数理解析研究所講究録
1492 巻 2006 年 100-103

UPPER SEMICONTINUOUS SELECTIONS ON FINITISTIC SPACES

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Throughout this paper it is assumed that spaces are T_1-spaces, closed mappings and perfect mappings are continuous, and paracompact spaces are paracompact Hausdorff spaces. The purpose of this note is to introduce some results in [15].

1. Characterizations of finitistic spaces

Let ω denote the first infinite ordinal number. For a cover $\mathcal{U}$ of a space X and a positive integer n, we say that the *order of* $\mathcal{U}$ *is at most* n ($\mathrm{ord}\mathcal{U} \leq n$ for short) if every $n+1$ distinct members of $\mathcal{U}$ have empty intersection. A cover $\mathcal{U}$ of a space X is of finite order if $\mathrm{ord}\mathcal{U} \leq n$ for some positive integer n. The *dimension* of a space X ($\dim X$ in notation) is the least number n such that any finite open cover of X is refined by a finite open cover of X of order at most $n+1$. In case $\dim X \leq n$ for an integer n, we say X is finite-dimensional ($\dim X < \infty$ for short).

Recall some notations and terminology on set-valued mappings. Let X and Y be topological spaces and 2^Y the set of all non-empty subsets of Y. The symbol $\mathcal{F}(Y)$ (respectively, $\mathcal{C}(Y)$) denotes the set of all non-empty closed (respectively, non-empty compact) subsets of Y. A mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (respectively, *upper semicontinuous*) or l.s.c. (respectively, u.s.c.) if $\varphi^{-1}[A] = \{x \in X \mid \varphi(x) \cap A \neq \emptyset\}$ is open (respectively, closed) in X for every open (respectively, closed) subset A of Y. A mapping $\psi : X \to 2^Y$ is called a *set-valued selection* of a mapping $\varphi : X \to 2^Y$ if $\psi(x) \subset \varphi(x)$ for each $x \in X$.

M. M. Čoban [1] established the following characterization of finite-dimensional spaces, which is fundamental in our study. For an infinite cardinal τ, a Hausdorff space X is said to be *τ-paracompact* if every open cover of X of cardinality $\leq \tau$ is refined by a locally finite open cover of X.

THEOREM 1 (Čoban [1, Theorem 11.1]). *A T_1-space X is normal, τ-paracompact and $\dim X \leq n$ if and only if for every completely metrizable space Y of weight $\leq \tau$, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ such that $\psi(x)$ consists of at most $n+1$ points for every $x \in X$.*

A space X is said to be *finitistic* if every open cover is refined by an open cover of finite order. As a generalization of compact spaces and finite-dimensional spaces, the notion of finitistic spaces was introduced by R. Swan [14] in the study of fixed point theory of transformation group. Some of their properties have been investigated from the dimensional viewpoint ([2], [3], [6] and [8]). In particular, Y. Hattori [6, Proposition] proved that a paracompact Hausdorff space X is finitistic if and only if X satisfies the condition (K); there is a compact subset C of X such that $\dim F < \infty$ for every closed subset F of X with $F \cap C = \emptyset$ ([13]).

In [15], we proved the following characterization of finitistic spaces in terms of upper semicontinuous selections. By $\mathcal{D}(Y)$ we denote the set of all closed discrete subsets of X. For a space X, a subset A of X and an open cover $\mathcal{U}$ of X, we write $l(A,\mathcal{U}) = \min\{\mathrm{Card}\, \mathcal{U}_0 \mid \mathcal{U}_0 \subset \mathcal{U} \text{ and } A \subset \bigcup \mathcal{U}_0\}$.

THEOREM 2. *For a T_1-space X, the following are equivalent.*

(a) *X is paracompact and finitistic.*

(b) *For every completely metrizable space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ such that* $\sup\{l(\psi(x), \mathcal{V}) \mid x \in X\} < \omega$ *for every open cover $\mathcal{V}$ of Y.*

(c) *For every completely metrizable space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{F}(Y)$ such that* $\sup\{l(\psi(x), \mathcal{V}) \mid x \in D\} < \omega$ *for each $D \in \mathcal{D}(X)$ and every open cover $\mathcal{V}$ of Y.*

The following theorem due to K. Morita [11] is fundamental in dimension theory.

THEOREM 3 (Morita [11, Theorem 4]). *A T_1-space X is metrizable and* $\dim X \leq n$ *if and only if there exist a zero-dimensional metrizable space Z and a closed mapping f of Z onto X such that $f^{-1}(x)$ consists of at most $n+1$ points for every $x \in X$.*

The proof of Theorem 2 is based on the following finitistic analogue of Morita's theorem.

THEOREM 4. *For a T_1-space X, the following are equivalent.*

(a) *X is metrizable and finitistic.*

(b) *There exist a compact subset C of X, a zero-dimensional metrizable space Z and a perfect mapping f of Z onto X such that* $\sup\{\mathrm{Card}\, f^{-1}(x) \mid x \in F\} < \omega$ *for every closed subset F of X with $F \cap C = \emptyset$.*

(c) *There exist a zero-dimensional metrizable space Z and a perfect mapping f of Z onto X such that* $\sup\{l(f^{-1}(x), \mathcal{U}) \mid x \in X\} < \omega$ *for every open cover $\mathcal{U}$ of Z.*

(d) *There exist a zero-dimensional metrizable space Z and a closed mapping f of Z onto X such that* $\sup\{l(f^{-1}(x), \mathcal{U}) \mid x \in D\} < \omega$ *for each $D \in \mathcal{D}(X)$ and every open cover $\mathcal{U}$ of Z.*

2. INFINITE-DIMENSIONAL SPACES AND UPPER SEMICONTINUOUS SELECTION

Recall that a metrizable space is *countable-dimensional* if it can be expressed as a union of countably many zero-dimensional subspaces. The following theorem was established by V. Gutev [5].

THEOREM 5 (Gutev [5, Theorem 2.1]). *A metrizable space X is countable dimensional if and only if for every completely metrizable space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ such that $\psi(x)$ is finite for every $x \in X$.*

As applications of Theorem 2, we obtained the following characterizations of some kinds of infinite-dimensional spaces. For the definition of large transfinite dimension, see [4].

THEOREM 6. *A metrizable space X has large transfinite dimension if and only if for every completely metrizable space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a* *u.s.c. set-valued selection* $\psi : X \to \mathcal{C}(Y)$ *such that* $\sup\{\text{Card}\ \psi(x) \mid x \in D\} < \omega$ *for each* $D \in \mathcal{D}(X)$.

A normal space is *strongly countable-dimensional* if it can be expressed as a union of countably many finite-dimensional closed subspaces.

THEOREM 7. *A T_1-space X is normal, τ-paracompact and strongly countable-dimensional if and only if there exists a mapping $m : X \to \omega$ such that for every completely metrizable space Y of weight $\leq \tau$, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ satisfying* $\text{Card}\ \psi(x) \leq m(x)$ *for each $x \in X$.*

A normal space is *locally finite-dimensional* if every point has a finite-dimensional closed neighborhood. A mapping $m : X \to \omega$ is said to be *lower semicontinuous* if the set $\{x \in X \mid m(x) < k\}$ is open in X for each $k \in \omega$.

THEOREM 8. *A metacompact T_1-space X is normal, τ-paracompact and locally finite-dimensional if and only if there exists a lower semicontinuous mapping $m : X \to \omega$ such that for every completely metrizable space Y of weight $\leq \tau$, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ such that* $\text{Card}\ \psi(x) \leq m(x)$ *for each $x \in X$.*

A normal space X is *strong small transfinite dimension* (or is called a *shallow space*) if for every non-empty closed subset F of X there exists a nonempty open normal subset U of F such that $\dim U < \infty$ (see [4, 7.3.A]). A normal space is said to *have strong large transfinite dimension* ([7]) if it has both large transfinite dimension and strong small transfinite dimension.

THEOREM 9. *A metrizable space X has strong large transfinite dimension if and only if there exists a mapping $m : \mathcal{D}(X) \to \omega$ such that for every completely metrizable space Y, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. selection $\psi : X \to \mathcal{C}(Y)$ such that* $\sup\{\text{Card}\ \psi(x) \mid x \in D\} \leq m(D)$ *for each $D \in \mathcal{D}(X)$.*

3. A GENERALIZATION OF FINITISTIC SPACES

For normal finitistic spaces, we have the following.

PROPOSITION 10. *If X is normal and finitistic, then for every separable completely metrizable space Y every l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ such that $l(\psi, \mathcal{V}) < \infty$ for every open cover $\mathcal{V}$ of Y.*

Note that the converse of Proposition 10 does not hold. Indeed, the space G constructed by E. Michael [10, Example 1] is normal, countably paracompact and zero-dimensional, and hence satisfies the condition in Proposition 10 concerning a set-valued selection ([12, Theorem 4.6]). But G is not finitistic ([6, Theorem 1]). This disagreement is caused by the definition of finitistic spaces which starts with an arbitrary open cover of a given space. Applying the notion of finitistic spaces to normal spaces, we make a modification by restricting arbitrary open covers to normal ones. A space X is said to be *pseudofinitistic* if every normal open cover of X has a refinement which is normal and whose order is finite. V. Matijević [9] defined this notion under the name finitistic spaces. Every finitistic normal space is pseudofinitistic. In the realm of paracompact Hausdorff spaces these two notions are coincide. Concerning covering dimension, we have the following.

PROPOSITION 11. *For a normal space X, the following are equivalent.*

(a) *X is pseudofinitistic.*

(b) *Every locally finite open cover of X has a locally finite open refinement $\mathcal{V}$ such that the set $\{V \in \mathcal{V} \mid \dim \mathrm{cl} V > n\}$ is finite for some $n \in \omega$.*

(c) *Every locally finite open cover of X has a finite subcollection $\mathcal{W}$ such that $\dim(X \setminus \bigcup \mathcal{W}) < \infty$.*

For a subset A of Y and $\varepsilon > 0$, we write $l_d(A, \varepsilon) = \min\{\mathrm{Card}\, A_0 \mid A \subset B_d(A_0, \varepsilon)\}$, where $B_d(A, \varepsilon) = \{y \in Y \mid d(y, A) < \varepsilon\}$. For a mapping $\varphi : X \to 2^Y$ and $\varepsilon > 0$, we put $l_d(\varphi, \varepsilon) = \sup\{l_d(\varphi(x), \varepsilon) \mid x \in X\}$. As a pseudofinitistic analogue of Theorem 2 we proved the following.

THEOREM 12. *For a T_1-space X, the following are equivalent.*

(a) *X is normal, τ-paracompact and pseudofinitistic.*

(b) *For every complete metric space (Y, d) of weight $\leq \tau$, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ of φ such that $l_d(\psi, \varepsilon) < \infty$ for each $\varepsilon > 0$.*

(c) *For every complete metric space (Y, d) of weight $\leq \tau$, every l.s.c. mapping $\varphi : X \to \mathcal{F}(Y)$ admits a u.s.c. set-valued selection $\psi : X \to \mathcal{C}(Y)$ of φ such that $l_d(\psi|_D, \varepsilon) < \infty$ for each $D \in \mathcal{D}(X)$ and each $\varepsilon > 0$.*

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