic to $(\mathfrak{X}_0\setminus U_0)\times\Delta$ as a fiber bundle over Δ. Since Δ is contractible, the inclusion $\mathfrak{X}_t\setminus U_t \hookrightarrow \mathfrak{X}\setminus U$ induces an isomorphism $H_{2n}(\mathfrak{X}_t\setminus U_t,\mathbb{Z}) \cong H_{2n}(\mathfrak{X}\setminus U,\mathbb{Z})$. By (28), the inclusion $\mathfrak{X}_t\setminus U_t \hookrightarrow \mathfrak{X}$ induces an isomorphism $H_{2n}(\mathfrak{X}_t\setminus U_t,\mathbb{Z}) \to H_{2n}(\mathfrak{X},\mathbb{Z})$. Hence, for any $t\in\Delta$, any element of $H_{2n}(\mathfrak{X},\mathbb{Z})$ can be represented by a cycle contained in $\mathfrak{X}_t$. Therefore the intersection matrix of $H_{2n}(\mathfrak{X},\mathbb{Z})$ is trivial and $\mathrm{Sign}(\mathfrak{X}) = 0$. This completes the proof. □

Remark 8.3. When $g=1$, $\sigma_2\in\mathcal{M}_2$ is the Dehn twist along a separating simple closed curve on a Riemann suface of genus two. Since $\mathrm{Sign}(X_2) = -1$ and $B_2 = \frac{1}{30}$, we obtain $\phi_2(\sigma_2) = \Phi_2(\sigma_2) = -\frac{4}{5}$, which confirms a result of Matsumoto ([19]).

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