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Upper bounds of small transfinite compactness degree in metrizable spaces

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1 Introduction

In [1], Aarts and Nishiura investigated several types of dimensions modulo a class $\mathcal{P}$ of spaces. The small inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-ind has a natural transfinite extension as follows.

Definition 1.1 ([2]) Let X be a regular T_1-space, $\mathcal{P}$ a class of topologicdal spaces which is hereditary with respect to closed subsets. Let α be the integer -1 or an ordinal number. Then the *small transfinite dimension modulo* $\mathcal{P}$ of X, we denote $\mathcal{P}\text{-trind}\, X$, is defined as follows.

(i) $\mathcal{P}\text{-trind}\, X = -1$ if and only if $X \in \mathcal{P}$.

(ii) $\mathcal{P}\text{-trind}\, X \leq \alpha$ if each point in X has arbitrarily small neighbourhoods V with $\mathcal{P}\text{-trind}\, \mathrm{Bd}\, V < \alpha$.

(iii) $\mathcal{P}$-trind $X = \alpha$ if $\mathcal{P}$-trind $X \leq \alpha$ and $\mathcal{P}$-trind $X \leq \beta$ does not hold for every $\beta < \alpha$.

(iv) $\mathcal{P}$-trind $X = \infty$ if $\mathcal{P}$-trind $X \leq \alpha$ does not hold for every ordinal α.

Remark 1.1 Observe that if a space X has $\mathcal{P}$-trind $X \neq \infty$ then for any ordinal $\alpha \leq \mathcal{P}$-trind X there is a closed set $A \subset X$ such that $\mathcal{P}$-trind $A = \alpha$.

It is obvious that if $\alpha < \omega_0$, then the small transfinite dimension modulo $\mathcal{P}$, $\mathcal{P}$-trind, coincides with the small inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-ind. We also notice that if $\mathcal{P} = \{\emptyset\}$, then $\mathcal{P}$-trind X is the usual small transfinite dimension trind X. When we consider as $\mathcal{P}$ the class of compact spaces $\mathcal{K}$, $\mathcal{K}$-trind X is called the *small transfinite compactness degree* and denoted by trcmp X. It is clear that for a regular T_1-space X with weight $w(X) \leq \aleph_\alpha$, trcmp $X \leq$ trind $X \leq \omega_{\alpha+1}$ holds (cf. [3, Theorem 7.1.6]). It is well known that for every countable ordinal α there is a separable metrizable space Z_α such that trcmp $Z_\alpha = \alpha$ ([7]). On the other hand, there are metrizable spaces having arbitrarily high small transfinite dimension ([4], [6] or see [3, Problems 7.2.C]). In this note, we shall show the following theorem.

Theorem 1.1 *There is no upper bound for* trcmp *in the class of metrizable spaces, i.e., for each ordinal number α, there exists a metrizable space X_α such that* trcmp $X_\alpha = \alpha$.

We consider only metrizable spaces. For each ordinal number α, we denote $\alpha = \lambda(\alpha) + p(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $p(\alpha)$ is a non-negative integer. For the integer -1, let $\lambda(-1) = 0$ and $p(-1) = -1$. We refer the reader to [3] and [1] for dimension theory and dimensions modulo $\mathcal{P}$.

2 Upper bound for small transfinite compactness degree

We recall other two kinds of transfinite extensions of the dimension modulo $\mathcal{P}$.

Definition 2.1 ([2]) Let X be a metrizable space and $\mathcal{C}$ the class of completely metrizable spaces. Then the *small transfinite completeness degree* tricd X is defined as tricd $X = \mathcal{C}$-trind X.

We mean by a *complete extension* of a metrizable space X any completely metrizable space which contains X as a dense subset.

Definition 2.2 ([2]) Let X be a metrizable space. Then we define the *transfinite completeness deficiency* $\mathcal{C}$-trdef X of X as follows:

$$\mathcal{C}\text{-trdef}\, X = \min\{\operatorname{trind}(Y \setminus X) : Y \text{ is a complete extension of } X\}.$$

If $\alpha < \omega_0$, then the small transfinite completeness degree $\operatorname{tricd} X$ coincides with the small inductive completeness degree $\operatorname{icd} X$ for every metrizable space X, and the transfinite completeness deficiency $\mathcal{C}$-trdef X coincides with the completeness deficiency $\mathcal{C}$-def X for every metrizable space X with $\mathcal{C}$-trdef $X < \omega_0$ (cf. see [1, Section I.7] for the definition of $\operatorname{icd} X$ and $\mathcal{C}$-def X). Furthermore, it is shown that $\mathcal{C}\text{-def}\, X = \operatorname{icd} X \leq \operatorname{cmp} X$ holds for every separable metrizable space X ([1]).

We recall the construction of the metrizable spaces M^α in [4]. The spaces M^α is constructed by transfinite induction for each ordinal number α. Let $M^0 = \{0\}$ be the one-point space. Let $\alpha > 0$ and assume that M^β is constructed for each $\beta < \alpha$. If $\alpha = \beta + 1$, then we define $M^\alpha = M^\beta \times \mathbb{I}$ with the product topology, where $\mathbb{I}$ denotes the unit closed interval $[0, 1]$. If α is a limit ordinal number, let $M^\alpha = \oplus\{M^\beta : \beta < \alpha\} \cup \{x_\alpha\}$ be the space of the topological sum of the copies of all M^β, $\beta < \alpha$ adding a new point x_α. The neighbourhood base at x_α is defined as follows: For each natural numbers n, m, we put

$V_m(\alpha) = \oplus\{M^{\gamma+m} :\ \gamma \text{ is a limit ordinal with } \gamma < \alpha\}$, and

$U_n(x_\alpha) = \{x_\alpha\} \cup \bigcup\{V_m(\alpha) : m \geq n\}$.

Let $\{U_n(x_\alpha) : n = 1, 2, \dots\}$ be a neighbourhood base at x_α. Then we have the following (cf. [4] or [3, Problem 7.2.C]).

Lemma 2.1 *For each ordinal number α, M^α is a completely metrizable space with* $\operatorname{trind} M^\alpha \neq \infty$ *(in addition, for $\alpha < \omega_1$, the space M^α is separable). Moreover, if $\alpha = \omega_0 \cdot \beta$ for some ordinal number β then* $\operatorname{trind} M^\alpha \geq \beta$.

From Lemma 2.1 and Remark 1.1, we get

Theorem 2.1 *For each ordinal number α there is a completely metrizable space Y_α with* $\operatorname{trind} Y_\alpha = \alpha$.

Let $\mathbb{Q}$ denote the space of rational numbers. We have the following theorem which together with Lemma 2.1 and Remark 1.1 implies Theorem 1.1.

Theorem 2.2 *For each ordinal number α we have*

$$\operatorname{trind} M^\alpha = \operatorname{trcmp}(\mathbb{Q} \times M^\alpha) = \operatorname{tricd}(\mathbb{Q} \times M^\alpha).$$

It is clear that $\operatorname{trcmp} X \geq \operatorname{tricd} X$ and $\operatorname{trind} X = \operatorname{trind}(\mathbb{Q} \times X)$ hold for every metrizable space X. Hence, to show Theorem 2.2, it suffices to show the following two propositions.

Proposition 2.1 *For each ordinal number α,* $\operatorname{tricd}(\mathbb{Q} \times M^\alpha) \geq \mathcal{C}\text{-trdef}(\mathbb{Q} \times M^\alpha)$ *holds.*

Proposition 2.2 *For each ordinal number α,* $\mathcal{C}\text{-trdef}(\mathbb{Q} \times M^\alpha) \geq \operatorname{trind} M^\alpha$ *holds.*

To prove Proposition 2.1, we prove that for each closed set F of $\mathbb{Q} \times M^\alpha$, $\mathcal{C}\text{-trdef} F \leq \operatorname{tricd} F$ by the induction on α and by use of the following lemmas.

Lemma 2.2 ([1, Lemma 1.7.5]) *Let X be a metriable space. If X has a complete extension Y with* $\operatorname{trind}(Y - X) \leq \alpha$*, then every complete extension of X contains a complete extension Y' of X with* $\operatorname{trind}(Y' - X) \leq \alpha$.

Lemma 2.3 *Let α be an ordinal number, F a closed subset of $\mathbb{Q} \times M^\alpha$ and $\mathcal{B}$ a base for F. Then there is a subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that $\mathcal{B}'$ is a base for F, and for each $B \in \mathcal{B}'$ there is an open set $\tilde{B}$ of $\mathbb{I} \times M^\alpha$ with $\tilde{B} \cap F = B$ which satisfies the following condition:*

(*) *Let G be a complete extension of F in $\mathbb{I} \times M^\alpha$ and $E(B)$ an F_σ-set of G such that $E(B) \subset \operatorname{Bd}_G(\tilde{B} \cap G)$ for each $B \in \mathcal{B}'$. Then $\cup\{E(B) : B \in \mathcal{B}'\}$ is an F_σ-set of G.*

One can show Proposition 2.2 by use of the following lemma.

Lemma 2.4 *Let α be an ordinal, U a non-empty open set of $\mathbb{I}$ and X_α a complete extension of $\mathbb{Q} \times M^\alpha$ in $\mathbb{I} \times M^\alpha$. Then, there is a closed set Y_α of $\mathbb{I} \times M^{\lambda(\alpha)}$ such that Y_α is homeomorphic to $M^{\lambda(\alpha)}$ and $Y_\alpha \times \mathbb{I}^{p(\alpha)} \subset X_\alpha \cap (U \times M^\alpha) - (\mathbb{Q} \times M^\alpha)$.*

Remark 2.1 From Theorem 2.2, Lemma 2.1 and Remark 1.1 we have also that there is no upper bound for tricd in the class of metrizable spaces, i.e., for each ordinal number α, there exists a metrizable space X_α such that $\operatorname{tricd} X_\alpha = \alpha$.

3 Relations between the dimension and the small compactness degree

Throughout this section all spaces are assumed to be separable metrizable. We will present characterizations of the dimension dim of a space Y via the small compactness degree cmp of the product $X \times Y$, where the factor X is a nowhere locally compact zero-dimensional space.

The class of all σ-compact spaces will be denoted by $\mathcal{S}$. The small inductive dimension modulo $\mathcal{S}$, $\mathcal{S}$-ind, is called *the small inductive σ-compactness degree.* Evidently, for any space X we have $\mathcal{S}\text{-ind}\, X \leq \operatorname{cmp} X$. Then we have the following two propositions.

Proposition 3.1 *Let X be a zero-dimensional space containing a closed subset A which is a nowhere locally compact Baire space with respect to the induced topology. Then for any space Y we have*

$$\dim Y = \operatorname{cmp}(X \times Y) = \mathcal{S}\text{-ind}\,(X \times Y).$$

Proposition 3.2 *Let X be a zero-dimensional space containing a closed subset A which is a nowhere locally compact space with respect to the induced topology. Then for any compact space Y we have*

$$\dim Y = \operatorname{cmp}(X \times Y).$$

In the connection with the previous statement one can ask the following question.

Question 3.1 Is there a space X such that $\dim X > \operatorname{cmp}(X \times \mathbb{Q})$?

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