数理解析研究所講究録
1492 巻 2006 年 94-99

Rudin's Dowker space is base-normal — a direct proof —

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The theorem 'Rudin's Dowker space is base-normal' was proved in [7] by using some results of K. P. Hart in [3]. In this report, we give a direct proof to this theorem.

Throughout this paper, all spaces are assumed to be T_1 topological spaces. The symbol $\mathbb{N}$ denotes the set of all natural numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. The cardinality of a set X is denoted by $|X|$. For a space X, $w(X)$ stands for the weight of X. For a space X and a subspace A of X, the closure of A in X is denoted by $\overline{A}$.

Motivated by base-paracompactness of J. E. Porter [4], we introduced in [6] the notion of base-normality. Recall that a space X is said to be *base-normal* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ satisfying that every pair of disjoint closed subsets F_0, F_1 of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that, for every $B \in \mathcal{B}'$, either $\overline{B} \cap F_0 = \emptyset$ or $\overline{B} \cap F_1 = \emptyset$ holds. A space X is said to be *base-collectionwise normal* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ satisfying that every discrete closed collection $\{F_\alpha : \alpha \in \Omega\}$ of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that, for every $B \in \mathcal{B}'$, $|\{\alpha \in \Omega : \overline{B} \cap F_\alpha \neq \emptyset\}| \leq 1$. Note that every base-normal space is normal, and G. Gruenhage constructed in [2] a ZFC example of a countably compact zero-dimensional LOTS which is not base-normal.

Recall that a *Dowker space* is a normal space X for which $X \times [0,1]$ is not normal. In [6] we pointed out that a base-normal Dowker space can be constructed by using a technique of Porter in [4]. Indeed, let Y be any Dowker space. Then, the direct sum $Y \oplus (\kappa+1)$, where κ is the cardinality of all open subsets of Y and $\kappa+1$ has the usual order topology, is a base-normal Dowker space (although Y itself is not necessarily assumed to be base-normal) ([6]). Thus, it seems to be an interesting problem to find base-normal spaces among Dowker spaces which have been obtained so far. In fact, on the 3rd Japan-Mexico Joint Meeting on Topology and its Applications held in December, 2004, a participant asked a question if Rudin's Dowker space is base-normal or not, and in [7] this question is affirmatively answered.

Let us first recall the construction of Rudin's Dowker space in [5]. The symbol $cf(\lambda)$ stands for the cofinality of λ. Let

$$F = \left\{ f : \mathbb{N} \to \omega_\omega \ : \ f(n) \leqslant \omega_n \ \text{ for all } \ n \in \mathbb{N} \right\}$$

and

$$X = \left\{ f \in F \ : \ \exists i \in \mathbb{N} \ \text{ such that } \ \omega < cf(f(n)) < \omega_i \ \text{ for all } \ n \in \mathbb{N} \right\}.$$

Let $f, g \in F$. Then, we define $f < g$ if $f(n) < g(n)$ for every $n \in \mathbb{N}$, and define $f \leqslant g$ if $f(n) \leqslant g(n)$ for every $n \in \mathbb{N}$. Moreover, define

$$U_{f,g} = \{h \in X : f < h \leqslant g\}.$$

The set $\{U_{f,g} : f, g \in F\}$ is a base for a topology of X. The space X is *Rudin's Dowker space.* We set $\mathcal{B} = \{U_{f,g} : f, g \in F\}$. Note that $w(X) = \omega_\omega^\omega = |\mathcal{B}|$. For $U \subset F$, define a map $t_U \in F$ by $t_U(n) = \sup\{f(n) : f \in U\}$ for each $n \in \mathbb{N}$. For undefined terminology, see [1].

To prove base-normality of Rudin's Dowker space, we give a more strict result as follows.

Theorem. *Let X be Rudin's Dowker space, and $\mathcal{B}$ the base for X defined as above. For every discrete closed collection $\{F_\alpha : \alpha \in \Omega\}$ of X, there is a disjoint cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ satisfying that, for every $B \in \mathcal{B}'$, $|\{\alpha \in \Omega : B \cap F_\alpha \neq \emptyset\}| \leqslant 1$.*

This theorem was proved in [7, Theorem 3.4] by using results in [3]. As was announced in the introduction, we directly prove this.

Proof of Theorem. First show the following statements are valid.

(i) $X \in \mathcal{B}$.
(ii) *If $U(1), U(2) \in \mathcal{B}$, then $U(1) \cap U(2) \in \mathcal{B}$.*
(iii) *If $U(i) \in \mathcal{B}$, $i \in \mathbb{N}$, then $\bigcap_{i \in \mathbb{N}} U(i) \in \mathcal{B}$.*

Indeed, (i) is easy to see and (ii) follows from (i) and (iii), so we only give a proof of (iii). To prove (iii), let $U(i) \in \mathcal{B}$, $i \in \mathbb{N}$. Then, each $U(i)$ is expressed as $U(i) = U_{f_i, g_i}$ for some $f_i, g_i \in F$. Define $f, g \in F$ by $f(n) = \sup_{i \in \mathbb{N}} f_i(n)$, $n \in \mathbb{N}$, and $g(n) = \min_{i \in \mathbb{N}} g_i(n)$, $n \in \mathbb{N}$. Notice that $f \notin X$. Hence, we have $\bigcap_{i \in \mathbb{N}} U_{f_i, g_i} = U_{f,g}$. Thus, $\bigcap_{i \in \mathbb{N}} U(i) \in \mathcal{B}$.

Next, we show the following:

Claim. *For every disjoint closed subsets F_0, F_1 of X, there is a disjoint cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that, for every $B \in \mathcal{B}'$, either $\overline{B} \cap F_0 = \emptyset$ or $\overline{B} \cap F_1 = \emptyset$ holds.*

To show this, let F_0 and F_1 be disjoint closed subsets of X. The proof in [5] makes for each countable ordinal α a disjoint open collection $\mathcal{J}_\alpha$ of X which covers $F_0 \cup F_1$. We modify the proof in [5] so as to make disjoint open covers $\mathcal{J}_\alpha$ of X (consisting of members of $\mathcal{B}$).

Inductively, we construct disjoint open covers $\mathcal{J}_\alpha$ of X, $0 \leqslant \alpha < \omega_1$, with $\mathcal{J}_\alpha \subset \mathcal{B}$ having the following property:

For every $\beta < \alpha$ and every $V \in \mathcal{J}_\alpha$, there exists $U \in \mathcal{J}_\beta$ such that

(1) $V \subset U$,

(2) if $V \cap F_0 \neq \emptyset \neq V \cap F_1$, then $t_V \neq t_U$,

(3) if $U \cap F_0 = \emptyset$ or $U \cap F_1 = \emptyset$, then $U = V$.

First, set $\mathcal{J}_0 = \{X\}$. By (i), it follows that $X \in \mathcal{B}$, hence $\mathcal{J}_0 \subset \mathcal{B}$.

Next, assume that $\mathcal{J}_\beta$ has been constructed for every $\beta < \alpha$.

Case 1. α is limit. For every $f \in X$ and every $\beta < \alpha$, choose a unique $U(f)_\beta$ such that $f \in U(f)_\beta \in \mathcal{J}_\beta$. Define

$$U_f = \bigcap_{\beta<\alpha} U(f)_\beta \quad \text{for every } f \in X, \quad \text{and} \quad \mathcal{J}_\alpha = \{U_f : f \in X\}.$$

Then, by (iii), it follows that $\mathcal{J}_\alpha \subset \mathcal{B}$. Moreover, $\mathcal{J}_\alpha$ is a disjoint cover of X because each $\mathcal{J}_\beta$ is a disjoint cover of X. Fix $\beta < \alpha$. We shall show that U_f and $U(f)_\beta$ satisfying conditions (1), (2) and (3) above. Since $U_f \subset U(f)_\beta$, (1) holds. To show (2), assume $U_f \cap F_0 \neq \emptyset \neq U_f \cap F_1$. Then, $U(f)_{\beta+1} \cap F_0 \neq \emptyset \neq U(f)_{\beta+1} \cap F_1$. Hence, it follows from the assumption of induction that $t_{U(f)_{\beta+1}} \neq t_{U(f)_\beta}$. Since $t_{U_f} \leqslant t_{U(f)_{\beta+1}} \leqslant t_{U(f)_\beta}$, we have $t_{U_f} < t_{U(f)_\beta}$, so (2) holds. To show (3), assume either $U(f)_\beta \cap F_0 = \emptyset$ or $U(f)_\beta \cap F_1 = \emptyset$ holds. Then, since $U(f)_\beta = U(f)_{\beta'}$ for every β' with $\beta < \beta' < \alpha$, we have $U(f)_\beta = U(f)_{\beta'}$. It follows that $U_f = U(f)_\beta$. So, (3) holds.

Case 2. $\alpha = \beta + 1$. Fix $U \in \mathcal{J}_\beta$. We shall construct a disjoint cover $\mathcal{J}(U)$ of U with $\mathcal{J}(U) \subset \mathcal{B}$ so as to have the following property:

For every $V \in \mathcal{J}(U)$,

(2)' if $V \cap F_0 \neq \emptyset \neq V \cap F_1$, then $t_V \neq t_U$,

(3)' if $U \cap F_0 = \emptyset$ or $U \cap F_1 = \emptyset$, then $U = V$.

Case A. $U \cap F_0 = \emptyset$ or $U \cap F_1 = \emptyset$. Define

$$\mathcal{J}(U) = \{U\}.$$

Then, $\mathcal{J}(U) \subset \mathcal{B}$, and U satisfies conditions (2)' and (3)'.

Case B. $U \cap F_0 \neq \emptyset \neq U \cap F_1$, and there exists $i \in \mathbb{N}$ such that $cf(t_U(i)) \leqslant \omega$. Then, we select i_U so as to satisfy $cf(t_U(i_U)) \leqslant \omega$. Then, as in [5], we can show that $cf(t_U(i_U)) = \omega$. Choose an increasing sequence $\{\lambda_U(n) : n \in \mathbb{N}\}$ of terms of $t_U(i_U)$ cofinal with $t_U(i_U)$. Set

$$V(U,n) = \Big\{ f \in U : \lambda_U(n-1) < f(i_U) \leqslant \lambda_U(n) \Big\}$$

for each $n \in \mathbb{N}$. Define

$$\mathcal{J}(U) = \{V(U,n) : n \in \mathbb{N}\}.$$

Note that $V(U,n) = U_{f,g} \cap U$, where $f, g \in F$ is defined by $f(i_U) = \lambda_U(n-1)$ and $f(n) = 0$ if $n \neq i_U$, and $g(i_U) = \lambda_U(n)$ and $g(n) = \omega_n$ if $n \neq i_U$. Since $U_{f,g}, U \in \mathcal{B}$, it follows from (ii) that $V(U,n) \in \mathcal{B}$. Thus, $\mathcal{J}(U) \subset \mathcal{B}$. For every $V' \in \mathcal{J}(U)$, we can express as $V' = V(U,n)$ for some $n \in \mathbb{N}$, and we have $t_{V'}(i_U) = \lambda_U(n) = t_U(i_U)$, which shows $t_{V'} \neq t_U$. Hence, V' and U satisfy conditions (2)' and (3)'.

Case C. $U \cap F_0 \neq \emptyset \neq U \cap F_1$, and $cf(t_U(n)) > \omega$ for every $n \in \mathbb{N}$. By the quite similar proof to those of [5, Lemmas 5 and 6], we can select $f_U \in F$ such that $f_U < t_U$ and such that either $\{h \in U : f_U < h\} \cap F_0 = \emptyset$ or $\{h \in U : f_U < h\} \cap F_1 = \emptyset$ holds. For every $M \subset \mathbb{N}$, set

$$V(U,M,f_U) = \left\{ h \in U \ : \ \begin{array}{lll} h(n) \leqslant f_U(n) & \text{for every} & n \in M, \ \text{and} \\ h(n) > f_U(n) & \text{for every} & n \in \mathbb{N} - M \end{array} \right\}$$

Define

$$\mathcal{J}(U) = \Big\{ V(U,M,f_U) : M \subset \mathbb{N} \Big\}.$$

Likewise the proof of Case B, by (ii), we can show that $V(U,M,f_U) \in \mathcal{B}$ for each $M \subset \mathbb{N}$. Thus, $\mathcal{J}(U) \subset \mathcal{B}$. Also, we can show that $\mathcal{J}(U)$ is a disjoint cover of U. Finally, it is not difficult to show $V(U,M,f_U)$ and U satisfy conditions (2)' and (3)'.

Set

$$\mathcal{J}_\alpha = \bigcup_{U \in \mathcal{J}_\beta} \mathcal{J}(U).$$

By using conditions (2)' and (3)' above and the assumption of induction, we can show that $\mathcal{J}_\alpha$, $0 \leqslant \alpha < \omega_1$, have the required property.

For every $f \in X$ and every α with $0 \leqslant \alpha < \omega_1$, there exists a unique $U(f)_\alpha \in \mathcal{J}_\alpha$ such that $f \in U(f)_\alpha$. Let β and α with $\beta < \alpha < \omega_1$. Then, we have $U(f)_\alpha \subset U(f)_\beta$, hence $t_{U(f)_\alpha} \leqslant t_{U(f)_\beta}$. If $U(f)_\alpha \cap F_0 \neq \emptyset \neq U(f)_\alpha \cap F_1$, then $t_{U(f)_\alpha}(n) < t_{U(f)_\beta}(n)$ for some $n \in \mathbb{N}$. As in [5], for every $n \in \mathbb{N}$ one can move backward in ω_n only finitely many steps. Hence, there exists $\alpha(f) < \omega_1$ such that

$$U(f)_{\alpha(f)} \cap F_0 = \emptyset \quad \text{or} \quad U(f)_{\alpha(f)} \cap F_1 = \emptyset.$$

By (3), if $\alpha(f) < \beta < \omega_1$ then $U(f)_\beta = U(f)_{\alpha(f)}$. Clearly, $\{U(f)_{\alpha(f)} : f \in X\}$ is a cover of X consisting of elements of $\mathcal{B}$. To prove $\{U(f)_{\alpha(f)} : f \in X\}$ is pairwise disjoint, assume $U(f)_{\alpha(f)} \cap U(g)_{\alpha(g)} \neq \emptyset$. Take $\beta < \omega_1$ so as to satisfy $\alpha(f) < \beta$ and $\alpha(g) < \beta$. It follows from $U(f)_\beta = U(f)_{\alpha(f)}$ and $U(g)_\beta = U(g)_{\alpha(g)}$ that $U(f)_\beta \cap U(g)_\beta \neq \emptyset$. Since $\mathcal{J}_\beta$ is pairwise disjoint, we have $U(f)_\beta = U(g)_\beta$, hence $U(f)_{\alpha(f)} = U(g)_{\alpha(g)}$. This shows that $\{U(f)_{\alpha(f)} : f \in X\}$ is pairwise disjoint, and this is the required $\mathcal{B}'$ in Claim.

Finally, to complete the proof, let $\{F_\alpha : \alpha \in \Omega\}$ be a discrete closed collection of X. Since X is collectionwise normal, there is a discrete open collection $\{U_\alpha : \alpha \in \Omega\}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Omega$. Due to the fact shown above, for every $\alpha \in \Omega$, there is a disjoint cover $\mathcal{B}_\alpha$ of X by members of $\mathcal{B}$ such that, for every $B \in \mathcal{B}_\alpha$, either $B \cap F_\alpha = \emptyset$ or $B \subset U_\alpha$ holds. For every $\alpha \in \Omega$, define

$$\mathcal{B}^*_\alpha = \{B \in \mathcal{B}_\alpha : B \subset U_\alpha\}.$$

Note that $F_\alpha \subset \bigcup \mathcal{B}^*_\alpha \subset U_\alpha$ for every $\alpha \in \Omega$. Set

$$B^* = \bigcup_{\alpha \in \Omega} \bigcup \mathcal{B}^*_\alpha$$

Since $\bigcup \mathcal{B}^*_\alpha$ is clopen for each $\alpha \in \Omega$, and $\{\bigcup \mathcal{B}^*_\alpha : \alpha \in \Omega\}$ is discrete in X, it follows that B^* is clopen in X. Hence, by the fact shown in the above, there is a disjoint cover $\mathcal{C}$ of X by members of $\mathcal{B}$ such that, for each $C \in \mathcal{C}$, either $C \cap B^* = \emptyset$ or $C \subset B^*$ holds. Then, $\{C \in \mathcal{C} : C \cap B^* = \emptyset\} \cup \bigcup_{\alpha \in \Omega} \mathcal{B}^*_\alpha$ is the required disjoint cover of X by members of $\mathcal{B}$. This completes the proof. □

The notion of base-normality is motivated by the well-known fact that X is normal if and only if every pair of disjoint closed subsets F_0, F_1 of X admits a locally finite open cover $\mathcal{U}$ of X such that, for every $U \in \mathcal{U}$, either $\overline{U} \cap F_0 = \emptyset$ or $\overline{U} \cap F_1 = \emptyset$ holds. On the other hand, it is easy to see that "locally finite" in the above fact can be replaced by "star-finite"; a collection $\{U_\alpha : \alpha \in \Omega\}$ of subsets of X is said to be *star-finite* if $|\{\beta \in \Omega : U_\beta \cap U_\alpha \neq \emptyset\}| < \omega$ holds for every $\alpha \in \Omega$. In order to consider a base version of this fact, we define a space X to be *strongly base-normal* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ satisfying that every pair of disjoint closed subsets F_0, F_1 of X admits a star-finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that, for every

$B \in \mathcal{B}'$ either $\overline{B} \cap F_0 = \emptyset$ or $\overline{B} \cap F_1 = \emptyset$ holds. The Theorem in the above shows that Rudin's Dowker space possesses this property. Also, note that there is a base-normal space (in fact, a metric space) which is not strongly base-normal ([7]). Related results on strongly base-normal spaces, see [7].

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