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Stability of direct images by Frobenius morphisms

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Abstract

In this paper, we study stability of direct images by Frobenius morphisms. First, we compute the first Chern classes of direct images of vector bundles by Frobenius morphisms up to the numerical equivalence. Next, introducing the canonical filtrations, we prove that if X is a nonsingular projective surface with semistable Ω_X^1 with respect to an ample line bundle H and $K_X H > 0$, then the direct images of line bundles on X by Frobenius morphisms are semistable.

1 Introduction

This is a joint work with H. Sumihiro. Please refer [5] for details.

Let k be an algebraically closed field of characteristic $p > 0$, X a nonsingular projective variety over k of dimension n, $F = F_X$ the absolute Frobenius morphism of X and H be an ample divisor on X. Then one can define the *slope* of a torsion free sheaf $\mathscr{E}$ on X with respect to H by

$$\mu(\mathscr{E}) = \frac{c_1(\mathscr{E})H^{n-1}}{\mathrm{rk}(\mathscr{E})},$$

where $\mathrm{rk}(\mathscr{E})$ is the rank of $\mathscr{E}$. Then a torsion free sheaf $\mathscr{E}$ on X is called *semistable* (respectively, *stable*) with respect to H if for all nonzero torsion free subsheaves $\mathscr{F}$ of $\mathscr{E}$, $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$ (respectively, $\mu(\mathscr{F}) < \mu(\mathscr{E})$).

H. Lange and C. Pauly proved the following theorem:

Theorem 1.1 (H. Lange, C. Pauly [7]). *Let X be a nonsingular projective curve over k of genus $g(X) \geq 2$ and $\mathscr{L}$ a line bundle on X. Then $F_*\mathscr{L}$ is stable.*

Hence we can consider a following natural question for higher dimensional cases:

Problem 1.2. *Let X be a nonsingular projective variety of general type over k of dimension $n \geq 2$, $\mathscr{L}$ a line bundle on X, and H be an ample line bundle on X. Is $F_*\mathscr{L}$ semistable with respect to H?*

2 A formula for first Chern classes of the direct images by Frobenius morphisms

Let X be a nonsingular projective variety of dimension n over k and let $\mathscr{E}$ be a vector bundle of rank r on X. In this section, we compute the first Chern class $c_1(F_{X*}\mathscr{E})$ to compute its slope.

First we shall prepare the following. Let Y be a nonsingular irreducible divisor on X and let G be the kernel of the natural surjection $F_{X*}\mathscr{E}|_Y \to F_{Y*}(\mathscr{E}|_Y)$. Then G has the following filtration $G^\bullet$.

Theorem 2.1. $\mathrm{Gr}^i(G^\bullet) = F_{Y*}(\mathscr{E} \otimes \mathscr{O}_X(-iY)|_Y)$ $(1 \leq i \leq p-1)$.

Now we can compute first Chern classes of direct images by Frobenius morphisms.

Theorem 2.2. *Let $\mathscr{E}$ be a vector bundle on X of rank r. Then*

$$c_1(F_*\mathscr{E}) \underset{\text{num}}{\equiv} \frac{p^n - p^{n-1}}{2} rK_X + p^{n-1}c_1(\mathscr{E}),$$

where $\underset{\text{num}}{\equiv}$ denotes numerical equivalence and K_X is the canonical divisor of X.

By the way, K. Kurano proved a similar formula ring-theoritically:

Theorem 2.3 (K. Kurano[6]). *Let k be a perfect field,* $\mathrm{char}(k) = p > 0$ *and X be a normal algebraic variety of dimension n over k. Then*

$$c_1(F_*\mathscr{O}_X) = \frac{p^n - p^{n-1}}{2} K_X$$

in $A_{n-1}(X)_{\mathbf{Q}}$.

3 Canonical filtrations

In this section, we introduce a useful filtration on $F^*F_*\mathcal{O}_X$. Let I be the kernel of the natural surjection $F^*F_*\mathcal{O}_X \to \mathcal{O}_X$. Since $F^*F_*\mathcal{O}_X$ is an $\mathcal{O}_X$-algebra, we get a descending filtration

$$I^0 := F^*F_*\mathcal{O}_X \supset I^1 := I \supset I^2 \supset I^3 \supset \cdots$$

on $F^*F_*\mathcal{O}_X$. Here we consider $F^*F_*\mathcal{O}_X$ as an $\mathcal{O}_X$-module from right.

Definition 3.1. We call this filtration $I^\bullet$ the *canonical filtration* on $F^*F_*\mathcal{O}_X$.

Consider the following commutative diagram:

$$\begin{tikzcd}
X \times_{X^p} X \arrow[dr, dashed, "j"] \arrow[drrr, bend left, "p'_2"] \arrow[dddr, "p'_1"'] & & & \\
& X \times_k X \arrow[rr, "p_2"] \arrow[dd, "p_1"] & & X \arrow[dl, "F"'] \arrow[dd, "\varphi"] \\
& & X \arrow[dr, "\psi"] & \\
& X \arrow[ur, "F"] \arrow[rr, "\varphi"'] & & \operatorname{Spec} k
\end{tikzcd}$$

where φ is the structure morphism of X, ψ is the morphism induced from the map taking p-th root of elements of k and p_i, p'_i are natural projections. Then there exists the morphism j in the diagram which is a closed immersion.

Let J (resp. I') be the kernel of the natural surjection $\mathcal{O}_{X\times_k X} \to \mathcal{O}_X$ (resp. $\mathcal{O}_{X\times_{X^p} X} \to \mathcal{O}_X$). Then there exists the following commutative diagram with exact rows of sheaves on $X \times_k X$:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O}_{X\times_k X} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & j_*I' \arrow[r] & j_*\mathcal{O}_{X\times_{X^p} X} \arrow[r] & j_*\mathcal{O}_X \arrow[r] & 0.
\end{tikzcd}$$

and we have $I = p'_{2*}I'$ because $F^*F_*\mathcal{O}_X = p'_{2*}\mathcal{O}_{X\times_{X^p}X}$ and $p'_2 = p_2 \circ j$ is an affine morphism. Hence the morphism $J^i/J^{i+1} = S^i(\Omega^1_X) \twoheadrightarrow j_*(I'^i/I'^{i+1}) \cong I^i/I^{i+1}$ is surjective on X, where Ω^1_X is the vector bundle of regular differential forms of degree 1.

Let $U = \operatorname{Spec} A \subset X$ be a nonempty affine open subset. Then the exact sequence

$$0 \to I \to F^*F_*\mathcal{O}_X \to \mathcal{O}_X \to 0$$

is locally expressed in the following way:

$$0 \to I \to A \otimes_{A^p} A \to A \to 0$$

and $I = \langle a \otimes 1 - 1 \otimes a \mid a \in A \rangle A$. We consider $A \otimes_{A^p} A$ as an A-module from right. Let $\{x_1, \ldots, x_n\}$ be a regular system of parameters. For any element $a \in A$, write $a = \sum_{0 \le i_1, \ldots, i_n \le p-1} a^p_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}$, where $a_{i_1, \ldots, i_n} \in A$. Then we have

$$\begin{aligned} a \otimes 1 - 1 \otimes a &= \sum a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \otimes 1 - 1 \otimes \sum a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \\ &= \sum (x_1^{i_1} \cdots x_n^{i_n} \otimes a^p_{i_1,\ldots,i_n} - 1 \otimes a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}) \\ &= \sum (x_1^{i_1} \cdots x_n^{i_n} \otimes 1 - 1 \otimes x_1^{i_1} \cdots x_n^{i_n}) a^p_{i_1,\ldots,i_n}. \end{aligned}$$

Therefore, $I = \langle x_1^{i_1} \cdots x_n^{i_n} \otimes 1 - 1 \otimes x_1^{i_1} \cdots x_n^{i_n} \mid 0 \le i_1, \ldots, i_n \le p-1 \rangle A$ locally.

Next, we will calculate the filtration for curve and surface cases.

3.1 Curve case

Assume that X is a curve. Let x be a regular parameter. Then $I = \langle x^i \otimes 1 - 1 \otimes x^i \mid 1 \le i \le p-1 \rangle A$ by the above. Let us put $\omega := x \otimes 1 - 1 \otimes x$. Then I is a free A-module with basis $\{\omega^1, \ldots, \omega^{p-1}\}$.

Lemma 3.2. $I = \bigoplus_{1 \le i \le p-1} \omega^i A$.

Thus, we observe $I^i/I^{i+1} \cong \omega^i A$ locally, which implies I^i/I^{i+1} is a line bundle on X. Therefore the surjection $J^i/J^{i+1} = K_X^{\otimes i} \to I^i/I^{i+1}$ is an isomorphism $(0 \le i \le p-1)$. Hence we obtain

Proposition 3.3. *Let X be a nonsingular projective curve over k and $I^\bullet$ the canonical filtration on $F^*F_*\mathcal{O}_X$. Then it follows that*

$$F^*F_*\mathcal{O}_X \supset I \supset I^2 \supset \cdots \supset I^{p-1} \supset I^p = (0)$$

and $I^i/I^{i+1} = K_X^{\otimes i}$ $(0 \le i \le p-1)$.

3.2 Surface case

Assume X is a surface. Let $\{x, y\}$ be a regular system of parameters. Then $I = \langle x^i y^j \otimes 1 - 1 \otimes x^i y^j \mid 0 \leq i, j \leq p-1 \rangle A$. Let $\omega := x \otimes 1 - 1 \otimes x$ and $\eta := y \otimes 1 - 1 \otimes y$. Similarly to the curve case, we can see that $I = \langle \omega^k \eta^l \mid 0 \leq k, l \leq p-1 \rangle A$ ($\omega^p = \eta^p = 0$), $I^i = \langle \omega^k \eta^l \mid k + l \geq i, 0 \leq k, l \leq p-1 \rangle A$ and

$$F^* F_* \mathcal{O}_X \supset I \supset I^2 \supset \cdots \supset I^{2p-2} \supset I^{2p-1} = (0),$$

$$I^i / I^{i+1} = \bigoplus_{k+l=i,\ 0 \leq k, l \leq p-1} \omega^k \eta^l A.$$

Lemma 3.4. $\binom{p-1}{n} = (-1)^n$ *in positive characteristic* p.

Lemma 3.5. $I^{2p-2} \cong K_X^{\otimes(p-1)}$.

If $i \leq p-1$, then J^i / J^{i+1} and I^i / I^{i+1} are vector bundles on X of the same rank and so it follows that $I^i / I^{i+1} \cong J^i / J^{i+1} \cong S^i(\Omega_X^1)$. On the other hand, there exists the following perfect pairing:

$$\begin{array}{ccc} I^i / I^{i+1} \otimes_{\mathcal{O}_X} I^{2p-2-i} / I^{2p-1-i} & \longrightarrow & I^{2p-2} / I^{2p-1} = I^{2p-2} \cong \omega_X^{\otimes(p-1)}. \\ \uplus & & \uplus \\ \omega^k \eta^l \otimes \omega^{k'} \eta^{l'} & \longmapsto & \omega^{k+k'} \eta^{l+l'} \end{array}$$

Thus combining the above, we obtain

Proposition 3.6. *Let X be a nonsingular projective surface over k and $I^\bullet$ the canonical filtration on $F^* F_* \mathcal{O}_X$. Then it follows that*

$$F^* F_* \mathcal{O}_X \supset I \supset I^2 \supset \cdots \supset I^{2p-2} \supset I^{2p-1} = (0)$$

and

$$I^i / I^{i+1} = \begin{cases} S^i(\Omega_X^1) & (0 \leq i \leq p-1), \\ K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega_X^1) & (p \leq i \leq 2p-2). \end{cases}$$

4 Canonical connections

Let $\mathcal{E}$ be a quasi-coherent sheaf on X. Then there exists a connection $\nabla : F^* \mathcal{E} \to F^* \mathcal{E} \otimes \Omega_X^1$, which is called the *canonical connection*([4]). This is locally written as

$$\begin{array}{ccccc} M \otimes_{A^p} A & \to & M \otimes_{A^p} A \otimes_A \Omega^1_{A/k} & \cong & M \otimes_{A^p} \Omega^1_{A/k} \\ \cup\!\!\!| & & & & \cup\!\!\!| \\ m \otimes f & & \mapsto & & m \otimes df \end{array}$$

where $A = \Gamma(U, \mathcal{O}_X)$ and $M = \Gamma(U, \mathcal{E})$ for an affine open subset U of X. In particular, we get a connection on $F^*F_*\mathcal{O}_X$

$$\nabla : F^*F_*\mathcal{O}_X \to F^*F_*\mathcal{O}_X \otimes \Omega^1_X.$$

4.1 Curve case

Here, we compute ∇ for curve case. Let x be a regular parameter. Again, put $\omega = x \otimes 1 - 1 \otimes x$. Straightforwardly computing, we get

$$\nabla(\omega^k f) = \left(-k\omega^{k-1} f + \omega^k \frac{\partial f}{\partial x}\right) \otimes dx \ (f \in A).$$

4.2 Surface case

Here, we shall compute ∇ for surface case. Let $\{x, y\}$ be a regular system of parameters. Again, putting $\omega = x \otimes 1 - 1 \otimes x$ and $\eta = y \otimes 1 - 1 \otimes y$ and taking $f \in A$, we get by computing straightforwardly

$$\begin{aligned} \nabla(\omega^k \eta^l f) &= \left(-k\omega^{k-1}\eta^l f + \omega^k \eta^l \frac{\partial f}{\partial x}\right) \otimes dx \\ &\quad + \left(-l\omega^k \eta^{l-1} f + \omega^k \eta^l \frac{\partial f}{\partial y}\right) \otimes dy. \end{aligned}$$

5 Main results

Using the canonical filtrations (Theorem 3.6), we can prove the following theorem, which is a generalization of Theorem 1.1 to surface case.

Theorem 5.1. *Let X be a nonsingular projective surface over k and H an ample line bundle on X. Assume that $K_X H > 0$ and Ω^1_X is semistable with respect to H. Then $F_*\mathcal{L}$ is semistable with respect to H for any line bundle $\mathcal{L}$ on X.*

Using the canonical filtrations, we can also prove a similar result in the case that K_X is numerically trivial.

Theorem 5.2. *Let X be a nonsingular projective surface over k and H an ample line bundle on X. Assume that $K_X \underset{\mathrm{num}}{\equiv} 0$ and Ω_X^1 is semistable with respect to H. Then $F_*\mathscr{L}$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

Example 5.3. If $X \subset \mathbb{P}^3$ is a general surface of degree $d \geq 4$, we can prove Ω_X^1 is strongly stable, i.e., $(F^k)^*\Omega_X^1$ is stable for every $k \in \mathbb{N}$, with respect to any ample divisor H. So X satisfies the conditions of above theorems.

Remark 5.4. We can give another simple proof of the Lange and Pauly's Theorem (Theorem 1.1) using the canonical filtrations in the curve case.

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