

Weak topologies, and determining covers

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We assume that spaces are regular T_1 , and maps are continuous and onto.

For a cover \mathcal{P} of a space X , X is determined by \mathcal{P} [6], if X has the *weak topology* with respect to \mathcal{P} [3]; that is, $G \subset X$ is open in X if $G \cap P$ is open in P for each $P \in \mathcal{P}$. Here, we can replace “open” by “closed”. We call such a cover \mathcal{P} a **determining cover** in [20].

We recall that a space X is respectively a *sequential space* [4]; *k-space*; *quasi-k-space* [11] if X has a determining cover by (compact) metric subsets; compact subsets; countably compact subsets. Sequential spaces are *k-spaces*, and *k-spaces* are *quasi-k-spaces*.

As is well-known, every sequential space; *k-space*; *quasi-k-space* is respectively characterized as a quotient space of a (locally compact) metric space; locally compact (paracompact) space; *M-space*.

Let \mathcal{P} be a collection of subsets of a space X . Then, \mathcal{P} is *closure-preserving* (abbreviated by CP), if for any subfamily \mathcal{P}' of \mathcal{P} , $cl(\bigcup\{P : P \in \mathcal{P}'\}) = \bigcup\{clP : P \in \mathcal{P}'\}$. Also, \mathcal{P} is *hereditarily closure-preserving* (abbreviated by HCP), if for any subcollection $\mathcal{P}' = \{P_\alpha : \alpha\}$ of \mathcal{P} , and any $\{A_\alpha : \alpha\}$ such that $A_\alpha \subset P_\alpha$, the collection $\{A_\alpha : \alpha\}$ is CP.

For a closed cover \mathcal{F} of a space X , X is *dominated by* \mathcal{F} [7] if \mathcal{F} is a CP cover, and any $\mathcal{P} \subset \mathcal{F}$ is a determining cover of the union of \mathcal{P} . (Sometimes, we also say that X has the *Whitehead weak topology*; *Morita weak topology* (in the sense of [9]); or *hereditarily weak topology*, with respect to \mathcal{F}). We call such a closed cover \mathcal{F} a **dominating cover** in [20].

A space X having an increasing determining cover $\{X_n : n \in N\}$ is called the *inductive limit* of $\{X_n : n \in N\}$. When the X_n are closed in X , $\{X_n : n \in N\}$ is a dominating cover of X . Also, every CW-complex has a dominating cover by compact metric subsets.

Open covers \Rightarrow *Determining covers* \Leftarrow *Dominating covers* \Leftarrow *HCP closed covers* \Leftarrow *Locally finite closed covers*.

Every space having a determining cover by sequential spaces (resp. *k-spaces*; *quasi-k-spaces*) is a sequential space (resp. *k-space*; *quasi-k-space*).

While, every space having a dominating cover by paracompact spaces (resp. normal spaces) is paracompact (resp. normal); see [7] or [10].

Concerning “*preservations*” of weak topologies, we have the following natural questions (Q1), (Q2) and (Q3), and the same questions which are replaced “determining” by “dominating”.

(Q1) Let $f : X \rightarrow Y$ be a map, and let \mathcal{P} be a determining cover of X (resp. Y). Under what conditions, is $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ (resp. $f^{-1}(\mathcal{P}) = \{f^{-1}(P) : P \in \mathcal{P}\}$) a determining cover of Y (resp. X) ?

(Q2) Let \mathcal{P} be a determining cover of a space X . For a (or any) subset $S \subset X$, under what conditions, is $\mathcal{P}|S = \{P \cap S : P \in \mathcal{P}\}$ a determining cover of S ?

(Q3): For each $i = 1, 2$, let \mathcal{P}_i be a determining cover of a space X_i . Under what conditions, is $\mathcal{P}_1 \times \mathcal{P}_2 = \{P_1 \times P_2 : P_i \in \mathcal{P}_i\}$ a determining cover of $X_1 \times X_2$?

In [20], we gave some related answers to the question (Q3) (containing countable products of weak topologies), and their applications to products of paracompact spaces. For products of weak topologies (determining covers), see [19]. In this paper, let us give some related answers to the questions (Q1) and (Q2) in Section 1 and 2, respectively. Related to (Q3), we also give some results on countable products of spaces having certain determining covers in Section 3, containing additional matters to [20].

We recall some elementary facts which will be used in this paper. For basic matters on weak topologies, see [17] or [18], for example.

Fact A: (1) Let \mathcal{C} be a determining cover of X . Let \mathcal{P} be a cover of X . If each element of \mathcal{C} is contained in some element of \mathcal{P} , then \mathcal{P} is a determining cover of X .

(2) Let $\{P_\alpha : \alpha\}$ be a determining cover of X . If each P_α has a determining cover \mathcal{P}_α , then $\bigcup\{\mathcal{P}_\alpha : \alpha\}$ is a determining cover of X .

(3) Let \mathcal{P} be a determining cover of X . If S is a closed or open subset of X , then $\mathcal{P}|S$ is a determining cover of S .

(4) For a determining cover \mathcal{P} of a space X^ω , $\mathcal{P}_1 \times \mathcal{P}_2 \times \dots$ is a determining cover of X^ω , where $\mathcal{P}_i = P_i(\mathcal{P})$ for the projection P_i from X^ω onto the i -th coordinate space X .

A cover \mathcal{P} of X is *point-countable* if every $x \in X$ is in at most countably many $P \in \mathcal{P}$. A decreasing sequence (A_n) of non-empty subsets of X is a q -

sequence (resp. k -sequence) [8], if $C = \bigcap \{A_n : n \in N\}$ is countably compact (resp. compact) in X , and each open subset U with $C \subset U$ contains some A_n (equivalently, for any $x_n \in A_n$, $\{x_n : n \in N\}$ has an accumulation point in C).

Fact B: (1) Let \mathcal{P} be a point-countable determining cover of X . Then, for each q -sequence (A_n) in X , some A_n is contained in a finite union of elements of \mathcal{P} ([14, Lemma 6]).

(2) Let $\mathcal{F} = \{X_\alpha : \alpha \leq \gamma\}$ be a dominating cover of X . For each $\alpha \leq \gamma$, let $L_\alpha = X_\alpha - \bigcup \{X_\beta : \beta < \alpha\}$. Then $\{cl L_\alpha : \alpha \leq \gamma\}$ is a determining cover of X such that, for each q -sequence (A_n) in X , some A_n meets only finitely many L_α (cf. [16, Lemma 2.5]).

1. Maps

Example 1.1. (1) An open map $f : X \rightarrow Y$ with each $f^{-1}(y)$ at most two points, and X has a discrete, closed and open cover \mathcal{F} by compact subsets, but $f(\mathcal{F})$ is not a CP cover (hence, not a dominating cover).

(2) An open map $g : X \rightarrow Y$ with each $g^{-1}(y)$ at most two points, and Y has a countable determining cover \mathcal{F} by convergent sequences (or, a dominating cover by compact metric subsets), but $g^{-1}(\mathcal{F})$ is not a determining cover of X .

Theorem 1.2. (1) Let $f : X \rightarrow Y$ be a quotient map. If \mathcal{P} is a determining cover of X , as is well-known, $f(\mathcal{P})$ is a determining cover of Y .

(2) Let $f : X \rightarrow Y$ be a closed map. Then the following hold.

(a) If \mathcal{F} is a dominating cover of X , $f(\mathcal{F})$ is a dominating cover of Y .

(b) If \mathcal{P} is a determining (resp. dominating) cover of Y , $f^{-1}(\mathcal{P})$ is a determining (resp. dominating) cover of X ([13, Lemma 1.2]).

Corollary 1.3. Let $f : X \rightarrow Y$ be a closed map such that each $f^{-1}(y)$ is compact (resp. countably compact; first countable). Then X is a k -space ([1]) (resp. quasi- k -space; sequential space ([13])) if (and only if) Y is so respectively.

Corollary 1.4. Let $f : X \rightarrow Y$ be a map. Then the following hold.

(1) Let X be a k -space. If \mathcal{P} is a determining cover of Y , then $f^{-1}(\mathcal{P})$ is a determining cover of X .

(2) Let X be a quasi- k -space. If \mathcal{F} is a dominating (resp. point-countable closed) cover of Y , then $f^{-1}(\mathcal{F})$ is a dominating (resp. point-countable closed) cover of X .

The author doesn't know whether the above (1) remains true under X being a quasi- k -space. This is affirmative if any countably compact subset of Y is closed (as Y is a sequential space, or a space whose points are G_δ -sets, for example).

2. Subsets

For an open (resp. HCP closed) cover \mathcal{P} of X , $\mathcal{P}|S$ is a determining (resp. dominating) cover of S for any $S \subset X$. But, we have the following example. Here, the Arens' space S_2 is the space obtained from the disjoint union $\Sigma\{L_n : n = 0, 1, \dots\}$ of the convergent sequence $\{1/n : n \in N\} \cup \{0\}$ by identifying each $1/n \in L_0$ with $0 \in L_n$ ($n \geq 1$). The quotient space S_2/L_0 is called the *sequential fan* which is denoted by S_ω .

Example 2.1. The Arens' space S_2 has the obvious increasing and dominating countable cover \mathcal{F} by compact metric subsets, but for $S = S_2 - \{1/n \in L_0 : n \in N\}$, $\mathcal{F}|S$ is not a determining cover of S .

A space is *Fréchet*, if whenever $a \in clA$, then there exists a sequence in A converging to the point a . We recall that a space X is a k' -space [1], if whenever $a \in clA$, then there exists a compact subset K of X such that $a \in cl(A \cap K)$. Let us recall other related spaces due to [8]. A space X is a *countably bi-quasi- k -space* if, whenever $x \in clA_n$ with $A_{n+1} \subset A_n$ ($n \in N$), there exists a q -sequence (B_n) such that $x \in cl(A_n \cap B_n)$ for all $n \in N$. If the A_n are all the same set, then such a space is a *singly bi-quasi- k -space*. Fréchet spaces and locally compact spaces are k' -spaces. k' -spaces, and M -spaces (generally, countably bi-quasi- k -spaces) are singly bi-quasi- k -spaces. Singly bi-quasi- k -spaces are quasi- k -spaces. For properties related to dominating or point-countable determining covers among singly bi- k -spaces (or, singly bi-quasi- k -spaces), see [15] or [21].

Theorem 2.2. (1) Let \mathcal{P} be a determining cover of X . For $S \subset X$, $\mathcal{P}|S$ is a determining cover of S if S has a determining cover by open or closed sets in X , in particular, S is a k -space. When \mathcal{P} is point-countable and closed, the same result holds if S is a quasi- k -space.

(2) Let \mathcal{F} be a dominating cover of X . For $S \subset X$, $\mathcal{F}|S$ is a dominating cover if S has a determining cover by open or closed subsets in X , or S is a quasi- k -space.

Corollary 2.3. Let \mathcal{F} be a dominating cover of X by Fréchet spaces. For $S \subset X$, the following are equivalent.

- (a) S has a dominating cover $\mathcal{F}|S$.
- (b) S has a determining cover $\mathcal{F}|S$.

- (c) S is a quasi- k -space.
- (d) S is a sequential space.

Theorem 2.4. (1) Let \mathcal{P} be a cover of X . Then the following are equivalent.

- (a) For any $S \subset X$, $\mathcal{P}|S$ is a determining cover of S .
- (b) For any $A \subset X$ and any $a \in clA$, there exists $P \in \mathcal{P}$ such that $a \in cl_P(A \cap P)$.
- (2) Let \mathcal{F} be a closed cover of X . Then the following are equivalent.
 - (a) For any $S \subset X$, $\mathcal{F}|S$ is a dominating cover of S .
 - (b) For any $S \subset X$, $\mathcal{F}|S$ is CP in X .

Corollary 2.5. Let X be a singly bi-quasi- k -space, and let \mathcal{F} be a dominating (resp. point-countable determining closed) cover of X . Then, for any $S \subset X$, $\mathcal{F}|S$ is a dominating (resp. determining) cover of S .

Corollary 2.6. (1) For a space X , the following are equivalent.

- (a) X is Fréchet.
- (b) X has a determining cover \mathcal{P} by compact metric subsets such that for any $S \subset X$, $\mathcal{P}|S$ is a determining cover of S .
- (c) X is a sequential space, and for any *determining* cover \mathcal{P} of X and any $S \subset X$, $\mathcal{P}|S$ is a determining cover of S .
- (2) For a space X , the following are equivalent.
 - (a) X is a k' -space.
 - (b) X has a determining cover \mathcal{P} by compact subsets such that for any $S \subset X$, $\mathcal{P}|S$ is a determining cover of S .

Corollary 2.7. Let X be a sequential space. If any subset of X is a quasi- k -space, then X is Fréchet (cf. [5]).

Corollary 2.8. (1) Let X have a determining cover \mathcal{P} by Fréchet spaces. Then the following are equivalent.

- (a) X is Fréchet.
- (b) For any $S \subset X$, $\mathcal{P}|S$ is a determining cover of S .
- (2) Let X have a *dominating* (or point-countable determining closed) cover \mathcal{F} by k' -spaces. Then the following are equivalent.
 - (a) X is a k' -space.
 - (b) For any $S \subset X$, $\mathcal{F}|S$ is a determining cover of S .

Remark 2.9. Not every compact sequential space is Fréchet (the space Ψ^* in [5, Example 7.1], for example). Thus, in (c) \Rightarrow (a) of Corollary 2.6(1), we can't replace "*determining*" by "*dominating*". While, under X being a k -space, (c) implies X is a k' -space, but the converse need not hold even if

X is compact sequential. Also, in (a) \Rightarrow (b) of Corollary 2.8(2), we can't replace "dominating" by "determining".

Question 2.10. (1) Let \mathcal{P} be a determining cover of X . Let $S \subset X$, and S be a quasi- k -space. Is $\mathcal{P}|S$ a determining cover of S ?

(2) Let \mathcal{F} be a dominating cover of X . For any $S \subset X$, let $\mathcal{F}|S$ be a determining cover of S . Is $\mathcal{F}|S$ a dominating cover of S ?

(3) Let X be a k -space. For any determining cover \mathcal{P} of X , and any $S \subset X$, let $\mathcal{P}|S$ be a determining cover of S . Is X Fréchet?

3. Countable products

In this section, we consider countable products of weak topologies, as additional matters to Section 4 in [20]. For finite products of weak topologies in terms of Question 3, see [19] or [20]. First, let us give the following notations.

For a cover \mathcal{P} of a space, let $[\mathcal{P}] = \{A : A \text{ is a finite union of elements of } \mathcal{P}\}$, $\mathcal{P}^* = \{P \cup F : P \in \mathcal{P}, F \text{ is finite}\}$, and let $\mathcal{P}^\circ = \{\text{int}P : P \in \mathcal{P}\}$.

Remark 3.1. (1) For a space $X = F_1 + F_2$, $\mathcal{F} = \{F_1, F_2\}$ is a determining cover of X , but $\mathcal{F}^\omega (= \mathcal{F} \times \mathcal{F} \times \dots)$ is not a determining cover of X^ω .

(2) Let X be the sequential fan S_ω (or the Arens' space S_2). Then, for any (countable) determining closed cover \mathcal{F} by (compact) metric subsets in X , $[\mathcal{F}]^\omega$ is not a determining cover of X^ω by means of Theorem 3.2(2) below.

As a generalization of sequential spaces, we recall that a space X has *countable tightness*, $t(X) \leq \omega$, if whenever $a \in \text{cl}A$, $a \in \text{cl}C$ for some countable $C \subset A$ (equivalently, X has a determining cover by countable subsets); see [8]. While, as a generalization of countably bi-quasi- k -spaces, let us consider the following property (P), referring to [6, (3.1)].

(P): For each decreasing sequence (A_n) in X with $\bigcap \{\text{cl}A_n : n \in N\} \neq \emptyset$, there exists a countably compact set K of X with $K \cap A_n \neq \emptyset$ for all $n \in N$.

Theorem 3.2. (1) Let X^ω be a sequential space. Let \mathcal{P} be a determining cover of X . Then $\mathcal{P}^{*\omega}$ (hence, $[\mathcal{P}]^\omega$) is a determining cover of X^ω ([13]).

(2) Let X^ω be a quasi- k -space. Let \mathcal{P} be a dominating or point-countable determining cover of X . Then the following hold.

(a) $[\mathcal{P}]^\omega$ is a determining cover of X^ω .

(b) If $t(X) \leq \omega$, then X has property (P), hence $[\mathcal{P}]^{\circ\omega}$ is a determining cover of X^ω .

A space X is a *bi- k -space* [8] if, whenever \mathcal{A} is a filterbase with $x \in clA$ for every $A \in \mathcal{A}$, there exists a k -sequence (B_n) in X such that $x \in cl(A \cap B_n)$ for all $A \in \mathcal{A}$ and $n \in N$. Locally compact spaces, first countable spaces, or paracompact M -spaces are *bi- k -spaces*. *Bi- k -spaces* are k -spaces which are countably *bi-quasi- k* . Every countable product of *bi- k -spaces* is a *bi- k -space* ([8]), hence a k -space.

Corollary 3.3. Let X be a *bi- k -space*, and let \mathcal{P} be a determining cover of X . Then the following hold.

- (a) $[\mathcal{P}]^\omega$ is a determining cover of X^ω if X is sequential, or \mathcal{P} is a point-countable cover.
- (b) $[\mathcal{P}]^{\circ\omega}$ is a determining cover of X^ω if \mathcal{P} is a dominating cover, a point-countable closed cover, or a point-countable cover with $t(X) \leq \omega$.

Corollary 3.4. Let X have a dominating or point-countable determining closed cover \mathcal{F} by first countable spaces. Then the following properties are equivalent ([20]).

- (a) X^ω is a quasi- k -space.
- (b) X^ω is a sequential space.
- (c) $\mathcal{F}^{\ast\omega}$ is a determining cover of X^ω .
- (d) $[\mathcal{F}]^\omega$ (actually, $[\mathcal{F}]^{\circ\omega}$) is a determining cover of X^ω .
- (e) $[\mathcal{F}]^\circ$ is an open cover of X .
- (f) X is first countable.

Corollary 3.5. Let X satisfy (a), (b), or (c) below. If X^ω is a quasi- k -space, then X is metric.

- (a) X has a dominating cover by metric spaces.
- (b) X is a paracompact space having a point-countable determining closed cover by metric spaces.
- (c) X has a point-countable determining cover by locally separable, metric spaces.

Corollary 3.6. Let X have a dominating or point-countable determining closed cover \mathcal{F} by locally compact spaces (resp. *bi- k -spaces*). Then the implications (a) \Leftrightarrow (b) \Leftrightarrow (c), and (d) \Leftrightarrow (e) \Rightarrow (b) hold. When $t(X) \leq \omega$, (a) \sim (e) are equivalent.

- (a) X^ω is a quasi- k -space.
- (b) X^ω is a k -space.
- (c) $[\mathcal{F}]^\omega$ is a determining cover of X^ω .
- (d) $[\mathcal{F}]^\circ$ is an open cover of X .
- (e) X is a locally compact space (resp. *bi- k -space*).

Remark 3.7. (CH). “ $t(X) \leq \omega$ ” is essential in Corollary 3.6 (the implication (b) \Rightarrow (d) or (e)), and so is in Theorem 3.2(2). Actually, under (CH), there exists a space X having a countable dominating cover \mathcal{F} by compact subsets, and X^ω is a k -space, but X is not locally compact ([2]) (hence, $[\mathcal{F}]^\omega$ is a determining cover of X^ω , but $[\mathcal{F}]^\circ$ is not an open cover of X , and X doesn't have property (P)).

Finally, let us give questions on products of weak topologies. First, let us review some related matters.

Remark 3.8. (1) Let X be a sequential space (resp. paracompact space). Then X^ω is a sequential space (resp. k -space) iff X is a quasi- k -space (see [12] for the finite products).

(2) Let \mathcal{P} be a determining cover of X . Then (a) and (b) below hold.

(a) Let X^2 be a k -space. Then \mathcal{P}^2 is a determining cover of X^2 if X is a sequential space, or each element of \mathcal{P} is a k -space (see [19] or [20]), in particular, \mathcal{P} is a closed cover.

(b) Let X^ω be a k -space. Then $[\mathcal{P}]^\omega$ is a determining cover of X^ω if \mathcal{P} is a dominating or point-countable cover, or X is sequential (for example, the elements of \mathcal{P} are sequential).

In view of Remark 3.8, the author has Question 3.9 below, in particular. For (1), $X^2 \in [\mathcal{P}]^2$. Also, the compactness of X is essential even if \mathcal{P} is a countable HCP closed cover by separable metric subsets. If (1) is affirmative, then so is the question for X being a bi- k -space (generally, space with X^ω a k -space). For (2), any \mathcal{F}^n ($n \in N$) is a determining cover of X^n . (3) is affirmative if X is sequential, or \mathcal{P} is dominating or point-countable. If X is paracompact, then any \mathcal{F}^n ($n \in N$) is a determining cover of X^n .

Question 3.9. (1) Let X be a compact space, and let \mathcal{P} be a countable determining cover of X . Is \mathcal{P}^2 a determining cover of X^2 ?

(2) Let X be a compact space (or space with X^ω a k -space), and let \mathcal{F} be a determining closed cover of X . Is $[\mathcal{F}]^\omega$ a determining cover of X^ω ?

(3) Let \mathcal{F} be a determining cover of X by compact subsets. Let X^ω be a quasi- k -space (in particular, let X be a countably compact space). Is $[\mathcal{F}]^\omega$ a determining cover of X^ω ?

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