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FERROMAGNETIC SPIN MODEL AND THE LANDAU-LIFSCHIZ EQUATION

小川 卓克 (Takayoshi Ogawa)
Mathematical Institute, Tohoku University

1. Landau-Lifschitz equation

1.1. σ spin model. As a model of the ferromagnetic spin, the following equation is known: For a sphere valued function $u(t,x):\mathbb{R}\times\mathbb{R}^n\to\mathbb{S}^2$,

$$
\begin{cases}
\partial_t u=\kappa u\wedge\Delta u+\varepsilon(u\wedge(u\wedge\Delta u)), & t>0, x\in\mathbb{R}^n,\\
|u(t,x)|=1, \quad t\ge 0, x\in\mathbb{R}^n,\\
u(0,x)=u_0(x), \quad x\in\mathbb{R}^n,
\end{cases}
\tag{1.1}
$$

where $\wedge$ denotes the cross product and $\varepsilon\ge 0$ and $\kappa>0$ are physical constants. This equation has a dispersive structure as well as the dissipative effect. To see this, we draw back to the most simple original model of the Ferromagnetic spin. The hyperbolic analogue was originally considered earlier by Sideris [60] and Shatah [62].

1.2. The dispersion case. The original model connecting the above equation is the σ spin model of ferromagnetics known as the Heisenberg σ spin model [66]. It is considered as the following discrete setting: Let $S(t,x):\mathbb{R}\times\mathbb{Z}^n\to\mathbb{S}^2$ denote the spin of the ferromagnetic atom located on $\mathbb{Z}^2$. Each spin moves by the reactant only from the closest neighbors. The dynamics is determined by the following equation: Let $h_k=(0,\cdots 0,h,0\cdots 0)$ be a distance vector between each lattice.

$$
\begin{cases}
\partial_t S(t,x_i)=\kappa\sum_{k=1}^{n}S(t,x_i)\wedge\{S(t,x_i+h_k)+S(t,x_i-h_k)\}, & t\in\mathbb{R}, x_i\in\mathbb{Z}^n,\\
|S(t,x_i)|=1, \quad t\in\mathbb{R}, x_i\in\mathbb{Z}^n,\\
S(0,x_i)=S_0(x_i), \quad x_i\in\mathbb{Z}^n,
\end{cases}
\tag{1.2}
$$

where $\wedge$ is the cross product, the positive parameter h is the distance of the each lattice point and κ is a coupling constant. Noting

$$
\partial_t S=\tilde\kappa\sum_{k=1}^{n}S(t,x_i)\wedge\{\frac{S(t,x_i+h_k)-2S(t,x_i)+S(t,x_i-h_k)}{h^2}\}
$$

the continuum approximation is introduced by passing $h\to 0$. One may find by changing the coupling constant appropriately,

$$
\begin{cases}
\partial_t S=\tilde\kappa S\wedge\Delta S, & t\in\mathbb{R}, x\in\mathbb{R}^n,\\
|S(t,x)|=1, \quad t\in\mathbb{R}, x\in\mathbb{R}^n,\\
S(0,x)=S_0(x), \quad x\in\mathbb{R}^n.
\end{cases}
\tag{1.3}
$$

This continuum limit of the spin is called Heisenberg's σ model is corresponding to the equation (1.1) in the case when $\varepsilon=0$ and this is pure dispersive case.

This equation has a strong connection with the approximation theory of the motion of the vortex filament. According to the localized inductive approximation (LIA), the motion of the vortex filament is described by the space-time curve $\gamma(t,x); \mathbb{R}\times\mathbb{R}\to\mathbb{R}^3$ which is governed by the following partial differential equations.

$$
(1.4) \qquad \begin{cases} \partial_t\gamma = \partial_x\gamma\wedge\partial_x^2\gamma & t\in\mathbb{R}, x\in\mathbb{R}, \\ |\gamma(t,x)| = 1, & t\in\mathbb{R}, x\in\mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x\in\mathbb{R}. \end{cases}
$$

This equation was discovered by Da Rios [17] and also re-discovered by Ricca and it has a relation with so called the compressible dispersive Navier-Stokes equations. One can observe that by differentiate the equation and letting $\partial_x\gamma = u$

$$
(1.5) \qquad \partial_t(\partial_x\gamma) = \partial_x^2\gamma\wedge\partial_x^2\gamma + \partial_x\gamma\wedge\partial_x^2(\partial_x\gamma)
$$

and we have

$$
\Rightarrow \partial_t u = u\wedge\partial_x^2 u,
$$

which yields (1.3). One of the remarkable property of this equation is that the equation can be transformed into a complete integrable nonlinear partial differential equation by the famous Hasimoto transform. Applying the Frenel-Serre frame, we may introduce the curvature and torsion along the vortex filament and we define the new unknown function $\psi(t,x)$ such that

$$
\psi(t,x) = \kappa(t,x)\exp\left\{i\int_0^x \tau(t,y)dy - i/2\int_0^t a(\tau)d\tau\right\},
$$

where $\kappa(t,x) = |\partial_x^2\gamma|$: the curvature,

$$
\tau(t,x) = \frac{1}{|\partial_x^2\gamma|^2}\partial_x\gamma\cdot(\partial_x^2\gamma\wedge\partial_x^3\gamma) : \text{the torsion.}
$$

Then $\psi(t,x)$ solves the canonical 1-dimensional nonlinear Schrödinger equation (c- NLS)

$$
i\partial_t\psi + \partial_x^2\psi = \frac{1}{2}|\psi|^2\psi
$$

(cf. [43], [37]). Therefore the case $\varepsilon = 0$ for (1.1) is considered as the 2-dimensional analogue of the dispersive equation. Since the last decade, the Mathematical research of the theory of the vortex filament developed extensively. Fukumoto-Miyazaki [24] derived the equation (1.4) directly from the fluid dynamics and the Biot Savard law and find the higher correction terms appearing when the axial flow or higher di-pole flow are taking into account (cf. Fukumoto-Moffat [25]). The existence and uniqueness theory to those newly discovered equation was done by Nishiyama-Tani [43], Tani-Nishiyama [68]. Further the corresponding equations by the Hasimoto transform are also studied. For the third order modified KdV-NLS equation (also called as Hirota equation), the well posedness problem is studied by Takaoka [67] and the forth order NLS by Segata [57], [58]).

1.3. **The dissipative case.** In contrast with the case $\varepsilon = 0$, the counter part of the limiting case $\varepsilon = 1$ and $\kappa = 0$ is considered as the dissipative case. One can easily observe that $(u\cdot\Delta u) = -|\nabla u|^2$ by u being sphere valued and the equation is exactly corresponding to the harmonic heat flow onto sphere:

$$
(1.6) \qquad \begin{cases} \partial_t u = |u|^2\Delta u - (u\cdot\Delta u)u = \Delta u + |\nabla u|^2 u, & t>0, x\in\mathbb{R}^n, \\ |u(t,x)| = 1, & t\ge 0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n. \end{cases}
$$

In the other word, if we let the coupling constant $\kappa\to 0$ then the equation is connect to the time dependent harmonic map (harmonic heat flow) onto a sphere. (see also [32], [38], [45], [41]).

In general, the harmonic map from the manifold to the manifold is defined by the minimizing problem of the Dirichlet integral and studied by many authors. If the target manifold is a unit sphere, by embedded function u, the harmonic map from a bounded domain $\Omega \subset \mathbb{R}^n$ is described by

$$
(1.7) \qquad \begin{cases} -\Delta u = u(\nabla u, \nabla u), & x \in \Omega \subset \mathbb{R}^n, \\ u(x) = \phi(x), & x \in \partial\Omega. \end{cases}
$$

The heat flow version of the above equation is introduced by Eells-Sampson [19] in order to construct a homotopy from general smooth data to the harmonic map.

$$
(1.8) \qquad \begin{cases} \partial_t u - \Delta u = u|\nabla u|^2, & t > 0, \quad x \in \Omega \subset \mathbb{R}^n, \\ u(x) = \phi(x), & x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}
$$

The above equation is directly obtained from the Landau-Lifshitz model (1.1) by simply erasing the dispersive term.

This equation satisfies formally the Energy inequality as is naturally deduced by its variational origin: Multiplying the equation by $\partial_t u$ and integrated by parts, we see

$$
\|\nabla u(t)\|_2^2 + 2\int_0^t \|\partial_t u(s)\|_2^2 ds \le \|\nabla u_0\|_2^2 \qquad \text{for a.a. } t > 0.
$$

This energy inequality enable us to construct a weak global solution of (1.8) like the Leray-Hopff weak solution to the Navier-Stokes equations (Chen-Struwe [14]). When $n \ge 3$, the solution started from a smooth initial data may develops a singularity with in a finite time (Coron-Gidaglia [16]). When $n = 2$, although the stational solution to (1.8) has a unique smooth solution, the time dependent problem surprisingly develops a singularity with a finite time from a smooth initial data (Struwe [63], Chen-Ding [11], Chang-Ding-Ye [12]).

By a formal observation, the following type of the energy inequality is immediately obtained:

$$
(1.9) \qquad \|\nabla u(t)\|_{L^2(M)}^2 + 2\int_0^t \|\partial_t u(\tau)\|_{L^2(M)}^2 d\tau \le \|\nabla u_0\|_{L^2(M)}^2 \equiv E_0, \quad t \in [0,T].
$$

Based on the above energy inequality, a weak solution is constructed in the space $u \in L^\infty(0,T;\dot{H}^1(M;\mathbb{S}^m))$ with $\partial_t u \in L^2(0,T;L^2(M;\mathbb{S}^m))$. When the dimension of the base manifold M is 2, then Struwe [63] constructed the weak solution which is piecewise smooth in time variable. On the other hand, the existence of a partially regular global weak solution was established by Chen-Struwe [14] by the penalty method. If the initial data is smooth, a smooth local solution exists by using the Bochner type formula (see for example Eells-Sampson [19] and Struwe [63]). This time-local smooth solution is belonging to $u \in W^{1,\infty}(M;\mathbb{S}^m)$ and the maximal existence time is characterized by $\|\nabla u_0\|_\infty$.

The regularity of the weak solution fails in general because of the existence of a blowing up solution for a large initial data. The example for the map from $B_1(0) \subset \mathbb{R}^n$ to a sphere was shown by Coron-Ghidaglia [16] for $n \ge 3$ and Chang-Ding-Ye [12] for $n = 2$. However, some smallness assumption on the initial data or integrability condition on the solution itself are capable to give the regularity.

In fact in [45], it is proved that for a time-local smooth solution $u : [0,T_0) \times \mathbb{R}^n \to \mathbb{S}^m$ of (2.1) for some T_0 can be extended over $[T_0, T_0 + T')$ for some $T' > 0$, provided

$$
(1.10) \qquad \int_0^{T_0} \|\nabla u(t)\|_{BMO}^2 dt < \infty.
$$

Here BMO is the space of a function having *bounded mean oscillations* defined by

$$
f \in L^1_{loc}(\mathbb{R}^n), \quad \|f\|_{BMO} \equiv \sup_{x,R} \frac{1}{|B_R|}\int_{B_R(x)} |f(y) - \bar{f}_{B_R(x)}| dy < \infty,
$$

where $\bar{f}_{B_R}$ is the average of f over $B_R(x) = \{y \in \mathbb{R}^n; |x-y| < R\}$.

The above results can be compared with the existing blow-up solutions for (2.1). Coron-Ghidaglia [16] and Chen-Ding [11] showed that there exists a finite time blowing up solution to (2.1) for $n \geq 3$. For $n = 2$, Chang-Ding-Ye [12] constructed a blowing up solution from a smooth data (see for the regularity of the stationary harmonic maps, Schoen-Uhlenbeck [56], Hélen [32], Evans [21] and for the non-stationary case, Feldman [22]). The solution satisfies

$$\int_0^T \|\nabla u(t)\|_\infty^\theta dt = \infty \quad (\theta > 1),$$

where $T > 0$ is the expected blow-up time.

An analogous situation can be observed in the theory of a weak solution to the incompressible fluid mechanics. For the viscous incompressible fluid governed by the Navier-Stokes equation;

$$(1.11) \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, & t > 0, x \in \mathbb{R}^n, \\ \quad \text{div } u = 0, \quad t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), \end{cases}$$

it is well known that there exists a global weak solution u based on an analogous energy inequality to (2.1) due to Leray [40];

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u_0\|_2^2.$$

Although a full regularity of the weak solution to (1.11) still remains open, there are some sufficient conditions for the regularity of the solution in terms of a semi-norm invariant under the scaling that maintain the equations invariant. For the Navier-Stokes case, the equation is invariant under the scaling; $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$, $p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$ $(\lambda > 0)$. Hence a criterion by the space-time norms such as

$$\int_0^T \||\nabla|^\alpha u(t)\|_p^\theta dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 1 + \alpha, \quad 2 \leq \theta < \infty$$

gives the regularity of the weak solution. This is known as the Serrin condition (Prodi [52], Ohyama [44], Serrin [61], Giga [26], Beirão da Veiga [3]). For non-viscid case, there are some corresponding conditions known as the Beale-Kato-Majda's blow up criterion [2] and extended by several authors [39], [38]). By observing the analogous scaling $u \to u_\lambda = u(\lambda^2 t, \lambda x)$ that preserves the equation (2.1), it is expected that there is a regularity criterion for (2.1) under the condition;

$$\nabla u \in L^\theta(0,T; L^p(\mathbb{R}^n)), \qquad \frac{2}{\theta} + \frac{n}{p} = 1, \quad n < p \leq \infty.$$

Those conditions is corresponding to the Serrin criterion and enough to show the regularity of the strong solution to (2.1).

In view of the limiting condition to (1.11) the Leray-Hopf weak solution to (1.11) is regular under the corresponding regularity assumption for vorticity:

$$\int_0^\infty \|\text{rot } u(\tau)\|_{BMO} d\tau < \infty.$$

Hence it is expected that under the analogous regularity condition such as (1.10), certain weak solutions to (2.1) are shown to be regular. This is shown in [45] as an extension problem for the smooth (strong) solution for (2.1). However to show (1.10) being the criterion for a weak solution to (2.1) is not so straight forward, indeed. For the case of the Navier-Stokes equation, the proof is heavily depending on the fact that any weak solution corresponds the smooth solution for certain time interval. This partial uniqueness result fails in general for a weak solution to (2.1) by Bertsch-Dal Passo-Pisante [5] (cf. Freire [23]).

2. The Harmonic Heat Flow

One of the regularity class for the weak solution to

$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u, & t > 0, x \in \mathbb{R}^n, \\ |u(t,x)| = 1, & t \geq 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

(2.1) is the class introduced by Struwe [63]: $V = \{u : M \to \mathbb{R}^2 : \nabla u \in L^\infty(0,T; L^2(M)), \partial_t u, \Delta u \in L^2(0,T; L^2(M))\}$. where M denote the 2-dimensional Riemannian manifold. Our aim here is to extend this class larger when $dim\, M = 2$ in terms of the *mean oscillation* of the solution. For this purpose, we recall the definition of the class of the *Bounded Mean Oscillation.*

Definition. Let u be a map from $\mathbb{R}^n$ to a unit sphere $\mathbb{S}^m$. A map u is in a bounded mean oscillation over $\mathbb{R}^n$; $BMO = BMO(\mathbb{R}^n; \mathbb{S}^m)$ if

$$\|u\|_{BMO(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n, R>0} \frac{1}{|B_R(x)|} \int_{B_R(x)} |u(y) - \bar{u}_{B_R}| dy < \infty,$$

where $B_R(x)$ is a ball on $\mathbb{R}^n$ with radius $R > 0$ and

$$\bar{u}_{B_R} = \frac{1}{|B_R|} \int_{B_R(x)} u(y) dy$$

with $|B_R|$ is the geodesic volume of the ball.

However, we may show certain kind of weak solutions to (2.1) are regular under the same assumption (1.10) when we restrict the base manifold as in 2 dimensions. To state this precisely, we introduce the definition of the weak solution:

Definition. A map $u : \mathcal{M} \to \mathbb{S}^m$ is a weak solution of (2.1) over $[0, T)$ if

(1) $\nabla u \in L^\infty(0,T; L^2(\mathcal{M}))$ and $\partial_t u \in L^2(0,T; L^2(M))$.
(2) $\|\nabla u(t)\|^2_{L^2(\mathcal{M})} \leq \|\nabla u_0\|^2_{L^2(\mathcal{M})} \equiv E_0$ holds for all $t \geq 0$.
(3) u satisfies the harmonic heat flow in the sense of distribution:
For all $\phi \in C^1_0([0,T); C^\infty_0(\mathcal{M})^n)$,

$$-\int_0^T u(\tau) \cdot \partial_t \phi(\tau) dx d\tau + \int_0^T (\nabla u(\tau), \nabla \phi(\tau))_g d\tau = \int_0^T u(\nabla u, \nabla u)_g \phi(\tau) dx d\tau + u_0 \cdot \phi,$$

where $(\cdot,\cdot)_g$ is the L^2 inner product on $\mathcal{M}$.

The existence of a weak solution satisfies the above first two conditions are proved in most general case by Chen-Struwe [14]. The strong solution that has finite point singularity has been discussed by Struwe [61], Schoen-Uhlenbeck [56].

We suppose an extra regularity condition to the weak solution which is associated with the scaling invariant norm involving BMO which is shown for weak solution in Misawa-Ogawa [41].

Theorem 2.1 (Limiting regularity criterion [41]). *Let u be a weak solution to (2.1) defined in the above. If, for some $T > 0$, the solution u satisfies*

$$\int_0^T \|\nabla u(\tau)\|^2_{BMO(\mathbb{R}^2)} d\tau < \infty, \tag{2.2}$$

then the solution is regular up to $t = T$. Namely,
$u \in C((0,T]; W^{1,\infty}(\mathbb{R}^2; \mathbb{S}^2)) \cap C^1((0,T]; W^{2,\infty}(\mathbb{R}^2; \mathbb{S}^2))$. *In the other words, if the solution blows up at some time $t \leq T$, then*

$$\int_0^T \|\nabla u(\tau)\|^2_{BMO(\mathbb{R}^2)} d\tau = \infty.$$

In particular, if for any $t > 0$ and some $T > 0$

$$\int_t^{t+T} \|\nabla u(\tau)\|^2_{BMO(\mathbb{R}^2)} d\tau < \infty, \tag{2.3}$$

then the weak solution is globally regular.

The key ingredients to show the regularity is twofold. One is to employ a critical type of the Sobolev inequalities. Brezis-Gallouet [6] and Brezis-Wainger [8] firstly showed the following inequality: For $s > n/p$,

$$\|f\|_\infty \le C\left(1 + \||\nabla|^{n/p} f\|_p (\log(e + \|f\|_{W^{s,p}}))^{1-1/p}\right) \tag{2.4}$$

for $f \in W^{s,p}(\mathbb{R}^n)$. Analogous but vector version of this inequality was found by Beale-Kato-Majda [2]: For $f \in W^{s,p}(\mathbb{R}^n; \mathbb{R}^n)$ with div $f = 0$,

$$\|\nabla f\|_\infty \le C\left(1 + \|\nabla f\|_2 + \|\mathrm{rot}\ f\|_\infty \log(e + \|f\|_{W^{s,p}})\right) \tag{2.5}$$

and used for the regularity theory of the fluid mechanics. Kozono-Taniuchi [39] generalized the above inequality involving BMO; for $s > n/p + 1$, $f \in W^{s,p}$ with div $f = 0$,

$$\|\nabla f\|_\infty \le C\left(1 + \|\mathrm{rot}\ f\|_{BMO} \log(e + \|f\|_{W^{s,p}})\right) \tag{2.6}$$

and Kozono-Ogawa-Taniuchi [38] in Besov spaces. We first introduce a generalized version of the critical Sobolev inequality in the Lizorkin-Triebel space (cf. Ogawa [45]) that includes all the above inequalities. We first give the sharp version of the inequality shown in [45].

Lemma 2.2 (Sharp version of logarithmic inequality [45]). *For any $p, \rho, \sigma \in [1, \infty]$, $q \in [1, \infty)$, $\nu \le \sigma_1, \sigma_2$, $\nu < \rho$ and $\gamma > 0$, there exists a constant C which is only depending on n, p such that for $f \in \dot{F}^{n/p+\gamma}_{p,\sigma_1}(\mathbb{R}^n) \cap \dot{F}^{n/p-\gamma}_{p,\sigma_2}(\mathbb{R}^n)$, we have for $\gamma < \gamma'$*

$$\|f\|_{\dot{F}^0_{\infty,\nu}} \le C\|f\|_{\dot{F}^0_{\infty,\rho}}\left(1 + \left(\frac{1}{\gamma}\log^+ \frac{\|f_+\|_{\dot{F}^{n/p+\gamma'}_{p,\sigma_1}} + \|f_-\|_{\dot{F}^{n/p-\gamma'}_{p,\sigma_2}}}{\|f\|_{\dot{F}^0_{\infty,\rho}}}\right)^{1/\nu - 1/\rho}\right), \tag{2.7}$$

where $f_+ = \sum_{j\ge 0} \phi_j * f$ and $f_- = \sum_{j \le 0} \phi_j * f$.

REMARK 1. In the theorem, the assumption $\gamma > 0$ is essential. The analogous version of the inequality (2.7) in the Besov space was proved in Ogawa-Taniuchi [48].

The relation between the Lizorkin-Triebel spaces and $BMO(\mathbb{R}^n)$ is well understood. Namely $\dot{F}^0_{\infty,2}(\mathbb{R}^n) \simeq BMO(\mathbb{R}^n)$. In another word, there exists a constants C such that

$$C^{-1}\|f\|_{\dot{F}^0_{\infty,2}} \le \|f\|_{BMO} \le C\|f\|_{\dot{F}^0_{\infty,2}}$$

which is is due to Peetre and Triebel [69] (see also Bui Hui Qui [9]).

From (2.7) and the equivalence between $\dot{F}^0_{\infty,2}(\mathbb{R}^n) \simeq BMO(\mathbb{R}^n)$ and $\dot{F}^0_{\infty,\infty}(\mathbb{R}^n) \simeq \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ it is explicitly shown that the difference between $L^\infty(\mathbb{R}^n)$, $BMO(\mathbb{R}^n)$ and the Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ as follows. This is a version of the sharp form of the Kozono-Taniuchi inequality (2.6).

Proposition 2.3. *If $\nabla f \in W^{1,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $n < q$, we have*

$$\|\nabla f\|_\infty \le C(q)\left(1 + \|\nabla f\|_{BMO}\left(\log^+ (\|\nabla f\|_{W^{1,q}} + \|f\|_\infty)\right)^{1/2}\right). \tag{2.8}$$

It then, turns out that the second exponent of those spaces gives an explicit dependence of the power of the logarithmic term to the higher regularity, which reflects hypotheses on the integral exponent in the time direction of those criteria. In the following section, we show a refined version of the Beale-Kato-Majda and Kozono-Taniuchi type inequalities and give some discussion. Then in the successive section, we recall the regularity criterion for the strong (smooth) solution to (2.1).

To extend the above observation into a general weak solution, we need to employ the second ingredient which is a version of the monotonicity formula and so called ε−regularity argument by means of *the mean oscillation* of the gradient of the solution.

Proposition 2.4 ([41]). *Let u be a smooth solution of (2.1). For any fixed $\delta > 0$, $T > 0$ and $r > 0$ we set a time interval $I_{\delta^{1/2}r}(T) = (T - \delta r^2, T)$. Then for any $x_0 \in \mathbb{R}^2$, there exists an absolute constant $C > 0$ such that for any $r \in (0, R)$, we have*

$$\begin{aligned}\int_{I_{\delta^{1/2}r}(T)} &\left(\frac{1}{\pi r^2}\int_{B_r(x_0)} |\nabla u(\tau) - \overline{\nabla u}_{B_r(x_0)}(\tau)|^2 dx\right) d\tau \\ &\le \int_{I_{\delta^{1/2}R}(T)} \left(\frac{1}{\pi R^2}\int_{B_R(x_0)} |\nabla u(\tau) - \overline{\nabla u}_{B_R}(\tau)|^2 dx\right) d\tau + C\,\delta\, E(u_0),\end{aligned}$$

where $B_R(x_0) = \{|x - x_0| < R\}$.

The above proposition is a variant of the known monotonicity formula for a smooth solution of the harmonic heat flow. The advantage of the above formula is the monotonicity is in fact realized in the level of the mean oscillation of the gradient of the solution so that it is suitable for our purpose. Using Proposition 2.4 we may derive so called ε regularity theorem by the mean oscillation. Namely there exist some small constants $\varepsilon_0 > 0$ and $R_0 > 0$ such that if for some $R < R_0$,

$$\frac{1}{R^2}\int_{t_0 - R^2}^{t_2}\int_{B_R(x_0)} |\nabla u(t,x) - \overline{\nabla u}_R|^2 dx dt < \varepsilon_0$$

with $\overline{\nabla u}_R$ is roughly speaking the average of ∇u over $(t_0 - R^2, t_0) \times B_R(x_0)$, then the solution is regular around the space time point (t_0, x_0). This is an improved version of the existing regularity criterion (see [64]) and generally true even for the higher dimensional case (cf. [41]).

3. The Schrödigner map

According to [66], the Heisenberg model (1.3) can be interpreted as a kind of a derivative nonlinear Schrödinger equations.

Let $\pi : \mathbb{S}^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$

$$S = (S_1, S_2, S_3) = \left(\frac{Reu}{1 + |u|^2}, \frac{Imu}{1 + |u|^2}, \frac{1 - |u|^2}{1 + |u|^2}\right)$$

be the standard stereo graphic projection and the solution of (1.3) transformed into the following semi-linear Schrödinger equation of the derivative type.

$$\begin{cases} i\partial_t u + \Delta u = \dfrac{2\bar{u}(\nabla u, \nabla u)}{1 + |u|^2}, & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(0,x) = \dfrac{S_{1,0}(x) + iS_{2,0}(x)}{1 + |S_{3,0}(x)|^2}, & x \in \mathbb{R}^n. \end{cases} \tag{3.9}$$

There are many research on the nonlinear Schrödinger type equation with the derivative nonlinear terms ([30], [50]). Among others, Sulem-Sulem-Bardos [66] has also considered this equation and showed the time local well-posendess in the Sobolev space $H^{n/2+1}(\mathbb{R}^n)$ with $(n \ge 3)$.

In fact, the above equation is originally derived from the σ spin model initially considered as the model of the nonlinear hyperbolic equation. The earliest work on this direction is due to Shatah [59] and Sideris [60] (cf. [65]). Later on, Cheng-Uhlenbeck-Shatah ([13]) re-formulated this equation with the geometric point of view and consider the equation as a map into the general Riemannian manifold. They considered the equation when $n = 1$ and $n = 2$ with the axially symmetric case.

Concerning the Schrödinger map with the target manifold as a unit sphere, it is formulated by using the covariant derivative

$$\mathcal{D}_{x_i} = \partial_i + \frac{2\bar{u}\partial_i u}{1+|u|^2},$$

then the σ spin model (3.9) is expressed by the following way.

$$\begin{cases} i\partial_t u = \mathcal{D}_i \partial_i u, \\ u(0,x) = u_0(x), \end{cases} \tag{3.10}$$

where the covariant derivative satisfies the condition as the well-known Levi-Cibita connection

$$\mathcal{D}_k \partial_j u = \mathcal{D}_j \partial_k u. \tag{3.11}$$

The nature of the solution to the Schrödinger map heritages the property of the solution to the harmonic heat flow as well as the dispersive structure of the solution from the Schrödinger part. There are several result that the case of the target manifold is not a sphere but some other particular manifolds.

- Grillakis-Stefanopoulos [29] considered the equation (3.9) corresponding to the one for the target is $\mathbb{S}^2$ and also $\mathbb{H}^2$.
- M.Tsutsumi [70] considered the one dimensional ferromagnetic spin model to the Lobachevski plain $\mathcal{L} = \{u = (u_1, u_2, u_3) | |u_1|^2 + |u_2|^2 - |u_3|^3 = -1, u_3 > 0\}$ and constructed a time global solution $S(t,x); \mathbb{R} \times \mathbb{T}^1 \to \mathcal{L}$ by showing the higher order conservation law of the energy.
- N. Koiso [37] generalized the vortex filament equation from a manifold to a Keher manifold and reduce the equation into the nonlinear Schrödinger equation..

4. 2-DIMENSIONAL CASE

In what follows we consider the initial value problem for the Schrödinger map (3.9) in the two special dimension $n = 2$. Practically this situation corresponds a model for a simulation of the magnetic tape of media.

For this special case, the function space for solving the equation required the larger space so that it is not included into L^∞. Since the principal part of the equation is the Schrödinger type, the suitable and the best possible choice of the function space is the Sobolev space based on L^2 namely $H^s(\mathbb{R}^2)$ and for the above mentioned purpose, H^1 is the critical space. Indeed, the smaller spaces than H^1, say H^s with $s > 1$ are all included into L^∞ so that the original spin can not reach the south pole under this setting of the problem. Considering the original problem, it is natural to consider the case when the map covers whole $\mathbb{S}^2$.

However the corresponding Schrödinger map in the Sobolev space $H^s(\mathbb{R}^2)$ $(s > 1)$ never can reach the South pole since this space is embedded into L^∞ and this shows that the image never reach the infinity point. This problem is closely related to the local well-posedness problem for the Schrödinger map and for the two dimensional case, it is critical to construct the local solution in the critical space $H^1(\mathbb{R}^2)$ since this space gives no restriction on the sides of solution by $H^1(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2)$. Indeed, this space is the critical space by the scaling point of view, namely $\dot{H}^1(\mathbb{R}^2)$ is the invariant space for the scaling $u(t,x) \to u(\lambda^2 t, \lambda x)$.

Nohmod-Stefanov-Uhlenbeck [42] has introduced a proper gauge transform (the Coulomb gauge) and considered the transformed equation called as the modified Schrödinger map from the above original Schrödinger map and discuss the time local well-posedness. Let $\psi(t,x)$ be a phase function of the gauge such that

$$\nabla_j \psi - 2Im \frac{u \nabla_j \bar{u}}{1+|u|^2} \equiv -a_j, \tag{4.1}$$

Coulomb (Hodge) gauge: div $a = 0$

and by using the solution u for (3.9), they define a new function $e^{i\psi}\nabla_j u \to u_j$ by the gauge transform. It then follows from the above definition that ψ is explicitly given by

$$\psi(t,x) = -2(-\Delta)^{-1}\text{div}\,\frac{Im(u\nabla_j\bar{u})}{1+|u|^2}.$$

and the corresponding equation to (3.9) is introduced as the following modified version of the Schrödinger map:

$$\begin{cases} i\partial_t u + \Delta u = -2ia\cdot\nabla u - Au + 2Im(\bar{u}\otimes u)u + a_0 u, & t\in\mathbb{R}, x\in\mathbb{R}^2, \\ u(0,x) = u_0(x), \quad x\in\mathbb{R}^2, \end{cases} \tag{4.2}$$

where

$$\vec{a} = (a_1,a_2) = 4Im\,\text{div}\,(-\Delta)^{-1}(u\otimes u), \qquad A = |\vec{a}|,$$
$$a_0 = 4(-\Delta)^{-1}\left\{\nabla_i\nabla_j Re(u_i u_j) - \frac{1}{2}\Delta|u|^2\right\}.$$

In [42], they treat this new equation (4.2) and established the time-local well posedness of this equation by using the Bourgain method of the restriction norm. Namely they showed that for the initial data $u_0\in H^s(\mathbb{R}^2)$ $s>0$, there exists a time local solution in the same Sobolev class. The result is corresponding to the solution in $H^{2+\varepsilon}(\mathbb{R}^2)$ for the original Schrödinger map.

Recently, J. Kato [33] (and [34]) investigate the above modified equation and give a existence and uniqueness of the solution in the larger function space. Namely the weak solution in the class $H^{3/2+\varepsilon}(\mathbb{R}^2)$ is unique. He used the argument due to Koch-Tzvetkov [36] for the Benjamin-Ono equation.

5. Solvability in the Energy class

In what follows we consider the Schrödinger map (3.9) under the different type of gauge from the one used in [42].

The corresponding new equation to (3.9) is obtained by a new gauge transform that basically obtained the following strategy. We choose ta new gauge phase function so that t he worst nonlinear term appeared in the modified Schrödinger map is canceled. This is along the idea due to Hayashi [30] and Doi [18] (see also [31], [50] and [49]), however since the problem is nonlinear, this new gauge may cause a new nonlinear term that may be worse than the original one. First of all we differentiate the equation (3.9) and let $v=\nabla u$ as a new unknown vector function of (t,x). Then the equation can be read as the system such that

$$\begin{cases} i\partial_t v + \Delta v = \dfrac{4\bar{u}}{1+|u|^2}v\cdot\nabla v + \dfrac{2(v\cdot v)(\bar{v}_j - \bar{u}^2 v_j)}{(1+|u|^2)^2}, & t>0, x\in\mathbb{R}^2, \\ v(0,x) = \nabla u_0(x), \quad x\in\mathbb{R}^2. \end{cases} \tag{5.1}$$

The choose a new gauge as $\theta(t,x)$ and for $E(u,v) = e^{i\theta(t,x)}$, we let

$$w_j = E^{-1}v_j. \tag{5.2}$$

The equation that w solves is

$$\begin{cases} i\partial_t w + \Delta w = F(v,w)w - 2i(\nabla\theta\cdot\nabla)w + \dfrac{4\bar{u}E}{1+|u|^2}(w\cdot\nabla)w \\ \qquad + \dfrac{4i\bar{u}wE}{1+|u|^2}(w\cdot\nabla)\theta + H(u,w,E), \quad t>0, x\in\mathbb{R}^2, \\ w_j(0,x) = E^{-1}\nabla_j u_0(x), \quad x\in\mathbb{R}^2, \end{cases} \tag{5.3}$$

where

$$F(u,v) = \partial_t\theta - i\Delta\theta + (\nabla\theta,\nabla\theta),$$
$$H_j(u,w,E) = \frac{2(w\cdot w)}{1+|u|^2}\bar{w}_j|\bar{E}|^2 - \frac{2\bar{u}^2(w\cdot w)}{1+|u|^2}w_jE^2.$$

We then choose the phase of the gauge θ so that the most difficult term the second term of the right hand side of the equation (5.4) can be canceled:

$$2i\nabla_k\theta = \frac{4\bar{u}}{1+|u|^2}w_kE = \frac{4\bar{u}}{1+|u|^2}v_k.$$

Certainly this chose of gauge can cancel the worst term, however it may appear more complex term $F(u,w)$ that may things more complicated. The essential fact here is that we may show the following fact:

Lemma 5.1. *The nonlinear term $F(u,w)$ appeared in (5.4) is expressed as follows.*

$$F(v,w) = -\frac{6\bar{u}^2}{(1+|u|^2)^2}(v\cdot v) + 4\nabla_k\nabla_l(-\Delta)^{-1}[w_k\otimes w_l].$$

Therefore the transformed equation (5.4) has no term that may cause the derivative loss. The original equation (3.9) can be solved as regarding the solution of the system:

$$\text{(5.4)}\qquad \begin{cases} i\partial_t w_j + \Delta w_j = 2(w\cdot w)\bar{w}_j + 4w_j\nabla_k\nabla_l(-\Delta)^{-1}[w\otimes\bar{w}], & t>0, x\in\mathbb{R}^2,\\ u(0,x) = u_0(x), & x\in\mathbb{R}^2,\\ w_j(0,x) = E^{-1}\nabla_j u_0(x), & x\in\mathbb{R}^2. \end{cases}$$

This system is essentially decoupled and can be solved for the second equation in the space $C(0,T;L^2(\mathbb{R}^2))$ and we can obtain the time local wellposedness. By this observation we are able to show the following theorem:

Theorem 5.2. *For $u_0\in H^1(\mathbb{R}^2)$, the corresponding equation (5.4) to (3.9) that is obtained by the transform (5.2) is time locally well-posed in the class $(L^2(\mathbb{R}^2))$ and satisfies the L^2 conservation law:*

$$\|w(t)\|_2 = \|E(u_0)^{-1}\nabla u_0\|_2$$

for all $t\in(0,T)$, where $T>0$ is the maximal existence time. If the data $E(u_0)^{-1}\nabla u_0$ is small in L^2, then the solution exists globally in time.

The above theorem states that the transformed equation is time locally wellposed in the corresponding class where the original Schrödinger map (3.9) is considered in the energy class $H^1(\mathbb{R}^2)$. Especially the equation (3.9) has a unique time local solution in $H^1(\mathbb{R}^2)$ and if the data in this class is small then the solution globally exists. In view of the equation (5.4) the worst derivative term is just canceled out and therefore the transform (5.2) may considered as the two dimensional Hasimoto transform for the Schrödinger map. We should note that for $n=2$, the nonlinear term of the second equation is the critical order for solvability in L^2 space. Yet one may derive the time local well posedness for the above equation in this situation by the method of Y.Tsutsumi [71] (see also Cazenave-Weisslar [10]). The transform (5.2) is somehow corresponding to the 2-dimensional Hasimoto transform as it can be canceled the nonlinear term that involving the derivative term.

Let us recall the fundamental result on the linear Schrödinger equation. That is so called the Strichartz-Brenner space time estimate of L^p type.

Definition. Let $e^{i\Delta t}$ be two dimensional linear Schrödinger evolution group. If a pair of the exponents (θ,p) verifies

$$\frac{2}{\theta}+\frac{2}{p}=1,\quad 2\le p<\infty,$$

then it is called as L^2-admissible. See for example, Ginibre-Velo [27], [28], Keel-Tao [39].

For a general nonlinear term $F(u)$, the corresponding integral equation:

$$u(t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-s)\Delta}F(u(s))ds$$

yields a map from a certain complete metric space $\Xi; X_T \to X_T$ where

$$\Xi[u](t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-s)\Delta}F(u(s))ds$$

and existence and wellposedness problem can be derived from the existence of the unique fixed point of the above map. Underlying fact is that the space X_T is chosen so that the map is closed in the metric by the Strichartz estimate.

If the nonlinear term $F(u)$ is expressed as the power of u of order p, there is a standard argument by choosing L^2 admissible pair as $(\theta, q) = (\theta, p+1)$ (Ginibre-Velo, Lin-Strauss, Baillon-Cazenave-Fuguira). For our case, let $n = 2$ and choose L^2 admissible as $(\theta, q) = (4, 4)$ and

$$X_T = \left\{ f; [0,T] \times \mathbb{R}^2 \to \mathbb{C}; \|f\|_{L^4(I;L^4(\mathbb{R}^2))} \le \frac{1}{2} \right\},$$

where $I = [0,T]$ and $M = C\|u_0\|_2$ with the metric

$$d(u,v) = \|u - v\|_{L^4(I;L^4)},$$

then X_T is a complete metric space.

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