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Various Gauss fibers

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ABSTRACT. We find examples and constructions of nontrivial fiber structures of Gauss maps in positive characteristic. 3 types of constructions of projective varieties were announced in this talk: (A) Gauss fibers are the given projective variety, (B) Gauss fibers are hyperplane sections of the given projective variety, and (C) Gauss map is the given rational map $g : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ with $dg \equiv 0$. Additional matters to the talk are included in this paper, for example, discussion of the differences of the constructions (A)-(C), and a generalization of Kaji and Rathmann's construction of Gauss map, related to (C), which is the given inseparable morphism $\mathbf{P}^1 \to \mathbf{P}^1$.

1. INTRODUCTION

In this paper, the base field K is an algebraically closed field and varieties are integral algebraic schemes over K.

Precisely, "Gauss fibers" mean (general) fibers of the Gauss maps. Definition of the Gauss map is as follows:

Definition 1.1. *Let $X \subset \mathbf{P}^N$ be a projective variety. The* Gauss map *γ on X is the rational map from X to the Grassmannian $\mathbf{G}(\dim X, N)$ such that $\gamma(p) = \mathbf{T}_pX$ for any smooth point $p \in X$, where $\mathbf{T}_pX$ is the projective embedded tangent space.*

Example 1.2. *If $X \subset \mathbf{P}^N$ is the hypersurface given by F, then*

$$\gamma = \left(\frac{\partial F}{\partial X_0} : \cdots : \frac{\partial F}{\partial X_N}\right) : X \dashrightarrow \mathbf{G}(N-1, N) \cong \mathbf{P}^{N*}.$$

Remark 1.3. *The following facts are known.*

(1) *If* $\text{char}K = 0$ *then general fibers of* γ *are linear spaces (*[1],[5],[14]*).* *(They are one-points when* $\dim = 0$.*)*
(2) *If* $\text{char}K > 0$ *then there is a variety whose general fibers of* γ *are two or more distinct points.*

The fact (1), when X is a curve, implies that multiple tangent lines (which have two or more distinct tangential points at X) are only finitely many. If X is a surface with $\dim \gamma(X) = 1$ then we can classify X to the two kinds of ruled surfaces, a cone or a tangent surface. (A cone surface is the join of a curve and one point, and a tangent surface is covered by tangent lines of some curve.) These surfaces are called developable surfaces.

A. H. Wallace gave examples of the kind of (2) ([13]). Kleiman-Laksov also found interesting examples ([10]). It seems to be difficult to construct smooth varieties of this kind, but H. Kaji ([7],[8]), J. Rathmann ([12]) and A. Noma ([11]) constructed such varieties.

The author found the following example.

Example 1.4. $XZ^6 - (Y^6 + W^6)W = 0 \subset \mathbf{P}^3$. *If* $\text{char}K = 2$ *(resp.* $\text{char}K = 3$*) then general Gauss fibers are plane elliptic curves (resp. plane smooth conics) (*[2]*).*

In the author's best knowledge, this is the first example whose general Gauss fibers are not finite unions of linear spaces. Furthermore, the author found constructions of varieties with non-linear smooth Gauss fibers in positive characteristic.

(A) Construction of a projective variety whose general fibers of the Gauss map are the given projective variety ([3]).
(B) Construction of a projective variety whose general fibers of the Gauss map are hyperplane sections of the given (general) projective variety ([4]).
(C) Construction of a projective variety whose Gauss map is the given rational map $g : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ with $dg \equiv 0$.

The main purpose of this paper is an introduction of these constructions and explanation of differences of each constructions.

2. Construction (C)

Let $g = (g_0, \dots, g_n) : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ be the given rational map such that $\frac{\partial g_j}{\partial x_i} \equiv 0$ for all i, j, and let X be the closure of image of the rational map $i : \mathbf{A}^n \dashrightarrow \mathbf{P}^{n+1}$; $(x_1, \dots, x_n) \dashrightarrow (1 : x_1 : \cdots : x_n : -g_0 - x_1 g_1 - \cdots - x_n g_n)$. Then, $\mathbf{T}_{i(x)} X$ is spanned by the row vectors of the following matrices;

$$\begin{pmatrix} 1 & x_1 & \dots & x_n & -g_0 - x_1 g_1 - \cdots - x_n g_n \\ 0 & 1 & \dots & 0 & -g_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -g_n \end{pmatrix} \sim \begin{pmatrix} & -g_0 \\ I_{n+1} & -g_1 \\ & \vdots \\ & -g_n \end{pmatrix}$$

where I_{n+1} is the $(n+1) \times (n+1)$ unit matrix. Hence $\gamma : X \dashrightarrow \mathbf{P}^{n+1*}$ is given by $(g_0 : \cdots : g_n : 1)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{A}^n & \overset{g}{\dashrightarrow} & \mathbf{A}^{n+1} \\ \downarrow & & \downarrow \\ X & \overset{\gamma}{\dashrightarrow} & \mathbf{P}^{n+1*} \end{array}$$

Remark 2.1. *Example 1.4 is given by this construction:* $n = 2$, $g_0 = g_2 = g_3 = 0$ *and* $g_1 = x_1^6 + x_2^6$ *(with suitable coordinates).*

If we consider the rational map $g : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ as the rational map g' from $\mathbf{P}^n$ to $\mathbf{P}^{n+1}$, then the above varieties can be got as the image of a suitable linear projection of the graph $\Gamma_{g'} \subset \mathbf{P}^{n^2+3n+1}$ of g' which is embedded by Segre embedding.

Now we study the graph Γ_g of a rational map $g : \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ with $dg \equiv 0$. Let $X \subset \mathbf{P}^{nm+n+m}$ be the image of the graph Γ_g by Segre embedding $\mathbf{P}^n \times \mathbf{P}^m \subset \mathbf{P}^{nm+n+m}$. Then we have the commutative diagram

$$\begin{array}{ccccc} & & \Gamma_g & \cong & X \\ & \swarrow{\scriptstyle p_1} & \downarrow{\scriptstyle p_2} & & \vdots{\scriptstyle \gamma} \\ \mathbf{P}^n & \underset{g}{\dashrightarrow} & \mathbf{P}^m & \overset{h}{\hookrightarrow} & \mathbf{G}(n, nm+n+m) \end{array}$$

where h is an embedding given by $h(t) = \mathbf{P}^n \times t$. This implies that the Gauss map γ can be identified with the projection p_2, hence generically identified with g. Precisely, X is the closure of the image of the rational map $\mathbf{P}^n \dashrightarrow \mathbf{P}^{nm+n+m}$:

$$(1 : x_1 : \cdots : x_n) \mapsto (1 : x_1 : \cdots : x_n : g_1 : \cdots : g_m : \cdots : x_i g_j : \cdots).$$

We can check easily that varieties given by (C) can be got as the image of a suitable linear projection of the graph of $\mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}$.

The latter construction is a generalization of Kaji and Rathmann's for inseparable morphisms $\mathbf{P}^1 \to \mathbf{P}^1$ ([7],[12]).

In the latter construction, it is very interesting that the Gauss map can be defined at any point of X, hence it is a morphism, and the tangent variety of X is $\mathbf{P}^n \times \mathbf{P}^m$ if g is dominant. The second fact implies that any Segre variety of two projective spaces is the tangent variety of some variety, and that the classical fact $X \subset \mathrm{SingTan}X$ in characteristic 0 ([1]) does not hold in positive characteristic.

3. Construction (A)

3.1. Concept of (A) or (B). Let $Y \subset \mathbf{P}^k$ be a given projective variety of codimension r. We move $\mathbf{P}^k$ "inseparably" in $\mathbf{P}^N$ ($N >> k$). Then, Y moves in conformity to the projective space $\mathbf{P}^k$, and constructs X. We will have the diagram

$$\begin{array}{ccc} \mathbf{A}^r \times \mathbf{P}^k & \overset{\eta}{\to} & \mathbf{P}^N \\ \cup & & \cup \\ \mathbf{A}^r \times Y & \overset{\eta|_{\mathbf{A}^r \times Y}}{\to} & X \end{array}$$

such that η is inseparable (onto its image) and $\eta|_{\mathbf{A}^r \times Y}$ is birational.

The idea for our construction (A) or (B) is based on "circular surfaces" ([6]) studied in differential geometry or real singularity theory. Conceptually, our variety with (A) could be called a "developable" circular surface.

3.2. **Construction (A) (plane curve's case).** Let $p > 0$ be the characteristic, and let $\rho_0, \rho_1, \rho_2 : \mathbf{A}^1 \to \mathbf{P}^3$ be morphisms (which form a frame) as follows,

$$\begin{aligned} \rho_0 &= (1 \;\; 0 \;\; u \;\; u^p) \\ \rho_1 &= (0 \;\; 1 \;\; 0 \;\; 0) \\ \rho_2 &= (0 \;\; 0 \;\; 1 \;\; 0) \end{aligned}$$

Let $\eta : \mathbf{A}^1 \times \mathbf{P}^2 \to \mathbf{P}^3$ be

$$(u) \times (1 : y_1 : y_2) \mapsto [\rho_0 + y_1\rho_1 + y_2\rho_2] = (1 : y_1 : u + y_2 : u^p).$$

We may assume that $y_1 - a$ is a local parameter at a smooth point $(1 : a : b) \in Y$. (We can always take this coordinates by linear transforms of $\mathbf{P}^2$.) Let X be the closure of $\eta(\mathbf{A}^1 \times Y)$, and let $\tau := \eta|_{\mathbf{A}^1 \times Y} : \mathbf{A}^1 \times Y \to X$. Then the following proposition holds.

Proposition 3.1. *The morphism τ is birational, and $\mathbf{T}_{\tau(u,y)}X = \eta(u \times \mathbf{P}^2)$ for a general point (u, y).*

Proof. The differentials of τ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & dy_2/dy_1 & 0 \end{pmatrix}$$

(upper row is a list of the differentials by u, lower is the differentials by y_1). We find the separability of τ by this matrix and, because τ is generically one-to-one, birationality of τ.

$\mathbf{T}_{\tau(u,y)}X$ is spanned by the row vectors of the following matrices:

$$\begin{pmatrix} 1 & y_1 & u + y_2 & u^p \\ 0 & 0 & 1 & 0 \\ 0 & 1 & dy_2/dy_1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & u^p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This coincides with $\eta(u \times \mathbf{P}^2)$. □

3.3. **Generalized form.** Let $k \geq 2$ and $r < k$ be positive integers, and let $Y \subset \mathbf{P}^k$ be a closed subvariety of codimension r. We take the morphisms $\rho_0, \ldots, \rho_k : \mathbf{A}^r \to \mathbf{P}^{k+r}$ as follows,

$$\begin{array}{rl} \rho_0 = & (1 \;\; 0 \;\; \ldots \;\; 0 \;\; u_1 \;\; \ldots \;\; u_r \;\; u_1^p \;\; \ldots \;\; u_r^p) \\ \rho_1 = & (0 \;\; 1 \;\; \ldots \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; \ldots \;\; 0) \\ \vdots & \\ \rho_k = & (0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; \ldots \;\; 1 \;\; 0 \;\; \ldots \;\; 0) \end{array}$$

Remark 3.2. *The form can be more generalized (see* [3]*). The base space $\mathbf{A}^r$ which moves projective planes and the how of the moving $\{\rho_i\}$ are more free and formulated to some extent. Furthermore, we can also construct varieties whose general fibers are two or more Ys for the suitable moving $\{\rho_i\}$.*

4. Construction (B)

4.1. **Construction (B) (surface case).** Let $p > 0$ be the characteristic, and let $\rho_0, \rho_1, \rho_2, \rho_3 : \mathbf{A}^2 \to \mathbf{P}^6$ be morphisms (which form a frame) as follows:

$$\begin{array}{rcl} \rho_0 & = & (1 \;\; 0 \;\; 0 \;\; u \;\; u^p \;\; v \;\; 0) \\ \rho_1 & = & (0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; v) \\ \rho_2 & = & (0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0) \\ \rho_3 & = & (0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0) \end{array}$$

Let $\eta : \mathbf{A}^2 \times \mathbf{P}^3 \to \mathbf{P}^6$ be

$$\begin{aligned} (u, v) \times (1 : y_1 : y_2 : y_3) &\mapsto [\rho_0 + y_1\rho_1 + y_2\rho_2 + y_3\rho_3] \\ &= (1 : y_1 : y_2 : u + y_3 : u^p : v : vy_1). \end{aligned}$$

We may assume that $y_1 - a$, $y_2 - b$ are a local parameter at a smooth point $(1 : a : b : c) \in Y$. Let X be the closure of $\eta(\mathbf{A}^2 \times Y)$, and let $\tau := \eta|_{\mathbf{A}^2 \times Y} : \mathbf{A}^2 \times Y \to X$. Then the following proposition holds.

Proposition 4.1. *The morphism τ is birational, and γ is generically identified with the morphism $\mathbf{A}^2 \times Y \to \mathbf{A}^3 : (u, v) \times (y_1, y_2, y_3) \mapsto (u^p, v, y_1)$.*

Proof. The differentials of τ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & y_1 \\ 1 & 0 & dy_3/dy_1 & 0 & 0 & v \\ 0 & 1 & dy_3/dy_2 & 0 & 0 & 0 \end{pmatrix}$$

(the first row is a list of the differentials by u, the second is by v, the third, fourth are the differentials by y_1, y_2 respectively). We find the separability of τ by this matrix and, because τ is generically one-to-one, birationality of τ.

$\mathbf{T}_{\tau(u,y)}X$ is spanned by the row vectors of the following matrices:

$$\begin{pmatrix} 1 & y_1 & y_2 & u+y_3 & u^p & v & vy_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & y_1 \\ 0 & 1 & 0 & dy_3/dy_1 & 0 & 0 & v \\ 0 & 0 & 1 & dy_3/dy_2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & u^p & 0 & -vy_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & y_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This implies our 2nd assertion. □

Corollary 4.2. *The set* $\{\gamma^{-1}(p)\}_{p\in X_{sm}}$ *almost coincides with* $\{Y_\lambda\}_{\lambda\in K}$, *where* Y_λ *is the hyperplane section* $Y \cap \{Y_1 - \lambda Y_0 = 0\}$ *(except the line* $Y_0 = Y_1 = 0$*).*

Example 4.3. *Let* $\mathrm{char}K > 2$. *Let* $Y \subset \mathbf{P}^3$ *be the surface given by* $Y_2^2Y_0 - Y_3(Y_3 - Y_0)(Y_3 - Y_1)$, *and let* Y_λ *be the hyperplane section* $Y \cap \{Y_1 - \lambda Y_0 = 0\}$. *Then, the set of all Gauss fibers of* X *almost coincides with* $\{Y_\lambda\}$.

4.2. **Generalized form.** Let $k \geq 2$ and $r < k$ be positive integers, and let $Y \subset \mathbf{P}^k$ be a closed subvariety of codimension r. We take the morphisms $\rho_0, \dots, \rho_k : \mathbf{A}^{r+1} \to \mathbf{P}^{k+r+2}$ as follows,

$$\begin{aligned} \rho_0 &= (1\ 0\ \dots\ 0\ u_1\ \dots\ u_r\ u_1^p\ \dots\ u_r^p\ u_{r+1}\ \ 0) \\ \rho_1 &= (0\ 1\ \dots\ 0\ 0\ \dots\ 0\ 0\ \dots\ 0\ 0\ u_{r+1}) \\ &\vdots \\ \rho_k &= (0\ 0\ \dots\ 0\ 0\ \dots\ 1\ 0\ \dots\ 0\ 0\ 0) \end{aligned}$$

5. Elliptic curves as Gauss fibers

In this section, we discuss the differences of our constructions. We recall some properties of our constructions.

Remark 5.1. *A variety X given by each constructions has the following properties:*

(A) *X is birational to the product of two some varieties and one of which varieties Y is the general fiber of the Gauss map.*
(B) *X is birational to the product of two some varieties and the differential $d\gamma$ of the Gauss map is not identically zero.*
(C) *X is rational and the differential $d\gamma$ of the Gauss map is identically zero.*

Let $\mathrm{char}K = p > 3$. Then we get projective varieties whose Gauss fibers are elliptic curves if we take Y or g as follows.

(A) Let $Y \subset \mathbf{P}^2$ be given by $Y_0^3 + Y_1^3 + Y_2^3 = 0$.
(B) Let $Y \subset \mathbf{P}^3$ be given by $Y_2^2 Y_0 - Y_3(Y_3 - Y_0)(Y_3 - Y_1) = 0$ (Example 4.3).
(C) Let $g : \mathbf{A}^2 \to \mathbf{A}^3$ be $(x_1, x_2) \mapsto (x_1^{3p} + x_2^{3p}, 0, 0)$.

We call X_a (resp. X_b, X_c) constructed by (A) (resp. (B), (C)) with the above Y (resp. Y, g).

X_a can not be constructed by (B) nor (C), because $d\gamma \equiv 0$ and this is not rational.

X_b can not be constructed by (A) nor (C), because isomorphic classes of Gauss fibers vary and $d\gamma$ is not identically 0.

X_c can not be constructed by (A) nor (B), because Gauss fibers are elliptic curves and this is rational, and $d\gamma \equiv 0$.

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