数理解析研究所講究録
1493 巻 2006 年 191-211

Virtual turning points and isomonodromic deformations

— On the observation of S. Sasaki for the creation of new Stokes curves of Noumi-Yamada systems —

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1 Introduction

In a series of papers ([KT1], [AKT2], [KT2] and [T1]) Aoki, Kawai and Takei developed the exact WKB analysis of Painlevé equations by making use of their relationship with isomonodromic deformations of linear differential equations. In this report we would like to discuss its generalization to the so-called Noumi-Yamada systems, a typical example of higher order Painlevé equations with affine Weyl group symmetry discovered by Noumi and Yamada (cf. [NY], see Section 2 below for their explicit form).

In the case of traditional (i.e. second order) Painlevé equations the underlying linear equations are also of second order, while the underlying linear equations are higher order for Noumi-Yamada systems. As is well-known, there is an essential difficulty in the WKB analysis of higher order linear ordinary differential equations; in addition to ordinary turning points and Stokes curves it is necessary to introduce virtual turning points and new Stokes curves to analyze the global behavior of solutions (cf. [BNR], [AKT1], [AKSST]). Hence it is expected that not only ordinary

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turning points and Stokes curves but also virtual turning points and new Stokes curves of the underlying linear equations should be relevant to the WKB analysis of Noumi-Yamada systems.

As a matter of fact, Sasaki has discovered in his master thesis and its successive paper that virtual turning points of the underlying linear equations play a crucially important role in the creation of new Stokes curves of Noumi-Yamada systems (cf. [S1, S2]). The purpose of this report is to explain the role of virtual turning points in the (isomonodromic) deformation theory of linear differential equations and to discuss Sasaki's observation for the creation of new Stokes curves of Noumi-Yamada systems.

The plan of this report is as follows: We first recall the explicit form of Noumi-Yamada systems and their underlying linear equations in Section 2. Then in Section 3 we review the definition of virtual turning points of higher order linear ordinary differential equations. Finally in Section 4 we explain the importance of virtual turning points in the deformation theory of higher order equations and discuss Sasaki's observation for the relevance of virtual turning points to the creation of new Stokes curves of Noumi-Yamada systems.

2 Noumi-Yamada systems of type $A_l^{(1)}$

The Noumi-Yamada system, denoted by $(NY)_l$ $(l = 2, 3, 4, \ldots)$ in what follows, is a higher order Painlevé equation with affine Weyl group symmetry of type $A_l^{(1)}$ (cf. [NY]). It may be considered as higher order analogue of the fourth Painlevé equation when l is even and that of the fifth Painlevé equation when l is odd, respectively. As we discuss the case where l is even in this report, let us give the explicit form of $(NY)_l$ only for $l = 2m$ here:

$$(NY)_{2m} \qquad \eta^{-1}\frac{du_j}{dt} = \Big[u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j\Big] \quad (j = 0, 1, \ldots, 2m),$$

where $\eta > 0$ denotes a large parameter, α_j are complex parameters satisfying

$$(1) \qquad \alpha_0 + \cdots + \alpha_{2m} = \eta^{-1},$$

and the independent variable t and the unknown functions u_j are normalized so that

$$(2) \qquad u_0 + \cdots + u_{2m} = t$$

may be satisfied. (It is also assumed that α_j and u_j are cyclic with respect to the index j with the cycle $N = l + 1$.) In view of (2) we find that $(NY)_{2m}$ contains $2m$ independent unknown functions and is equivalent to a $(2m)$-th order nonlinear ordinary differential equation with one unknown function, say, u_0. (For example, $(NY)_2$ is equivalent to the traditional fourth Painlevé equation.)

The system $(NY)_l$ with $l = 2m$ describes the compatibility condition of the following system of linear equations ("Lax pair") of the size $N \times N$ ($N = l + 1 = 2m + 1$):

$$\eta^{-1}\frac{\partial}{\partial x}\psi = A\psi, \tag{3}$$

$$\eta^{-1}\frac{\partial}{\partial t}\psi = B\psi, \tag{4}$$

where

$$A = -\frac{1}{x}\begin{pmatrix} \epsilon_1 & u_1 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \epsilon_{N-2} & u_{N-2} & 1 \\ x & & & \epsilon_{N-1} & u_{N-1} \\ xu_0 & x & & & \epsilon_N \end{pmatrix} \tag{5}$$

and

$$B = \begin{pmatrix} q_1 & -1 & & & \\ & q_2 & -1 & & \\ & & \ddots & \ddots & \\ & & & q_{N-1} & -1 \\ -x & & & & q_N \end{pmatrix}. \tag{6}$$

Here ϵ_j are parameters determined by the relations

$$\alpha_j = \epsilon_j - \epsilon_{j+1} + \eta^{-1}\delta_{j,0}, \quad \epsilon_1 + \cdots + \epsilon_N = 0 \tag{7}$$

($\delta_{j,k}$ denotes Kronecker's symbol), and $q_j = q_j(t)$ are functions of t satisfying

$$q_{j+2} - q_j = u_j - u_{j+1}, \quad q_1 + \cdots + q_N = -t/2. \tag{8}$$

The aim of this report is to analyze the Noumi-Yamada system $(NY)_{2m}$ from the viewpoint of the exact WKB analysis, making full use of the exact WKB analysis of the underlying Lax pair (3) and (4). Note that we have introduced the large parameter η into $(NY)_{2m}$ and its underlying Lax pair (3) and (4) through an appropriate scaling of the variables so that we may discuss the exact WKB analysis for them; the original Noumi-Yamada system is obtained by putting $\eta = 1$.

3 Virtual turning points of higher order linear ordinary differential equations

Before discussing the exact WKB analysis of the Lax pair (3) and (4), we review the definition of an (ordinary) turning point, a Stoks curve and a virtual turning point for a system of first order linear ordinary differential equations with size m ($m \geq 3$). (In this report, since we are discussing the Noumi-Yamada system and its underlying Lax pair, we deal with a system of differential equations instead of a higher order single differential equation. Note that fundamental notions and interesting phenomena for a higher order single differential equation discussed in [BNR], [AKT1], [AKSST] etc. can be easily translated into those for a system of differential equations, as we will see below.)

Let us consider the following system of linear ordinary differential equations:

$$(9) \qquad \eta^{-1}\frac{d}{dx}\psi = A(x,\eta)\psi,$$

where A is an $m \times m$ matrix ($m \geq 3$) of the form

$$(10) \qquad A_0(x) + \eta^{-1}A_1(x) + \eta^{-2}A_2(x) + \cdots$$

and $\eta > 0$ denotes a large parameter. For such a system (9) a polynomial (in λ)

$$(11) \qquad P(x,\lambda) \stackrel{\text{def}}{=} \det(\lambda - A_0(x)) = 0$$

of degree m is called the characteristic equation of (9) and a solution $\lambda_j(x)$ ($j = 1,\ldots,m$) of (11) is called a characteristic root of (9). For each characteristic root $\lambda_j(x)$ there exists a (formal) solution $\psi_j(x,\eta)$ of (9) of the following form:

$$(12) \qquad \psi_j(x,\eta) = \left(\exp \eta \int_{x_0}^{x} \lambda_j(x)dx\right) \sum_{l=0}^{\infty} \psi_{j,l}(x)\eta^{-(l+1/2)},$$

where x_0 is a fixed point and vector-valued functions $\psi_{j,l}(x)$ ($l = 0,1,\ldots$) are recursively determined (up to constants of integration). The solution (12) is called a WKB solution of (9).

Now let us first recall the definition of an ordinary turning point and a Stoks curve.

Definition 1. (i) When two characteristic roots $\lambda_j(x)$ and $\lambda_{j'}(x)$ coalesce at $x = a$, the point a is called an ordinary turning point (of type (j,j')). In particular, when $x = a$ is a simple (resp., double) zero of the discriminant of $P(x,\lambda)$, a is called a simple (resp., double) ordinary turning point. (In what follows the adjective "ordinary" is often omitted if there is no fear of confusions.)

(ii) An integral curve of the direction field

$$(13) \qquad \Im[(\lambda_j(x) - \lambda_{j'}(x))dx] = 0$$

that emanates from an ordinary turning point a of type (j,j') is called a Stokes curve of type (j,j'). That is, a Stokes curve is a real one-dimensional curve determined by the equation

$$(14) \qquad \Im \int_a^x (\lambda_j(x) - \lambda_{j'}(x))dx = 0.$$

Furthermore, a portion of a Stokes curve is labeled as $(j > j')$, or simply $j > j'$, if

$$\Re \int_a^x (\lambda_j(x) - \lambda_{j'}(x))dx > 0 \tag{15}$$

holds there.

In the exact WKB analysis an ordinary turning point and a Stokes curve are important to the effect that the so-called Stokes phenomenon for WKB solutions (to be more precise, for their Borel sums) is in general observed on a Stokes curve. In the case where $m \geq 3$, however, we need to take into account a virtual turning point and a new Stokes curve, which are defined as follows, in addition to ordinary turning points and Stokes curves.

Let $(x(s), y(s), \xi(s), \eta(s))$ be a bicharacteristic strip (to be more precise, a "null-bicharacteristic strip") of the Borel transform of (9), that is, a solution of the following Hamiltonian system determined by the principal symbol $P = P(x, \eta^{-1}\xi)$ of the Borel transform of (9):

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{\partial P}{\partial \xi}, & \dfrac{dy}{ds} = \dfrac{\partial P}{\partial \eta}, \\ \dfrac{d\xi}{ds} = -\dfrac{\partial P}{\partial x}, & \dfrac{d\eta}{ds} = -\dfrac{\partial P}{\partial y}(= 0), \end{cases} \tag{16}$$

with the constraint

$$P(x(s), \eta(s)^{-1}\xi(s)) = 0. \tag{17}$$

Note that, since $P = P(x, \eta^{-1}\xi)$ can be factorized as

$$P(x, \eta^{-1}\xi) = (\eta^{-1}\xi - \lambda_1(x)) \cdots (\eta^{-1}\xi - \lambda_m(x)) \tag{18}$$

except at a turning point, the characteristic set $\{P = 0\}$ (and hence a bicharacteristic strip itself also) has m branches locally. Now, a bicharacteristic strip is a curve in the cotangent bundle $T^*\mathbb{C}^2_{(x,y)}$. We call its projection to the base manifold $\mathbb{C}^2_{(x,y)}$ a bicharacteristic curve. The notion of a virtual turning point is then defined in terms of a bicharacteristic curve as follows:

Definition 2. (i) When a bicharacteristic curve of the Borel transform of (9) crosses itself at a point (x_0, y_0), the point x_0 is called a virtual turning point of (9) (cf. [AKT1], [AKSST]). When the crossing point is determined by a pair of Hamiltonians $(\eta^{-1}\xi - \lambda_j(x))$ and $(\eta^{-1}\xi - \lambda_{j'}(x))$, we say that the virtual turning point is of type (j, j').

(ii) An integral curve of the direction field

$$\Im[(\lambda_j(x) - \lambda_{j'}(x))dx] = 0 \tag{19}$$

that emanates from a virtual turning point $x = v$ of type (j, j') is called a new Stokes curve of type (j, j'). Just like an ordinary Stokes curve a portion of a new Stokes curve is labeled as $(j > j')$ or $j > j'$ if

$$\Re \int_v^x (\lambda_j(x) - \lambda_{j'}(x))dx > 0 \tag{20}$$

holds there. (If there is no fear of confusions the adjective "virtual" or "new" is sometimes omitted.)

As first observed by Berk et al ([BNR]), Stokes phenomena for WKB solutions are observed also on some portions of new Stokes curves for a higher order linear ordinary differential equation or, equivalently, a system of linear differential equations with size $m \geq 3$.

Now, not only an ordinary turning point and a Stokes curve but also a virtual turning point and a new Stokes curve of the underlying Lax pair (3) (and (4)) play an important role for the description of turning points and Stokes curves of the Noumi-Yamada system $(NY)_{2m}$, as we will see in the next section.

4 Sasaki's observation — The relevance of virtual turning points to the creation of new Stokes curves of Noumi-Yamada systems

In this section we discuss the Stokes geometry, i.e. the collection of turning points and Stokes curves, of the Noumi-Yamada system $(NY)_{2m}$ and its relationship with that of the underlying Lax pair (3) (and (4)).

4.1 Turning points and Stokes curves of $(NY)_{2m}$

We first recall the definition of turning points and Stokes curves of $(NY)_{2m}$ (cf. [T2]).

Using the fact that $(NY)_{2m}$ discussed here contains a large parameter η, we can construct a formal power series (in η^{-1}) solution of $(NY)_{2m}$ of the following form:

$$\widehat{u}_j = u_{j,0}(t) + \eta^{-1}u_{j,1}(t) + \eta^{-2}u_{j,2}(t) + \cdots \quad (j = 0, \ldots, 2m), \tag{21}$$

where $\{u_{j,0}(t)\}_{0 \leq j \leq 2m}$ satisfy a system of algebraic equations and, once $\{u_{j,0}(t)\}$ are fixed, the coefficients $\{u_{j,l}(t)\}_{0 \leq j \geq 2m, l \leq 1}$ of lower order terms are determined recursively. Such a formal solution is called a 0-parameter solution of $(NY)_{2m}$.

Let $(\Delta NY)_{2m}$ denote the Fréchet derivative (i.e. the linearized equation) of $(NY)_{2m}$ at the 0-parameter solution $\widehat{u}_j$. A turning point and a Stokes curve of $(NY)_{2m}$ are then defined in terms of $(\Delta NY)_{2m}$ in the following manner:

Definition 3. An (ordinary) turning point (resp., a Stokes curve) of $(NY)_{2m}$ is, by definition, an (ordinary) turning point (resp., a Stokes curve) of $(\Delta NY)_{2m}$.

Since the Fréchet derivative $(\Delta NY)_{2m}$ is a system of linear differential equations of the form

$$\eta^{-1}\frac{d}{dt}\Delta u = (C_0(t) + \eta^{-1}C_1(t) + \cdots)\Delta u, \tag{22}$$

where $\Delta u = {}^t(\Delta u_0, \ldots, \Delta u_{2m})$ denotes an unknown vector-valued function and $C_l(t)$ $(l = 0, 1, \ldots)$ are $(2m+1)\times(2m+1)$ matrices, the definition of its (ordinary) turning points and Stokes curves is given by Definition 1. In the case of $(\Delta NY)_{2m}$, as is shown in [T2], the characteristic equation $\det(\nu - C_0(t)) = 0$ becomes a polynomial of ν^2 of degree m (except for some trivial factor). Thus $(\Delta NY)_{2m}$ has essentially $(2m)$ characteristic roots, which are labeled as $\nu_{1,\pm}, \ldots, \nu_{m,\pm}$ so that they may satisfy $\nu_{j,+} + \nu_{j,-} = 0$ in what follows. Note that, thanks to this peculiar property of the characteristic equation, there are two kinds of (ordinary) turning points for the Noumi-Yamada system $(NY)_{2m}$; one is a turning point where $\nu_{j,+} = -\nu_{j,-} = 0$ holds for some j ("a turning point of the first kind"), and the other is a turning point where $\nu_{j,+} = \nu_{j',+}$ or $\nu_{j,+} = \nu_{j',-}$ holds for some $j \neq j'$ ("a turning point of the second kind").

Now, let us substitute a 0-parameter solution $\widehat{u}_j$ of $(NY)_{2m}$ into the coefficients of the underlying Lax pair (3) and (4) to obtain

$$(LNY)_{2m} \qquad \left\{ \begin{array}{rcl} \eta^{-1}\dfrac{\partial}{\partial x}\psi & = & (A_0(x,t) + \eta^{-1}A_1(x,t) + \cdots)\psi, \\ \eta^{-1}\dfrac{\partial}{\partial t}\psi & = & (B_0(x,t) + \eta^{-1}B_1(x,t) + \cdots)\psi. \end{array} \right.$$

The main results of [T2] claim that the Stokes geometry of $(NY)_{2m}$ is closely related to that of (the first equation of) $(LNY)_{2m}$. To state the relationship between the two Stokes geometries in a specific manner, we prepare some notation here. It is shown in [T2] that the first equation of $(LNY)_{2m}$ has, in general, several simple ordinary turning points and m double ordinary turning points; the former will be denoted by $a_1(t), \ldots, a_n(t)$, where n designates the number of simple turning points, and the latter by $b_1(t), \ldots, b_m(t)$ in what follows. Then the main results of [T2] can be stated as follows:

Proposition 1. *Let $t = \tau^{\mathrm{I}}$ be a turning point of the first kind of $(NY)_{2m}$, that is, $\nu_{j,\pm}(\tau^{\mathrm{I}}) = 0$ holds for some j. Then there exist a simple turning point $a_l(t)$ of the first equation of $(LNY)_{2m}$, a double turning point $b_j(t)$ of the first equation of $(LNY)_{2m}$, and two eigenvalues λ_k and $\lambda_{k'}$ of A_0 that merge both at $x = a_l(t)$ and $x = b_j(t)$, such that the following relations hold:*

$$a_l(\tau^{\mathrm{I}}) = b_j(\tau^{\mathrm{I}}), \tag{23}$$

(24) $$\frac{1}{2}\int_{\tau^{\mathrm{I}}}^{t}(\nu_{j,+}-\nu_{j,-})dt=\int_{a_l(t)}^{b_j(t)}(\lambda_k-\lambda_{k'})dx.$$

In particular, if t lies on a Stokes curve of $(NY)_{2m}$ emanating from τ^{I} and is sufficiently close to τ^{I}, the simple turning point $x=a_k(t)$ and the double turning point $x=b_j(t)$ are connected by a Stokes curve of the first equation of $(LNY)_{2m}$.

Proposition 2. *Let $t=\tau^{\mathrm{II}}$ be a turning point of the second kind of $(NY)_{2m}$, that is, $\nu_{j,+}(\tau^{\mathrm{II}})=\nu_{j',+}(\tau^{\mathrm{II}})$ holds for some j and j'. Then there exist two double turning points $b_j(t)$ and $b_{j'}(t)$ of the first equation of $(LNY)_{2m}$ and two eigenvalues λ_k and $\lambda_{k'}$ of A_0 that merge both at $x=b_j(t)$ and $x=b_{j'}(t)$, such that the following relations hold:*

(25) $$b_j(\tau^{\mathrm{II}})=b_{j'}(\tau^{\mathrm{II}}),$$

(26) $$\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,+}-\nu_{j',+})dt=\int_{b_{j'}(t)}^{b_j(t)}(\lambda_k-\lambda_{k'})dx.$$

In particular, if t lies on a Stokes curve of $(NY)_{2m}$ emanating from τ^{II} and is sufficiently close to τ^{II}, the two double turning points $x=b_j(t)$ and $x=b_{j'}(t)$ are connected by a Stokes curve of the first equation of $(LNY)_{2m}$.

Thus, as in the case of traditional (i.e. second order) Painlevé equations (cf. [KT1], [AKT2]) and hierarchies of the first and second Painlevé equations of higher order (cf. [KKNT1]), (ordinary) turning points and Stokes curves of the Noumi-Yamada system $(NY)_{2m}$ can be characterized near a turning point by the degeneracy of the configuration of ordinary turning points and Stokes curves of the underlying Lax pair $(LNY)_{2m}$. Note that such degeneracy of the Stokes geometry of $(LNY)_{2m}$ induces the Stokes phenomenon for $(NY)_{2m}$ (cf. [T1], where the Stokes phenomena for the traditional first Painlevé equation are explicitly computed by using the degeneracy of the Stokes geometry of its underlying Lax pair). However, in order to describe the Stokes geometry of $(NY)_{2m}$ globally, virtual turning points and new Stokes curves of $(LNY)_{2m}$ become also relevant.

4.2 Bifurcation of Stokes curves — an important role of virtual turning points in the theory of isomonodromic deformations

In the precedent subsection we have reviewed the fact that two ordinary turning points of the Lax pair $(LNY)_{2m}$ are connected by a Stokes curve when the parameter t lies on a Stokes curve of $(NY)_{2m}$ near an (ordinary) turning point. However, as is observed in [AKSST] and [S1], "bifurcation of Stokes curves" often occurs for the deformation of a higher order equation (i.e. a higher order equation containing a parameter) or the deformation of a system of first order equations with size $m \geq 3$. After such a bifurcation phenomenon occurs, the role of an ordinary Stokes curve and that of a new Stokes curve are interchanged, and consequently we may observe that an ordinary turning point and a virtual turning point of $(LNY)_{2m}$ are connected by a Stokes curve when the parameter t lies in some portion, which is rather distant from a turning point, of a Stokes curve of $(NY)_{2m}$. In this subsection we discuss this phenomenon more concretely by making use of the following example.

Example 1. ([S1]) Let us consider the following Noumi-Yamada system $(NY)_{2m}$ with $m = 1$:

$$(NY)_2 \qquad \begin{cases} \eta^{-1}\dfrac{du_0}{dt} &= u_0(u_1 - u_2) + \alpha_0, \\ \eta^{-1}\dfrac{du_1}{dt} &= u_1(u_2 - u_0) + \alpha_1, \\ \eta^{-1}\dfrac{du_2}{dt} &= u_2(u_0 - u_1) + \alpha_2, \end{cases}$$

where $\alpha_0 + \alpha_1 + \alpha_2 = \eta^{-1}$ and $u_0 + u_1 + u_2 = t$ are satisfied. Here, to do concrete numerical computations, we take the following particular value of parameters: $\alpha_0 = 1 + 0.6\sqrt{-1}$ and $\alpha_1 = 0.2 - 0.1\sqrt{-1}$.

This system has an ordinary turning point of the first kind at, for example, $\tau = -1.6276 - 0.0986\sqrt{-1}$. Hereafter we investigate the change of the Stokes geometry of (the first equation of) the underlying Lax pair $(LNY)_2$ along a Stokes curve Γ of $(NY)_2$ emanating from this turning point τ, as is shown in Figure 1.

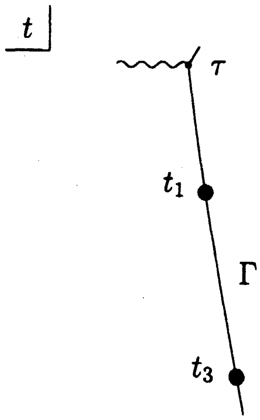

Fig. 1 : Stokes curve Γ of $(NY)_2$ emanating from $\tau = -1.6276 - 0.0986\sqrt{-1}$.

Figure 2 (i) and (iii) indicate the configuration of Stokes curves of $(LNY)_2$ for $t = t_1 = -1.6104 - 0.2268\sqrt{-1}$ and $t = t_3 = -1.5783 - 0.4130\sqrt{-1}$ on the Stokes curve Γ, respectively. In Figure 2 (i) a simple ordinary turning point s_1 and a double ordinary turning point d are connected by a Stokes curve γ_0; this is

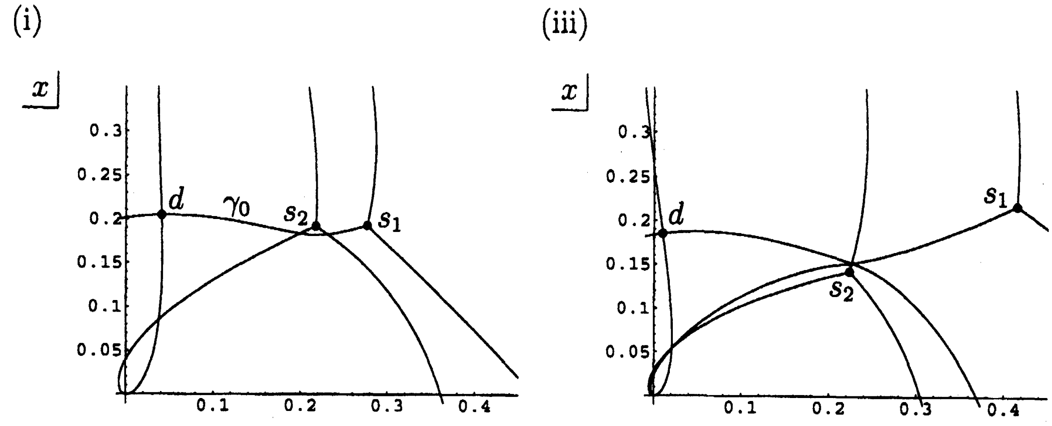

Fig. 2 : Configuration of Stokes curves of $(LNY)_2$ for (i) $t_1 = -1.6104 - 0.2268\sqrt{-1}$ and (iii) $t_3 = -1.5783 - 0.4130\sqrt{-1}$.

consistent with the claim of Proposition 1. (Here, instead of a_l and b_j, we use the symbol s_k and d respectively to denote a simple turning point and a double one for the sake of simplicity.) In Figure 2 (iii), however, these two turning points are no longer connected by a Stokes curve. This is an effect of the following bifurcation phenomenon of Stokes curves: It is readily surmised that a simple ordinary turning point s_2 should cross the Stokes curve γ_0 connecting d and s_1 as t moves from t_1 to t_3, say at $t = t_2$. This actually occurs. As a matter of fact, if we add relevant virtual turning points and new Stokes curves to Figure 2, we obtain Figure 3 which indicates the Stokes geometry of $(LNY)_2$ for $t = t_j$ $(j = 1, 2, 3)$. As is visualized in Figure 3 (ii), at $t = t_2$ the double turning point d is connected both with the simple turning point s_1 and with a virtual turning point v_1. (Similarly s_1 is connected both with d and with another virtual turning point v_2.) This is a typical "bifurcation of Stokes curves" discussed in [AKSST] and [S1]; at $t = t_2$ the relative location of an ordinary Stokes curve emanating from d and that of a new Stokes curve emanating from v_2 are interchanged on the right of their crossing point and consequently, when t reaches $t = t_3$, the target of the Stokes curve emanating from d is switched from s_1 to v_1. Thus at $t = t_3$ the double turning point d is connected with the virtual turning point v_1 and simultaneously the simple turning point s_1 is connected with v_2. In this way in some portion of the Stokes curve Γ of $(NY)_2$ a new kind of degeneracy of the Stokes geometry of $(LNY)_2$ is observed; an ordinary turning point and a virtual turning point are connected by a Stokes curve there.

We also note that, as is shown in Figure 3 (i), the two virtual turning points v_1 and v_2 are connected by a (new) Stokes curve at $t = t_1$, i.e. in a portion of Γ near the turning point $t = \tau$ of $(NY)_2$, in addition to the already-mentioned degeneracy that d and s_1 are connected by the Stokes curve γ_0.

Bifurcation of Stokes curves is a commonly observed phenomenon for the deformation of a higher order equation or a system of first order equations with size

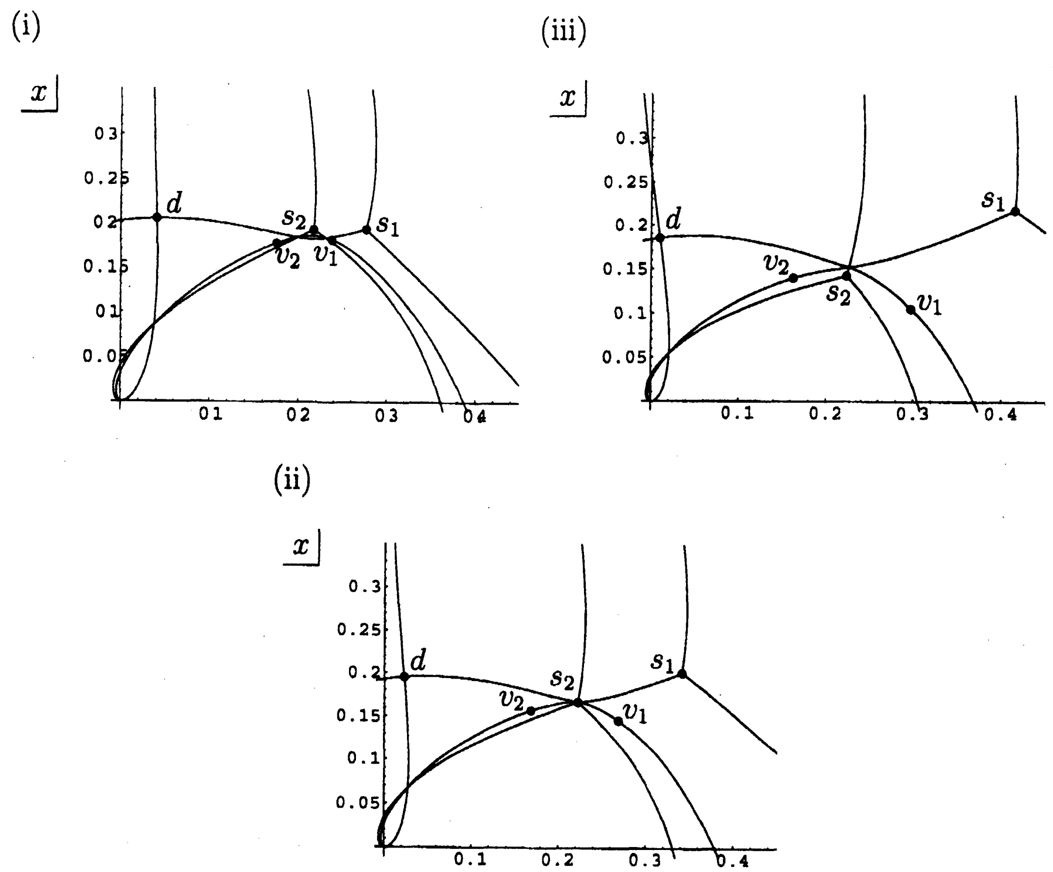

Fig. 3 : Stokes geometry of $(LNY)_2$ with virtual turning points added. (Figure (i) is for $t = t_1$, (ii) for $t = t_2$ and (iii) for $t = t_3$.)

$m \geq 3$. In particular, in the case of $(LNY)_{2m}$ that underlies the Noumi-Yamada system $(NY)_{2m}$, as an effect of these phenomena degenerate configurations of Stokes curves of $(LNY)_{2m}$, i.e. two ordinary and/or virtual turning points being connected by a Stokes curve, can be found simultaneously at several places when the parameter t lies on a Stokes curve of $(NY)_{2m}$, as is exemplified by Example 1. This shows an important role of virtual turning points in the theory of isomonodromic deformations of higher order linear equations or systems of first order linear equations with size $m \geq 3$; virtual turning points are indispensable for the complete description of such degeneracy of the Stokes geometry.

4.3 Sasaki's observation for the creation of new Stokes curves of $(NY)_{2m}$

In his master thesis and its successive paper (cf. [S1, S2]) Sasaki has made an intriguing observation for the creation of new Stokes curves of Noumi-Yamada systems

$(NY)_{2m}$. Virtual turning points of the underlying Lax pair $(LNY)_{2m}$ play an important role also there (in a complicated and somewhat mysterious way). In the final section of this report we discuss Sasaki's observation by using $(NY)_4$ and its underlying Lax pair $(LNY)_4$.

Example 2. ([S2]) Let us consider the following Noumi-Yamada system $(NY)_4$:

$$(NY)_4 \qquad \begin{cases} \eta^{-1}\dfrac{du_0}{dt} &= u_0(u_1-u_2+u_3-u_4)+\alpha_0, \\ & \cdots \\ \eta^{-1}\dfrac{du_4}{dt} &= u_4(u_0-u_1+u_2-u_3)+\alpha_4. \end{cases}$$

Here $\alpha_0+\cdots+\alpha_4=\eta^{-1}$ and $u_0+\cdots+u_4=t$ are satisfied and we take the following particular value of parameters: $\alpha_0=1-0.35\sqrt{-1}$, $\alpha_1=0.45-0.7\sqrt{-1}$, $\alpha_2=-0.5-0.2\sqrt{-1}$ and $\alpha_3=-1.05+0.25\sqrt{-1}$.

Since the system $(NY)_4$ is equivalent to a nonlinear differential equation of fourth order, the so-called Nishikawa phenomenon ([N], [KKNT1]) is expected to occur. That is, two Stokes curves of $(NY)_4$ may cross and a new Stokes curve may appear from such a crossing point of Stokes curves. As a matter of fact, in our case we observe, for example, that a Stokes curve $\Gamma^{(1)}$ of $(NY)_4$ emanating from a turning point of the first kind $\tau^{(1)}=-0.0347+0.1545\sqrt{-1}$ crosses with a Stokes curve $\Gamma^{(2)}$ emanating from another turning point of the first kind $\tau^{(2)}=0.3094+0.4662\sqrt{-1}$ at a point $T=0.3101+0.2789\sqrt{-1}$, as is shown in Figure 4.

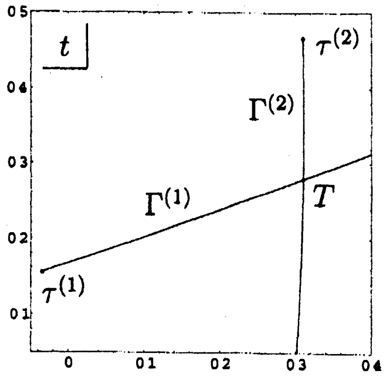

Fig. 4 : Crossing of two Stokes curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of $(NY)_4$ emanating respectively from $\tau^{(1)}=-0.0347+0.1545\sqrt{-1}$ and $\tau^{(2)}=0.3094+0.4662\sqrt{-1}$.

To examine if a new Stokes curve of $(NY)_4$ appears from this crossing point T, we investigate the change of the Stokes geometry of (the first equation of) the underlying Lax pair $(LNY)_4$ around the crossing point. First of all, the Stokes geometry of $(LNY)_4$ at $t=T$ is provided in Figure 5. Although it is a quite complicated figure,

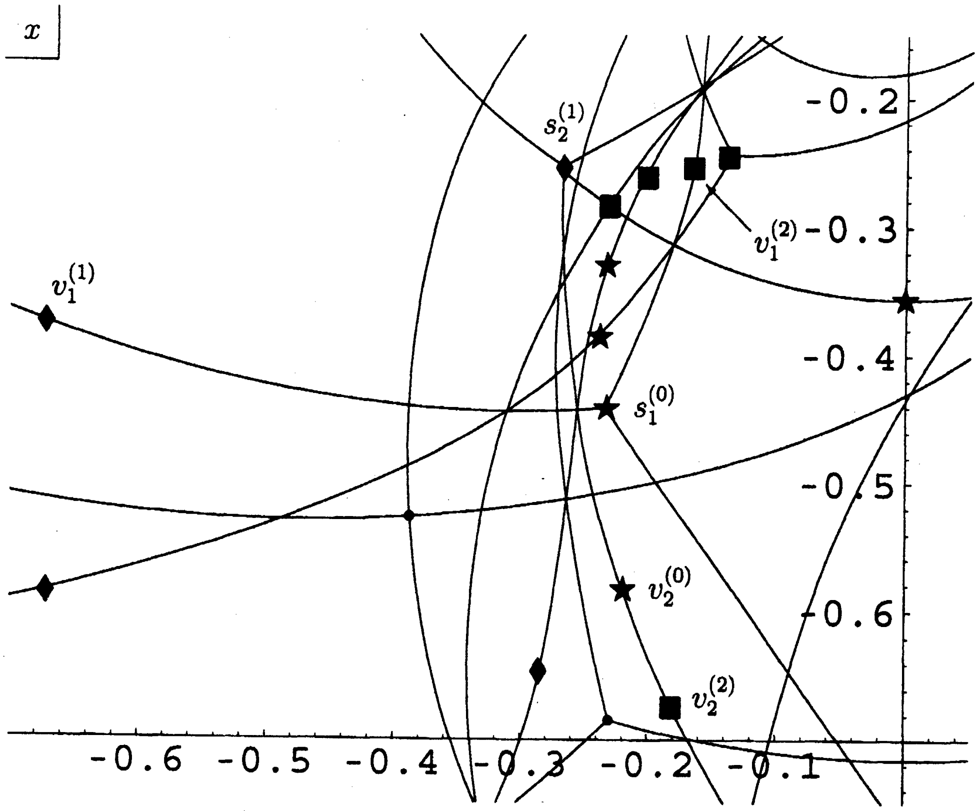

Fig. 5 : Stokes geometry of $(LNY)_4$ at the crossing point $t = T$.

we can recognize that there exist several triplets of (ordinary and/or virtual) turning points that are connected by a Stokes curve of $(LNY)_4$. This is actually effected by the fact that $t = T$ is a crossing point of the two Stokes curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of $(NY)_4$. For example, as $t = T$ lies in $\Gamma^{(1)}$ an ordinary simple turning point $s_1^{(0)}$ is connected with a virtual turning point $v_1^{(1)}$ by a Stokes curve, and simultaneously $s_1^{(0)}$ is connected with another virtual turning point $v_1^{(2)}$ as $t = T$ lies in $\Gamma^{(2)}$. That is, $\{s_1^{(0)}, v_1^{(1)}, v_1^{(2)}\}$ forms such a triplet of turning points. Similarly, a virtual turning point $v_2^{(0)}$ is connected both with an ordinary simple turning point $s_2^{(1)}$ and with a virtual turning point $v_2^{(2)}$; $\{v_2^{(0)}, s_2^{(1)}, v_2^{(2)}\}$ gives another triplet. In Figure 5 we may find at least five such triplets of turning points connected by a Stokes curve. On $\Gamma^{(1)}$ (resp., $\Gamma^{(2)}$) a turning point designated by the symbol ♦ (resp., ■) is connected with a turning point designated by ★. A turning point designated by ★ hinges two degeneracies of the Stokes geometry of $(LNY)_4$ and it is called a "hinging (or, shared) turning point" in what follows. Of course, the existence of (not one, but) several triplets of such turning points is an effect of the bifurcation phenomena of Stokes curves discussed in the precedent subsection.

Next, let us trace the change of the Stokes geometry of $(LNY)_4$ around the crossing point $t=T$. To avoid presenting complicated figures, we see the change of configurations of Stokes curves relevant to each triplet of turning points separately. Let us start with a triplet $\{s_1^{(0)}, v_1^{(1)}, v_1^{(2)}\}$. Figure 6 indicates the change of configurations for this triplet $\{s_1^{(0)}, v_1^{(1)}, v_1^{(2)}\}$. As is clearly visualized there, the relative location of a Stokes curve passing through $v_1^{(1)}$ and that passing through $s_1^{(0)}$ are interchanged both between Figures (ii) and (iii) and between Figures (v) and (vi). This is due to the fact that these two turning points $v_1^{(1)}$ and $s_1^{(0)}$ are connected by a Stokes curve when the parameter t lies in $\Gamma^{(1)}$. Similarly, the topological configurations are different both between Figures (iii) and (iv) and between Figures (vi) and (i) since on $\Gamma^{(2)}$ the two turning points $v_1^{(2)}$ and $s_1^{(0)}$ are connected by a Stokes curve. However, in addition to these differences, we can also observe another difference; the relative locations of Stokes curves passing through $v_1^{(1)}$ and $v_1^{(2)}$ are interchanged between Figures (i) and (ii). This difference means that between the two points t_1 and t_2 there should pass a new Stokes curve $\tilde{\Gamma}$ of $(NY)_4$ where the two virtual turning points $v_1^{(1)}$ and $v_1^{(2)}$ are connected by a Stokes curve of $(LNY)_4$. Note that on the other side of this new Stokes curve, that is, between the two Figures (iv) and (v) we cannot observe any difference of topological configurations; the new Stokes curve $\tilde{\Gamma}$ of $(NY)_4$ is inactive there. The change of configurations described in Figure 6 is exactly the same as that in the case of a higher order member of the (P_J) hierarchy $(J = \mathrm{I}, \mathrm{II}\text{-}1, \mathrm{II}\text{-}2)$ which we discussed in [KKNT1] (see also [N]). Thus we may conclude that the Nishikawa phenomenon is occurring at this crossing point T of Stokes curves of the Noumi-Yamada system $(NY)_4$.

The situation, however, is not so simple as in the case of the (P_J) hierarchy $(J = \mathrm{I}, \mathrm{II}\text{-}1, \mathrm{II}\text{-}2)$. In fact, if we were to guess simple-mindedly the change of configurations of Stokes curves relevant to another triplet $\{v_2^{(0)}, s_2^{(1)}, v_2^{(2)}\}$ in parallel with Figure 6, we should obtain Figure 7. In Figure 7 we readily find that the topological configurations are also different between Figures (iv) and (v). This does not seem consistent with Figure 6. What is a problem? Where does this inconsistency come from?

If we trace the change of configurations for the triplet $\{v_2^{(0)}, s_2^{(1)}, v_2^{(2)}\}$ more precisely and carefully with taking the global structure of the Stokes geometry of $(LNY)_4$ into account, we obtain Figure 8. According to Figure 8, we can say the answer to the above question is that the virtual turning point $v_2^{(2)}$ should "disappear" in the left half plane of t-space (i.e. in the left side of the Stokes curve $\Gamma^{(2)}$ of $(NY)_4$) and this disappearance of $v_2^{(2)}$ recovers the consistency with Figure 6, i.e. the change of configurations for the other triplet $\{s_1^{(0)}, v_1^{(1)}, v_1^{(2)}\}$. As a matter of fact, in Figure 8 no difference is observed between the topological configurations of Figures (iv) and (v) since the virtual turning point $v_2^{(2)}$ and a new Stokes curve emanating from it disappear there. Consequently Figure 8 becomes completely consistent with Figure 6.

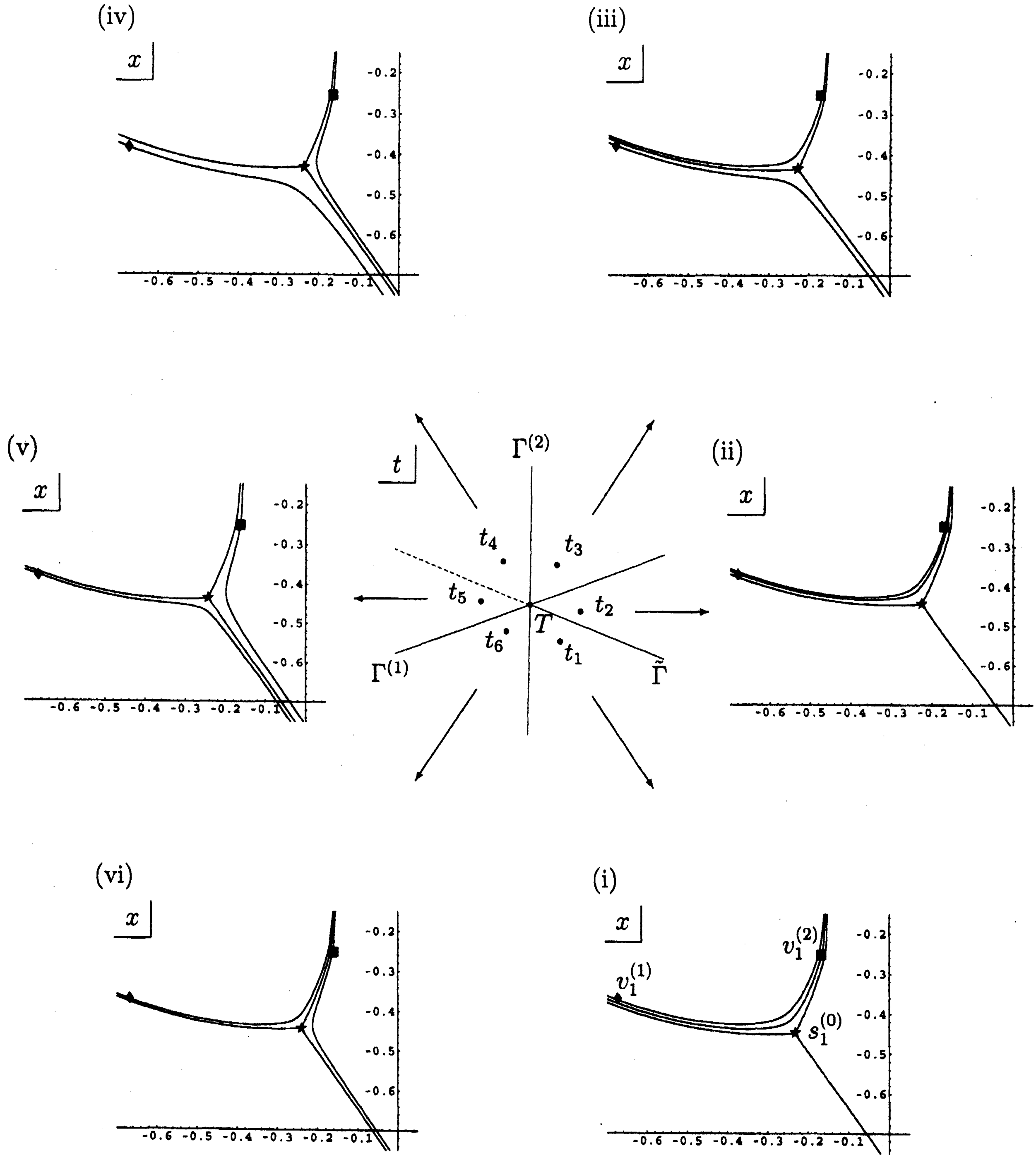

Fig. 6 : Change of configurations for $s_1^{(0)}$, $v_1^{(1)}$, and $v_1^{(2)}$.

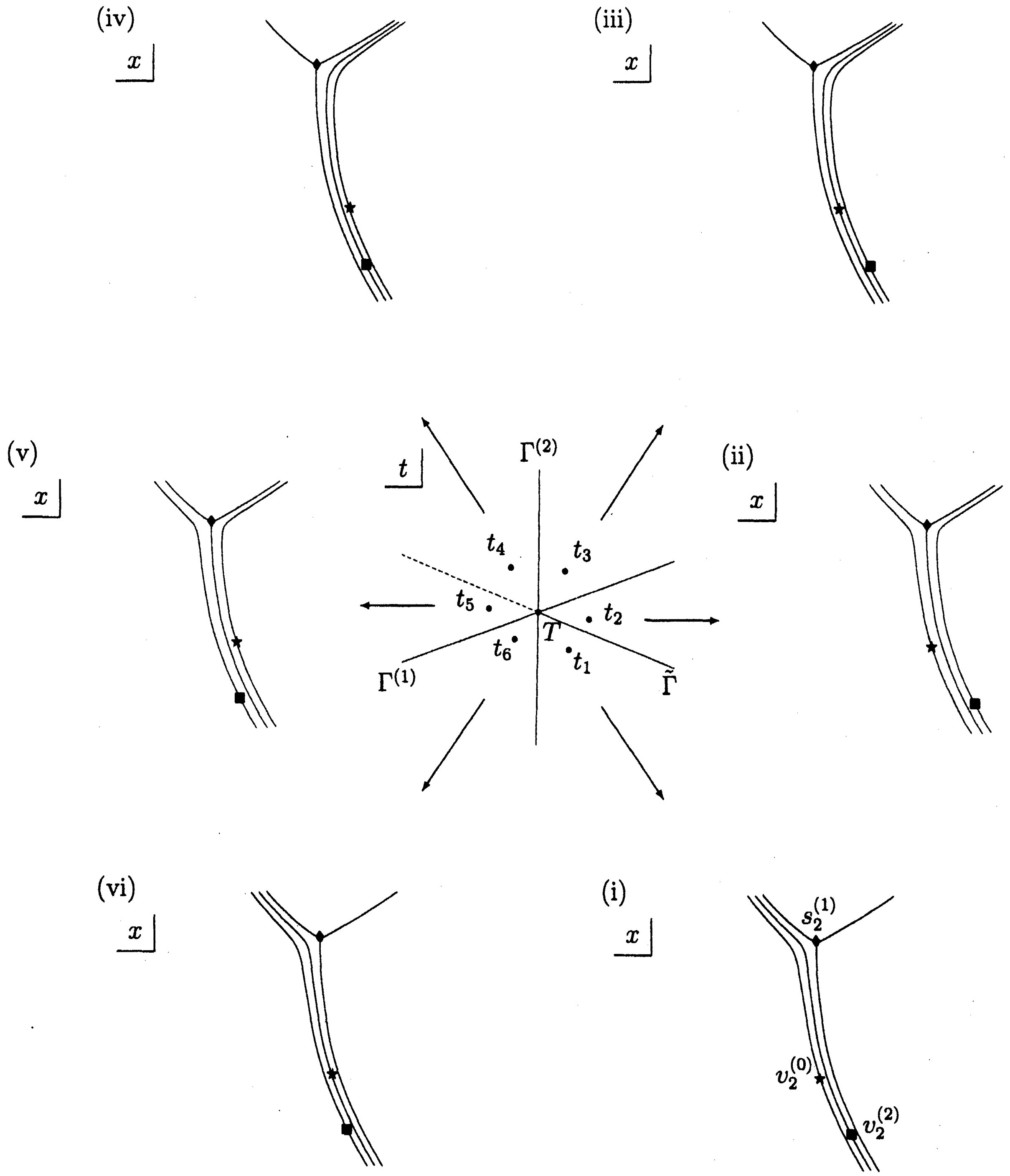

Fig. 7 : Change of configurations for $v_2^{(0)}$, $s_2^{(1)}$, and $v_2^{(2)}$ — Simple-minded guess.

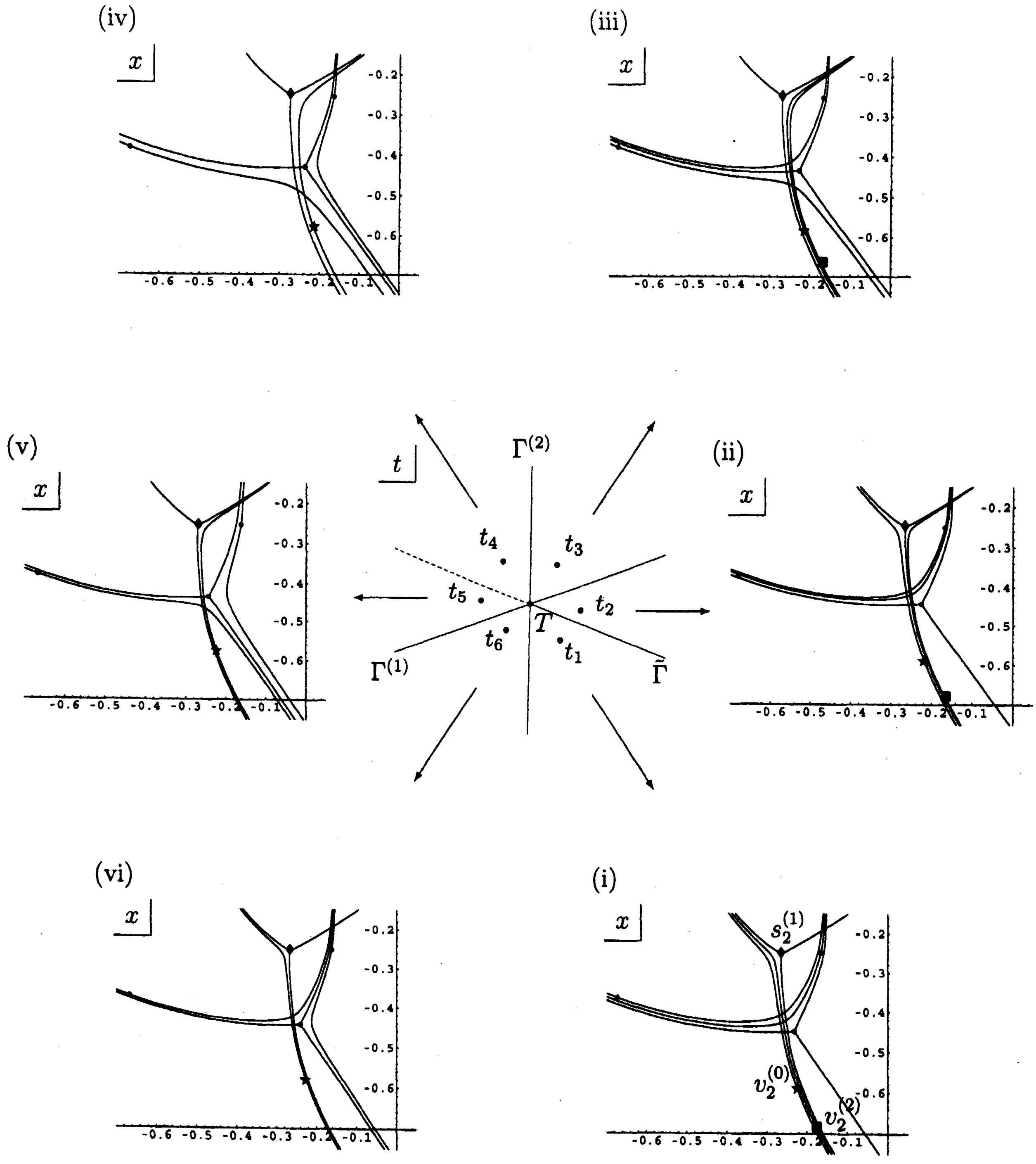

Fig. 8 : Change of configurations for $v_2^{(0)}$, $s_2^{(1)}$, and $v_2^{(2)}$ — Correct figure.

In [S2] such a virtual turning point as $v_2^{(2)}$ is called a "napping virtual turning point"; it appears only in a half plane of t-space ("waking region", the right side of $\Gamma^{(2)}$ in the case of $v_2^{(2)}$), while it disappears in the opposite half plane of t-space ("sleeping region", the left side of $\Gamma^{(2)}$ in this case). The existence of a napping virtual turning point saves us from the inconsistency mentioned above and confirms the appearance of a new Stokes curve $\tilde{\Gamma}$ of $(NY)_4$ at the crossing point $t = T$.

Here let us briefly explain the reason why the change of the state (i.e. "waking" or "sleeping") of $v_2^{(2)}$ occurs. Figure 9 (i) shows the configuration of all the relevant Stokes curves at a point (say, t_1) in the waking region of $v_2^{(2)}$. Figure 9 (i) tells us that a new Stokes curve γ_2 emanating from the virtual turning point $v_2^{(2)}$ in question appears in conjunction with the crossing of a new Stokes curve γ_1 emanating from a virtual turning point $v_1^{(2)}$ and a new Stokes curve γ_3 emanating from another virtual turning point $v_3^{(1)}$. On the other hand, the configuration at a point (say, t_{11}) in the sleeping region of $v_2^{(2)}$ becomes as is described in Figure 9 (ii). In Figure 9 (ii) the relative location of the new Stokes curve γ_1 emanating from $v_1^{(2)}$ and that of a Stokes curve emanating from an ordinary turning point $s_1^{(0)}$ are interchanged so that γ_1 goes downward to the right. As its consequence γ_1 no longer crosses with γ_3 in Figure 9 (ii). Hence the new Stokes curve γ_2 and its starting point $v_2^{(2)}$ disappear there. This is the mechanism that induces the change of the status of the napping virtual turning point $v_2^{(2)}$. Such a subtle mechanism related to the global structure of the Stokes geometry produces a napping virtual turning point.

We finally note that in the case of $\{v_2^{(0)}, s_2^{(1)}, v_2^{(2)}\}$ a hinging (or, shared) turning point is a virtual turning point $v_2^{(0)}$. This caused the above apparent inconsistency between the change of configurations for $\{v_2^{(0)}, s_2^{(1)}, v_2^{(2)}\}$ and that for the other triplet $\{s_1^{(0)}, v_1^{(1)}, v_1^{(2)}\}$ whose hinging turning point is an ordinary turning point $s_1^{(0)}$. In case a hinging (or, shared) turning point of a triplet in question is a virtual turning point, we believe that a napping virtual turning point should be contained in this triplet to avoid such apparent inconsistency.

In conclusion, at a crossing point of two Stokes curves of the Noumi-Yamada system $(NY)_{2m}$ $(m \geq 2)$ we can expect the following:

> Assume that two Stokes curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of $(NY)_{2m}$ cross at $t = T$. In this situation, as is discussed in the precedent subsection, for each Stokes curve $\Gamma^{(k)}$ $(k = 1, 2)$ there should exist several pairs of (ordinary and/or virtual) turning points $\{x_j^{(k)}(t), \tilde{x}_j^{(k)}(t)\}_{j=1,2,\ldots}$ of the underlying Lax pair $(LNY)_{2m}$ that are connected by a Stokes curve simultaneously. Then, if every pair $\{x_j^{(1)}(t), \tilde{x}_j^{(1)}(t)\}$ for $\Gamma^{(1)}$ and the corresponding pair $\{x_j^{(2)}(t), \tilde{x}_j^{(2)}(t)\}$ for $\Gamma^{(2)}$ share one turning point and "Lax-adjacency" (cf. [KKNT1], see also Probelm 3 below) holds there, a new Stokes curve of $(NY)_{2m}$ emanates from the crossing point $t = T$.
>
> Furthermore, if a shared turning point is a virtual turning point, a napping virtual turning point should be contained in these pairs.

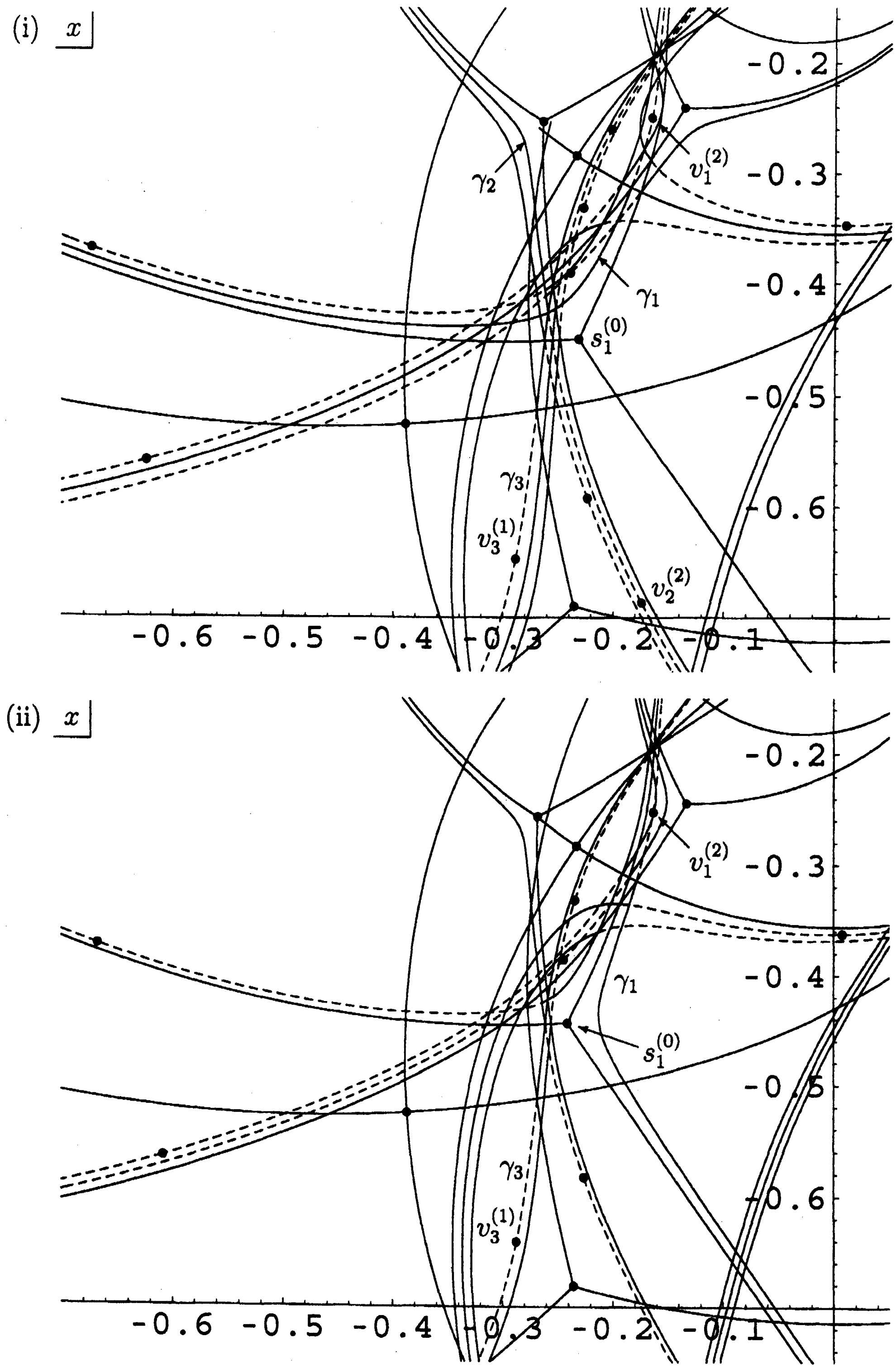

Fig. 9 : Configuration of Stokes curves of $(LNY)_4$ for (i) t_1, i.e. in the waking region of $v_2^{(2)}$, and (ii) t_{11}, i.e. in the sleeping region of $v_2^{(2)}$.

In this manner virtual turning points of $(LNY)_{2m}$ play an important role also for the creation of new Stokes curves of Noumi-Yamada systems $(NY)_{2m}$.

However, there still remain many things to be studied. In ending this report, we list up some problems concerning the creation of new Stokes curves of Noumi-Yamada systems $(NY)_{2m}$.

Problem 1. To study analytic properties (e.g. the connection formulas) for Stokes phenomena on both ordinary and new Stokes curves of $(NY)_{2m}$.

This is the most important problem for the global study of solutions of Noumi-Yamada systems. To discuss Problem 1 we need deeper understanding for the Stokes geometry of the underlying Lax pair $(LNY)_{2m}$ discussed in this report. For example, the following points should be clarified.

Problem 2. On each Stokes curve of $(NY)_{2m}$ there exist several pairs of (ordinary and/or virtual) turning points of $(LNY)_{2m}$ that are connected by a Stokes curve simultaneously, and at a crossing point of two Stokes curves of $(NY)_{2m}$ from which a new Stokes curve starts there exist several triplets of turning points of $(LNY)_{2m}$ connected by a Stokes curve simultaneously. Then, are all triplets of turning points equally important for the study of Stokes phenomena on the new Stokes curve, or only a part of them relevant to Stokes phenomena? If only a small number of triplets are concerned with Stokes phenomena, how should we choose them?

Problem 3. The "Lax-adjacency", i.e. the key property that determines whether a new Stokes curve does really appear or not at a crossing point of two Stokes curves of a higher order Painlevé equation, is defined in [KKNT1] for the Painlevé hierarchies (P_J) $(J = \mathrm{I}, \mathrm{II}\text{-}1, \mathrm{II}\text{-}2)$ where the size of the underlying Lax pair is 2×2. Then, what is the precise definition of the "Lax-adjacency" for Noumi-Yamada systems $(NY)_{2m}$?

To answer these problems we need to develop more systematic study of the Stokes geometry of $(LNY)_{2m}$. Recently Honda has been undertaking such systematization. The details of his study will be reported in [H].

References

[AKSST] T. Aoki, T. Kawai, S. Sasaki, A. Shudo and Y. Takei, Virtual turning points and bifurcation of Stokes curves for higher order ordinary differential equations, J. Phys. A, **38**(2005), 3317–3336.

[AKT1] T. Aoki, T. Kawai and Y. Takei, New turning points in the exact WKB analysis for higher-order ordinary differential equations, Analyse algébrique des perturbations singulières. I, Hermann, Paris, 1994, pp. 69–84.

[AKT2] ———, WKB analysis of Painlevé transcendents with a large parameter. II, Structure of Solutions of Differential Equations, World Scientific, 1996, pp. 1–49.

[BNR] H.L. Berk, W.M. Nevins and K.V. Roberts, New Stokes' line in WKB theory, J. Math. Phys., **23**(1982), 988–1002.

[H] N. Honda, Toward the complete description of the Stokes geometry, in preparation.

[KKNT1] T. Kawai, T. Koike, Y. Nishikawa and Y. Takei, On the Stokes geometry of higher order Painlevé equations, Astérisque, Vol. 297, 2004, pp. 117–166.

[KKNT2] ———, On the complete description of the Stokes geometry for the first Painlevé hierarchy, RIMS Kôkyûroku, Vol. 1397, 2004, pp. 74–101.

[KT1] T. Kawai and Y. Takei, WKB analysis of Painlevé transcendents with a large parameter. I, Adv. Math., **118**(1996), 1–33.

[KT2] ———, WKB analysis of Painlevé transcendents with a large parameter. III, Adv. Math., **134**(1998), 178–218.

[KT3] ———, Algebraic Analysis of Singular Perturbation Theory, Translations of Mathematical Monographs, Vol. 227, Amer. Math. Soc., 2005. (Originally published in Japanese by Iwanami, Tokyo in 1998.)

[N] Y. Nishikawa, Towards the exact WKB analysis of the P_{II} hierarchy, submitted.

[NY] M. Noumi and Y. Yamada, Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial Ekvac., **41**(1998), 483–503.

[S1] S. Sasaki, The role of virtual turning points in the deformation of higher order linear ordinary differential equations. I, RIMS Kôkyûroku, Vol. 1433, 2005, pp. 27–64. (In Japanese.)

[S2] ———, The role of virtual turning points in the deformation of higher order linear ordinary differential equations. II, ibid., pp. 65–109. (In Japanese.)

[T1] Y. Takei, An explicit description of the connection formula for the first Painlevé equation, Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto Univ. Press, 2000, pp. 271–296.

[T2] ———, Toward the exact WKB analysis for higher-order Painlevé equations — The case of Noumi-Yamada systems, Publ. RIMS, Kyoto Univ., **40**(2004), 709–730.