数理解析研究所講究録
1490 巻 2006 年 11-25

Classification of projective manifolds containing four-sheeted covers of projective space as very ample divisors

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ABSTRACT

The aim of this talk is to report a classification result of complex projective manifolds which contain finite covers of $\mathbf{P}^n$ of degree $d = 4$ as their very ample divisors. In the case where d is a small prime number, several classification results have been obtained by Lanteri-Palleschi-Sommese ($d = 2, 3$) and the speaker ($d = 5$).

This report is a exposition of the speaker's paper [2]. We first introduce our problem and background. Secondly, we talk about new problems arising in the case where d is a composite number, and state our result. Finally, we illustrate several key points of our proof.

1 Introduction

1.1 A problem in geometry of hyperplane sections

We consider a pair (X, L) consisting of a smooth projective variety X of dimension $n+1$ and a very ample line bundle L on it. In what follows, we assume that the ground field is the field of complex numbers $\mathbf{C}$.

Geometry of hyperplane sections has attracted several authors. In the late of 19-th century, in the course of studies of projective surfaces, G. Castelnuovo [5] classified the pairs (X, L) in the case where

$$n = 1 \text{ and } |L| \text{ contains a smooth hyperelliptic curve,}$$

which is a double cover of $\mathbf{P}^1$.

It is known that topological nature of X is intensively imposed by that of a member of $|L|$. In fact, A. J. Sommese expressed a philosophy of geometry of hyperplane sections as follows:

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A projective manifold is at least as special as any of its ample divisors (see [17, Introduction]).

Additionally, the revisions of the classification result made in 1987 ([16], [18]) called new attention to the following generalized problem.

Problem 1.1 ([12, §1]) *Fix an integer $d \geq 2$. Classify the pairs (X, L) with the following condition.*

$(*)_d$ *There exists a smooth member $A \in |L|$ endowed with a finite morphism $\pi: A \to \mathbf{P}^n$ of degree d.* □

From now on, we study this classification problem.

1.2 The settled cases

We begin with "obvious" examples of (X, L) with the condition $(*)_d$.

Examples 1.2 (Obvious pairs) $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d))$ and $(H_d^{n+1}, \mathcal{O}_{H_d^{n+1}}(1))$, where $H_d^{n+1} \subset \mathbf{P}^{n+2}$ is a smooth hypersurface of degree d, and π is a projection from a point.

We are interested in what kind of "non-obvious" pairs show up. In the cases where the covering degrees d are small prime numbers, several authors have studied the classification problems of (X, L) with $(*)_d$:

- For the case of $(n, d) = (1, 2)$, F. Serrano [16], A. J. Sommese-A. Van de Ven [18] classified the pairs (X, L) with $(*)_d$, completely. These are the revisions of Castelnuovo's classification result as mentioned above.
- For the case of $(n, d) = (1, 3)$, M. L. Fania [6] studied the structure of the pairs (X, L).
- As to the cases where (i) $(n \geq 2, d = 2)$ and (ii) $(n \geq 4, d = 3)$, A. Lanteri-M. Palleschi-A. J. Sommese (L-P-S for short) classified the pairs (X, L) in [12] and [13].
- For the case of $(n \geq 6, d = 5)$, Y. Amitani gave a complete classification of (X, L) in [1].

We wish to solve the problem for any n and d. But it seems to be difficult. So, **in what follows, we assume that $n > d$.** Under this assumption, a Barth-type theorem for branched coverings of $\mathbf{P}^n$ by R. Lazarsfeld [14, Theorem 1] asserts that

$$\mathrm{Pic}(A) \cong \mathrm{Pic}(\mathbf{P}^n) \cong \mathbf{Z} \text{ and } h^1(\mathscr{O}_A) = 0.$$

Note that $\pi^* \mathscr{O}_{\mathbf{P}^n}(1)$ is ample due to the finiteness of π. Taking the self-intersection number of the line bundle, we see that it is the ample generator of $\mathrm{Pic}(A)$. And, by the Lefschetz hyperplane section theorem, we have

$$\mathrm{Pic}(X) \cong \mathrm{Pic}(A) = \mathbf{Z}[\pi^* \mathscr{O}_{\mathbf{P}^n}(1)] \text{ and } h^1(\mathscr{O}_X) = 0.$$

Let $\mathscr{H} \in \mathrm{Pic}(X)$ be its ample generator. Then, by easy calculations, we also see that

$$\mathscr{H}_A = \pi^* \mathscr{O}_{\mathbf{P}^n}(1).$$

In the cases where $n > d = 2$ and 3, the classification results are quite simple.

Theorem 1.3 (Lanteri-Palleschi-Sommese) *Let X be a smooth projective variety with $\dim X = n + 1$ and L a line bundle on it. Suppose that $n > d$ for $d \in \{2, 3\}$. Then the following hold.*

(1) *There exists a very ample L with the condition $(*)_{d=2}$ if and only if (X, L) is an "obvious" pair.*

(2) *There exists a very ample L with $(*)_{d=3}$ if and only if (X, L) is an "obvious" pair or $(Y, 3\mathscr{L})$, where $(Y, \mathscr{L})$ is a Del Pezzo manifold of degree one.* □

For proofs, we refer to [12, (1.5)] and [13, (2.5)].

Definition and terminology We introduce the definitions of polarized manifolds and their important invariants.

- A *polarized manifold* is a pair (M, D) of a smooth projective variety M and an ample line bundle D on it.
- The *Δ-genus of* (M, D) is defined by $\Delta(M, D) := \dim M + D^{\dim M} - h^0(M, D)$, where we call the self-intersection number $D^{\dim M}$ the *degree* of a polarized manifold (M, D).
- The *sectional genus of* (M, D) is defined by

  $$g(M, D) := 1 + \frac{1}{2}(K_M + (\dim M - 1)D) \cdot D^{\dim M - 1},$$

  where K_M denotes the canonical bundle of M.
- A *Del Pezzo manifold* (M, D) of degree b is a polarized manifold of degree b satisfying one of the following equivalent conditions.

  (1) $\Delta(M, D) = g(M, D) = 1$; or

(2) $-K_M = (\dim M - 1)D$.

It is known that $1 \le b \le 8$. Del Pezzo manifolds are classified completely (cf. [9, (8.11)]).

- A *weighted projective space* $\mathbf{P}(e_0, \dots, e_N)$ is of the form $\mathrm{Proj}(\mathbf{C}[s_0, \dots, s_N])$, where each weight $\mathrm{wt}(s_i) = e_i$. In general, it is known that $\mathbf{P}(e_0, \dots, e_N)$ is irreducible, normal, Cohen-Macaulay, and has at most cyclic quotient singularities (see [3, Theorem 3A.1]). And $\mathcal{O}_{\mathbf{P}(e_0,\dots,e_N)}(1)$ may not be invertible in general (cf. [3, 3D 3]). If one sets $S = \bigcup_{1<k}(s_j = 0 \mid k \nmid e_j)$, then $\mathcal{O}_{\mathbf{P}(e_0,\dots,e_N)}(1)$ is always invertible on $\mathbf{P}(e_0, \dots, e_N) \setminus S$.
- A *weighted complete intersection* (w.c.i. for short) V of type $(a_1, \dots, a_c)$ in $\mathbf{P}(e_0, \dots, e_N)$ is of the form $V = V_+(F_1, \dots, F_c)$, where $(F_1, \dots, F_c)$ is a regular sequence of $\mathbf{C}[s_0, \dots, s_N]$ with $a_i = \deg F_i$ for each $1 \le i \le c$, and $V \cap S = \emptyset$. When $c = 1$, we call it a *weighted hypersurface* of degree a_1.

2 Result

2.1 Problems arising in the case where d is composite

In the small prime degree cases where $d = 2, 3$ and 5 ([12], [13] and [1], resp.), the following plays a key role in the classification problems although the proof is simple.

Key fact 2.1 *Let q be the morphism associated to $\pi^*\mathcal{O}_{\mathbf{P}^n}(1)$, and assume $t := h^0(A, \pi^*\mathcal{O}_{\mathbf{P}^n}(1)) - n - 1 > 0$. Then we have a factorization of π as follows:*

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[dr, "\pi"'] & q(A) \subset \mathbf{P}^{n+t} \arrow[d, "p"] \\ & \mathbf{P}^n, \end{tikzcd}$$

where p is a projection from a $\mathbf{P}^{t-1}$ in $\mathbf{P}^{n+t}$ with $q(A) \cap \mathbf{P}^{t-1} = \emptyset$. In particular, if d is a prime, then q is birational onto its image $q(A)$, which is a variety of degree d. □

Remark 2.2 In the cases where $d = 2, 3$ and 5, we can actually prove that $q(A)$ is isomorphic to A, hence it is smooth.

When d is a composite number, one can immediately obtain the following examples of (X, L).

Examples 2.3 If $d = \ell_1 \cdots \ell_e$, where each integer $\ell_i > 1$, then the following pairs satisfy $(*)_d$: $(H^{n+1}_{\ell_1,\ldots,\ell_e}, \mathscr{O}(1))$ and $(H^{n+1}_{\ell_1,\ldots,\ell_s}, \mathscr{O}(\ell_{s+1}\cdots\ell_e))$, where $H^{n+1}_{\ell_1,\ldots,\ell_s}$ is an $(n+1)$-dimensional complete intersection of type $(\ell_1,\ldots,\ell_s)$ in $\mathbf{P}^{n+1+s}$.

Here we mention new question and problem those we face in the cases where d is composite. First, note that there might exist a pair (X, L) with a non-birational morphism q. Studying these structures is quite significant for our original problem. If we take into consideration Remark 2.2, it is natural to ask the following.

Question 2.4 *Is the image $q(A)$ smooth for a non-birational morphism q?* □

Next, for a polarized manifold $(X, \mathscr{H})$ in question, we show the inequality

$$\Delta(X, \mathscr{H}) \le \mathscr{H}^{n+1}.$$

Indeed, from the definition of Δ-genus, we have

$$\Delta(A, \mathscr{H}_A) = n + d - h^0(A, \mathscr{H}_A).$$

And, since the Δ-genus of a polarized manifold is non-negative (see [9, Chapter I, (4.2)]), we obtain that

$$n + 1 \le h^0(A, \mathscr{H}_A) \le n + d.$$

By using the Kodaira vanishing theorem, we obtain the inequality

$$\Delta(X, \mathscr{H}) \le n + 1 + \mathscr{H}^{n+1} - h^0(\mathscr{H}_A) \le \mathscr{H}^{n+1}.$$

Now, reminding that $\mathrm{Pic}(X) = \mathbf{Z}[\mathscr{H}]$, we can write $L = \ell\mathscr{H}$ with some $\ell > 0$. Therefore the following seems to be crucial for a solution of Problem 1.1.

Problem 2.5 *Let (M, D) be an m-dimensional polarized manifold satisfying that $\Delta(M, D) \le D^m$, $\mathrm{Pic}(M) = \mathbf{Z}[D]$ and $h^1(\mathscr{O}_M) = 0$.*

(1) *If the line bundle ℓD is very ample for a fixed $\ell \ge 1$, then classify the polarized manifolds (M, D).*

(2) *For each $k \ge 1$, determine whether kD is very ample or not.* □

Studies in composite degree cases seem to be more difficult than those in prime degree cases. One of the reasons is as follows: Since

$$\ell\mathscr{H}^{n+1} = L \cdot \mathscr{H}^n = \mathscr{H}_A^n = d,$$

both ℓ and $\mathscr{H}^{n+1}$ divide the covering degree d. Hence we see that the the possibilities of Δ-genera in composite cases are more than those in prime cases by the above inequality. And it seems to come with some technical difficulties to determine the structures of polarized manifolds with large Δ-genera. Thus we realize that studies in composite cases are more complicated than those in prime cases.

2.2 Classification of (X, L) in the degree four case

As stated above, we know that specific problems arise in the case where d is a composite number. And now, in the degree $d = 4$ case, what kind of the pairs (X, L) show up? What can we say about Question 2.4 or Problem 2.5 in this case?

Our main result is a complete classification of (X, L) with $(*)_4$.

Theorem 2.6 [1]**([2, Theorem 1.1])** *Let X be a smooth projective variety with $\dim X = n + 1 > 5$. Then there exists a very ample line bundle L on X that satisfies the condition $(*)_4$ if and only if (X, L) is one of the following:*

(i) $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(4))$;

(ii) $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}^{n+1}}(2))$, *where $\mathbf{Q}^{n+1}$ is a smooth hyperquadric in $\mathbf{P}^{n+2}$*;

(iii) $(H_4^{n+1}, \mathcal{O}_{H_4^{n+1}}(1))$;

(iv) $(H_{2,2}^{n+1}, \mathcal{O}_{H_{2,2}^{n+1}}(1))$, *where $H_{2,2}^{n+1}$ is a smooth complete intersection of two hyperquadrics in $\mathbf{P}^{n+3}$*;

(v) $(Y, 4\mathscr{L})$, *where $(Y, \mathscr{L})$ is a Del Pezzo manifold of degree one*;

(vi) $(Z, 2\mathcal{L})$, *where $(Z, \mathcal{L})$ is a Del Pezzo manifold of degree 2; or*

(vii) $(W_{12}, \mathcal{O}_{W_{12}}(4))$, *where W_{12} is a smooth weighted hypersurface of degree 12 in the weighted projective space $\mathbf{P}(4, 3, 1^{n+1})$.* □

Remark 2.7 • The pairs (v)-(vii) show up newly. In particular, we see that (vi) is a unique polarized manifold with a non-birational morphism q. We deal with the structure of this pair in §2.3. And, for Question 2.4, it turns out that $q(A)$ is smooth in the degree 4 case.

[1] After the speaker has written up [2], he found that Lanteri ([11, Theorem 3.4]) had obtained a similar result, and had also proved Proposition 3.6 in this report. But the Lanteri's result contains one "doubtful" case. In fact, for the case (vii), it gives only some numerical invariants. In contrast, our classification result is perfect because it reveals the structure of a unique polarized manifold appearing in (vii).

- Our basic strategy is to reduce to Fujita's classification theory of polarized manifolds. However, one needs other techniques different from those in the theory in order to prove this theorem. In fact, we come across two possibilities of numerical invariants:

  (1) $g(X, \mathscr{H}) = 3$, $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 1$; and

  (2) $g(X, \mathscr{H}) = 3$, $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 2$.

  In general, polarized manifolds with these invariants are yet to be classified. We deal with these two possibilities in §3.2. Proposition 3.5 and 3.6 give an answer of Problem 2.5 in the degree 4 case.

2.3 The special example (vi)

Here we focus on the structure of the pair (vi) in Theorem 2.6. According to Fujita's classification of Del Pezzo manifolds of degree 2 ([9, (8.11)]), we see that $(Z, \mathcal{L})$ is a weighted hypersurface of degree 4 in $\mathbf{P}(2, 1^{n+2})$. Due to the smoothness of Z, its defining equation is given by the form of

$$x^2 + f(y_0, \ldots, y_{n+1}) = 0$$

with $(\mathrm{wt}(x), \mathrm{wt}(y_j)) = (2, 1)$ for each $0 \leq j \leq n+1$, where f is a homogenous polynomial of degree 4 in $\mathbf{C}[y_0, \ldots, y_{n+1}]$. Thus we can regard Z as a double covering of $\mathbf{P}^{n+1}$ branched along a quartic hypersurface defined by the equation $f(y_0, \ldots, y_{n+1}) = 0$.

Now, let q be the restriction of the morphism $\varphi_{\mathcal{L}}$ to a smooth member A of $|2\mathcal{L}|$. As a matter of fact, $2\mathcal{L}$ is very ample as we will see in §3.1. Therefore we obtain the following commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "\varphi_{\mathcal{L}}"] & \mathbf{P}^{n+1} \\
A \arrow[u, hook] \arrow[r, "q"] \arrow[dr, bend right, "\pi"'] & q(A) \arrow[u, hook] \arrow[d, "p"] \\
 & \mathbf{P}^n
\end{tikzcd}$$

and see that $\deg q = \deg q(A) = 2$. We see that $q(A)$ is smooth: Were $q(A)$ singular, then we would have $\dim q(A) \leq 3$ (see [7, (4.1)-(4.4)]). This contradicts our assumption $n > d = 4$.

3 Proof of Theorem 2.6

3.1 The key point in proof of the 'if' part

Proving the 'if' part is to show that each L of the pairs (i)-(vii) is very ample and that it satisfies the condition $(*)_4$. In the cases (i)-(iv), it is clear. In the case (v), it is already proved in [13, (1.2)]. In the case of (vi), we use Laface's theorem below.

Lemma 3.1 (Laface) *Let (M, D) be a polarized manifold. Suppose that $h^0(M, D) > 0$ and that the graded ring $R(M, D) := \bigoplus_{i \geq 0} H^0(M, iD)$ is generated in degrees $\leq k$. Then the map φ_{kD} associated to the linear system $|kD|$ is an embedding outside the base locus* $\mathrm{Bs}|D|$,

$$\varphi_{kD} \colon M \setminus \mathrm{Bs}|D| \hookrightarrow \mathbf{P}(|kD|).$$

In particular, if $|D|$ is free, then kD is very ample. □

For a proof of the lemma, refer to [10, Theorem 2.2]. In fact, since we see that $R(Z, \mathcal{L})$ is generated in degrees ≤ 2 and that $|\mathcal{L}|$ is free as observed in §2.3, the lemma implies that $2\mathcal{L}$ is very ample and that it satisfies $(*)_4$.

The most important thing is to consider the case of (vii). Specifically, we prove the following, which is the key.

Proposition 3.2 ([2, Lemma 2.1]) *Let W_{12} be a smooth weighted hypersurface of degree 12 in $\mathbf{P}(4, 3, 1^{n+1})$. Then $\mathcal{O}_{W_{12}}(4)$ is very ample.* □

We explain that $\mathcal{O}_{W_{12}}(4)$ fulfills the condition $(*)_4$ automatically. By easy calculations (use [15, Proposition 3.2 and 3.3]), we have

$$\Delta(W_{12}, \mathcal{O}_{W_{12}}(1)) = \mathcal{O}_{W_{12}}(1)^{n+1} = 1.$$

In general, as to a polarized manifold with these invariants, the following hold.

Fact 3.3 (Fujita [8, (13.1)]) *For an m-dimensional polarized manifold (M, D) of $\Delta(M, D) = D^m = 1$, the base locus $\mathrm{Bs}|D|$ consists of only one point, which we denote by p.* □

Lemma 3.4 *Let (M, D) be an m-dimensional polarized manifold of $\Delta(M, D) = D^m = 1$. Assume that kD is very ample. Then the polarized manifold (M, kD) satisfies the condition $(*)_k$.* □

For a proof, we refer to [1, Proposition 3.2]. Thus it suffices to prove Proposition 3.2.

Sketch of proof of Proposition 3.2. Since $R(W_{12}, \mathcal{O}_{W_{12}}(1))$ is generated in degrees ≤ 4, Lemma 3.1 and Fact 3.3 imply that $\varphi_{\mathcal{O}_{W_{12}}(4)}$ is an embedding outside p. So we have to verify that $\varphi_{\mathcal{O}_{W_{12}}(4)}$ gives an embedding at p. What we want to show is the following.

(a) $\mathrm{Bs}|\mathcal{O}_{W_{12}}(4)| = \emptyset$;

(b) The morphism $\varphi := \varphi_{\mathcal{O}_{W_{12}}(4)}$ associated to $\mathcal{O}_{W_{12}}(4)$ is injective;

(c) The linear system $|\mathcal{O}_{W_{12}}(4)|$ separates the tangent vectors.

We choose a coordinate system of $\mathbf{P}(4, 3, 1^{n+1})$ to verify that (a)-(c) hold. For details, refer to [2, Lemma 2.1].

(a) We see that $\mathrm{Bs}|\mathcal{O}_{W_{12}}(4)|$ is contained in the singular locus of $\mathbf{P}(4, 3, 1^{n+1})$. On the other hand, a weighted hypersurface does not meet the singular locus by its definition. Thus (a) holds.

(b) This holds because $\mathrm{Bs}|\mathcal{O}_{W_{12}}(1)|$ is single point.

(c) We can show this by using that general members of $|\mathcal{O}_{W_{12}}(1)|$ intersect at p transversally, which follows from $\mathcal{O}_{W_{12}}(1)^{n+1} = 1$. ■

3.2 Two key points in proof of the 'only if' part

Here we mention the key points in proof of the 'only if' part. We begin with an outline of the proof. By the arguments as in §2.1, we see that the possibilities of pairs $[\ell, \mathcal{H}^{n+1}]$ are as follows:

$$(\mathrm{p}_1)\ [1,4],\quad (\mathrm{p}_2)\ [2,2],\quad (\mathrm{p}_3)\ [4,1].$$

And we have the following table.

$h^0(\mathcal{H}_A)$	$\Delta(A, \mathcal{H}_A)$
$n+4$	0
$n+3$	1
$n+2$	2
$n+1$	3

Table 1.

We proceed with proof of the 'only if' part case by case. We apply Fujita's classification results of polarized manifolds of Δ-genera zero, one and two (see [9, Chapter I, (5.10), (8.11) and (10.8.1), resp.]), and investigate the types (p_1) –

(p_3). As a matter of fact, if $h^0(\mathscr{H}_A) = n+4$, then we can immediately show that there does not exist any pair $(A, \mathscr{H}_A)$ by taking the self-intersection number of $\mathscr{H}_A$.

If $h^0(\mathscr{H}_A) = n+3$, then we can prove that $g(A, \mathscr{H}_A) = 1$, hence $(A, \mathscr{H}_A)$ is a Del Pezzo manifold of degree 4. Thus, by using Fujita's classification ([9, (8.11)]), we are enable to show that (ii) and (iv) exactly appear in our classification.

If $h^0(\mathscr{H}_A) = n+2$, then we can also prove that (i), (iii) and (vi) appear in our classification, exactly.

In the case of $h^0(\mathscr{H}_A) = n+1$, we consider the pair $(X, \mathscr{H})$ instead of $(A, \mathscr{H}_A)$ since polarized manifolds with Δ-genera 3 are not still classified completely. Noting that $\ell \neq 1$, we obtain the following possibilities:

$$\Delta(X, \mathscr{H}) = \begin{cases} 1 & \text{for } \mathscr{H}^{n+1} = 1; \\ 2 & \text{for } \mathscr{H}^{n+1} = 2. \end{cases}$$

In fact, we can show that $g(X, \mathscr{H}) = 1$ or 3. If $g(X, \mathscr{H}) = 1$, then we apply a classification result of polarized manifolds of sectional genera one ([9, (12.3)]). In this way, it turns out that (v) actually shows up.

The difficulty arise in the case of $g(X, \mathscr{H}) = 3$. There are two keys in proof of the 'only if' part.

One is to determine the structure of a certain polarized manifold with $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 1$ and $g(X, \mathscr{H}) = 3$. Strictly speaking, we show the proposition below. In general, polarized manifolds with these invariants are yet to be classified for no less than two decades (cf. [9, (6.18)]).

Proposition 3.5 ([2, Proposition 3.1]) *Let $(X, \mathscr{H})$ be a polarized manifold with $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 1$ and $g(X, \mathscr{H}) = 3$. Suppose that $4\mathscr{H}$ is very ample and that* $\dim X > 5$. *Then $(X, \mathscr{H})$ is a smooth weighted hypersurface of degree* 12 *in* $\mathbf{P}(4, 3, 1^{n+1})$. □

The other is to rule out the possibilities of $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 2$ and $g(X, \mathscr{H}) = 3$. In general, polarized manifolds with these invariants are still difficult to study, and also have not been classified (cf. [9, (10.10)]). We prove the following in § 3.4.

Proposition 3.6 ([2, Proposition 3.2]) *Let $(X, \mathscr{H})$ be a polarized manifold with $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 2$. Suppose that $g(X, \mathscr{H}) = 3$ and that* $\dim X > 5$. *Then the line bundle $2\mathscr{H}$ cannot be very ample.* □

3.3 Sketch of proof of Proposition 3.5

We fix our notation. Let $X_r := \bigcap_{r\leq i\leq n} V_i$, where each $V_i \in |\mathscr{H}|$ is a general member. Due to $\mathscr{H}^{n+1} = 1$, we see that X_r is an r-dimensional submanifold of X_{r+1} by setting $X_{n+1} := X$. Hence, for every $1 \leq r \leq n+1$, we see that the adjunction formula implies $g(X_r, \mathscr{H}_{X_r}) = 3$ and that $\mathscr{H}_{X_r}^r = 1$.

First, note that X_1 is isomorphic to a plane quartic curve since $g(X_1) = g(X, \mathscr{H}) = 3$. Next, we are showing that

(A) $R(X_1, \mathscr{H}_{X_1}) \cong \mathbf{C}[x,y,z]/(F_{12})$, where $\mathrm{wt}(x,y,z) = (4,3,1)$ and $F_{12} = x^3 + y^4 + z\psi_{11}$ for some homogeneous polynomial $\psi_{11} \in \mathbf{C}[x,y,z]$ of degree 11; and

(B) The restriction map $\rho\colon R(X_2, \mathscr{H}_{X_2}) \to R(X_1, \mathscr{H}_{X_1})$ is surjective.

As a matter of fact, (A) and (B) imply the assertion. To explain this implication, we quote results by S. Mori.

Fact 3.7 (Mori) *Let D be an effective ample divisor of an m (≥ 3)-dimensional smooth projective variety M. Suppose that D is a w.c.i. of type $(a_1, \ldots, a_c)$ in $\mathbf{P}(e_0, \ldots, e_N)$. Assume that*

($\dagger$) *there exists a positive integer a such that $\mathscr{O}_M(D) \otimes \mathscr{O}_D \cong \mathscr{O}_D(a)$.*

Then M is a w.c.i. of type $(a_1, \ldots, a_c)$ in $\mathbf{P}(e_0, \ldots, e_N, a)$. In particular, if $m \geq 4$, then the assumption ($\dagger$) is satisfied. □

For a proof, see [15, Proposition 3.10]. Now, by combining (A) and (B), we see that X_2 is a weighted hypersurface of degree 12 in $\mathbf{P}(4,3,1^2)$. And, by using Fact 3.7, we obtain that X_3 is a weighted hypersurface of the same degree in $\mathbf{P}(4,3,1^3)$ since $a = 1$. Iterating to use Fact 3.7, we get the assertion.

From now on, we are going to outline the proofs of (A) and (B). For details to [2, Proposition 3.1].

(A) We find the generators of the graded algebra $R(X_1, \mathscr{H}_{X_1})$ and the relations among them.

The sectional genus $g(X, \mathscr{H}) = 3$ implies that $K_{X_1} = 4\mathscr{H}_{X_1}$. Therefore, by the Riemann-Roch theorem for X_1, we obtain the formula

$$h^0(i\mathscr{H}_{X_1}) = h^0((4-i)\mathscr{H}_{X_1}) + i - 2.$$

For all $i \geq 5$, we see $h^0(i\mathscr{H}_{X_1}) = i - 2$. For $i \leq 4$, we get the following table because a smooth plane quartic has no g_2^1:

i	$h^0(i\mathcal{H}_{X_1})$	i	$h^0(i\mathcal{H}_{X_1})$
1	1	3	2
2	1	4	3

Table 2.

Let z be a basis of the vector space $H^0(\mathcal{H}_{X_1})$. Choose $y \in H^0(3\mathcal{H}_{X_1})$ such that $H^0(3\mathcal{H}_{X_1}) = \langle y, z^3\rangle$. Similarly, choose $x \in H^0(4\mathcal{H}_{X_1})$ such that $H^0(4\mathcal{H}_{X_1}) = \langle x, yz, z^4\rangle$. Here we proceed in two steps.

Step 1 We show that $R(X_1, \mathcal{H}_{X_1})$ is generated by three elements x, y, z. Indeed, it suffices to show that there exist some monomials in x, y, z which form a basis of $H^0(i\mathcal{H}_{X_1})$ for each $i \geq 5$.

The proof of Step1 needs the following fact in elementary number theory.

Let a, b be coprime positive integers and l an integer. Suppose that $l \geq (a-1)(b-1)$. Then the equation $ai + bj = l$ has at least one solution (i, j) of non-negative integers.

We apply this fact to our proof by letting $(a, b) = (4, 3)$ and $l \geq 6$.

Due to the result of Step 1, we have a surjective homomorphism

$$\Phi\colon \mathbf{C}[x,y,z] \to R(X_1, \mathcal{H}_{X_1}).$$

Step 2 We show that there exists an irreducible homogeneous polynomial F_{12} of degree 12 in $\mathbf{C}[x,y,z]$ such that $\mathrm{Ker}(\Phi) = (F_{12})$. Indeed, here we compare $h^0(12\mathcal{H}_{X_1})$ with the number of monomials in x, y, z of degree 12 to find a relation among generators of $H^0(12\mathcal{H}_{X_1})$. We can conclude that there exist a unique generator of $\mathrm{Ker}(\Phi)$ from that $\mathrm{ht}(\mathrm{Ker}(\Phi)) \leq \dim \mathbf{C}[x,y,z] - \dim R(X_1, \mathcal{H}_{X_1}) = 1$. In this way, (A) is proved.

(B) We prove that $R(X_2, \mathcal{H}_{X_2})$ is Cohen-Macaulay. By doing so, we obtain the surjectivity of ρ because the Cohen-Macaulayness yields $H^1(i\mathcal{H}_{X_2}) = 0$ for each i. In fact, we find a regular sequence of length $\dim R(X_2, \mathcal{H}_2) = 3$ contained in $R(X_2, \mathcal{H}_{X_2})_+ := \bigoplus_{i>0} H^0(X_2, i\mathcal{H}_{X_2})$.

We fix our notation: Let $\mathbf{s} = \{s_0, \dots, s_N\}$ be a minimal set of generators of the graded algebra $R(X_2, \mathcal{H}_{X_2})$. Then one has an isomorphism

$$R(X_2, \mathcal{H}_{X_2}) \cong \mathbf{C}[s_0, \dots, s_N]/I_{\mathbf{s}},$$

where $I_{\mathbf{s}}$ denotes the (homogeneous) defining ideal of X_2.

First, we find a regular sequence of length 2 contained in $R(X_2, \mathcal{H}_{X_2})_+$. In fact, if we take sections $s, t \in H^0(\mathcal{H}_{X_2})$ such that

$$H^0(\mathscr{H}_{X_2}) = \langle s, t \rangle \text{ with } \rho(s) = z \text{ and } (t)_0 = X_1,$$

we can easily prove that s, t form the regular sequence. We may assume that $\mathbf{s}$ contains these two elements.

Next, we find an $R(X_2, \mathscr{H}_{X_2})/(t, s)$-regular element. One needs some information about generators of $I_{\mathbf{s}}$. For every $i \geq 0$, let

$$\rho_i \colon H^0(i\mathscr{H}_{X_2}) \twoheadrightarrow H^0(i\mathscr{H}_{X_2})/\langle t \rangle \hookrightarrow H^0(i\mathscr{H}_{X_1}).$$

denote the restriction map. We proceed in two steps.

Step 1 We show that the ideal $I_{\mathbf{s}}$ has no generators in degrees ≤ 4.

The crucial point is to show that

$$\mathrm{Im}(\rho_4) = H^0(4\mathscr{H}_{X_2}).$$

In order to show this, we need the following:

- The very ampleness of $4\mathscr{H}$; and
- X_1 is embedded by $\varphi_{4\mathscr{H}_{X_1}}$, and its image is a plane quartic curve.

By using the restriction maps ρ_i, we argue whether there exist relations among fixed generators of $H^0(i\mathscr{H}_{X_2})$ for each $i \leq 4$.

Step 2 We claim that there exists an $R(X_2, \mathscr{H}_{X_2})/(t, s)$-regular element. Let u denote a section of $H^0(4\mathscr{H}_{X_2})$ such that $\rho_4(u) = x$. We assert that u is $R(X_2, \mathscr{H}_{X_2})/(t, s)$-regular. Indeed, $\mathrm{Proj}(R(X_2, \mathscr{H}_{X_2})/(t, s))$ is an integral scheme p because of the assumption $\mathscr{H}_{X_2}^2 = 1$. Thus we see that $(R(X_2, \mathscr{H}_{X_2})/(t, s))_+$ has no zero-divisors. Moreover, by Step 1, u is $(R(X_2, \mathscr{H}_{X_2})/(t, s))_0$-regular. Thus we get the claim.

Consequently, since we obtain (A) and (B), Proposition 3.5 is proved. ■

3.4 Sketch of proof of Proposition 3.6

In this case, note that $K_X = (2 - n)\mathscr{H}$. We prove by contradiction. We are able to regard X_2 as a surface in $\mathbf{P}^4$ as follows: We can obtain that

$$h^0(X_2, 2\mathscr{H}_{X_2}) = h^0(X_1, 2\mathscr{H}_{X_1}) + 2 = 5$$

by using the fact that $H^1(X_3, i\mathscr{H}_{X_3}) = 0$ for all i. Here we assume that $L = 2\mathscr{H}$ is very ample. Then we see that L_{X_2} gives an embedding of X_2 into $\mathbf{P}^4$.

We use the double point formula for surfaces (see [4, Lemma 8.2.1])

$$L_{X_2}^2(L_{X_2}^2 - 5) - 10(g(X_2, L_{X_2}) - 1) + 12\chi(\mathscr{O}_{X_2}) - 2K_{X_2}^2 = 0.$$

Note that $K_{X_2} = \mathscr{H}_{X_2}$. Thus the formula implies that $-7 + 3p_g(X_2) = 0$, which is absurd. ■

Therefore we see that this case cannot occur, which completes the proof of Theorem 3.6.

Acknowledgements The speaker expresses gratitude to Profs. Mizuho Ishizaka, Hajime Kaji and Kazuhiro Konno, who are the organizers of the workshop "Algebraic Geometry and Topology" held in the RIMS, Kyoto University.

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