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Branner-Hubbard-Lavaurs deformation of parabolic cubic polynomials

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Since the wring deformation (or the Branner-Hubbard deformation) changes nothing on the filled-in Julia set, it does not deform polynomials with connected Julia sets. For polynomials with parabolic cycles, the Lavaurs maps enable us to deform complex structures also in the parabolic basins. This deformation, the *Branner-Hubbard-Lavaurs deformation*, can deform such polynomials. We will show that this happens for real cubic polynomials with parabolic fixed points of multiplier one. This is closely related to the non-landing of stretching rays. We will also show that the real BHL-deformation set coincides with the accumulation set of a stretching ray.

1 Stretching rays for real cubic polynomials

We consider a family of real cubic polynomials of the form :

$$\mathcal{P}_3 = \{P_{A,B}(z) = z^3 - 3Az + \sqrt{B}; \quad A, B > 0\}.$$

For $P \in \mathcal{P}_3$, let φ_P be its Böttcher coordinate. For a positive number $s > 0$, put $\ell_s(z) = z|z|^{s-1}$ and we define a P-invariant Beltrami form μ_s by

$$\mu_s := \begin{cases} (\ell_s \circ \varphi_P)^* \mu_0 \text{ in a nbd of } \infty , \\ \mu_0 \text{ on } K(P). \end{cases}$$

Then, by the Measurable Riemann Mapping Theorem, μ_s is integrated by a qc-map χ_s and $P_s := \chi_s \circ P \circ \chi_s^{-1} \in \mathcal{P}_3$. Thus we define a real analytic map $W_P : \mathbb{R}_+ \to \mathcal{P}_3$ by $W_P(s) = P_s$. The Böttcher coordinate φ_{P_s} of P_s is equal to $\ell_s \circ \varphi_P \circ \chi_s^{-1}$. Since P_s is hybrid equivalent to P, it holds $P_s \equiv P$ for $P \in \mathcal{C}_3$, the *connectedness locus*. For $P \in \mathcal{E}_3$, the *escape locus*, we define the *stretching ray* through P by

$$R(P) = W_P(\mathbb{R}_+) = \{P_s; s \in \mathbb{R}_+\}.$$

On the *shift locus*, where both critical points $\pm\sqrt{A}$ are escaping, we define the *Böttcher vector* by

$$\eta(P) := \frac{1}{\log 3} \log\log|\varphi_P(-\sqrt{A})| - \frac{1}{\log 3} \log\log|\varphi_P(\sqrt{A})|.$$

Lemma 1.1. *([KN]) The Böttcher vector is constant on each stretching ray in the shift locus.*

In the shift locus of our family $\mathcal{P}_3$, stretching rays are level curves of the Böttcher vector map $P \mapsto \eta(P)$.

2 Branner-Hubbard-Lavaurs deformation

Consider the locus $Per_1(1) = \{B = 4(A+1/3)^3; 0 < A < 1/9\}$ in $\mathcal{P}_3$, where the map Q has a parabolic fixed point β_Q of multiplier one whose immediate basin $\mathcal{B}_Q$ contains both critical points. Let $\phi_{Q,-}$ and $\phi_{Q,+}$ denote the attracting and repelling Fatou coordinates respectively of the parabolic fixed point β_Q for $Q \in Per_1(1)$. The *Lavaurs map* $g_{Q,\sigma} : \mathcal{B}_Q \to \mathbb{C}$ with *lifted phase* $\sigma \in \mathbb{R}$ is defined by $g_{Q,\sigma} = \phi_{Q,+}^{-1} \circ T_\sigma \circ \phi_{Q,-}$, where $T_\sigma(z) = z + \sigma$. For $Q \in Per_1(1)$ and $\sigma \in \mathbb{R}$, we define a $< Q, g_{Q,\sigma} >$-invariant Beltrami form $\mu_{s,\sigma}$ by

$$\mu_{s,\sigma} := \begin{cases} \mu_s \text{ in } \mathbb{C} - K(Q), \\ (g_{Q,\sigma}^n)^* \mu_s \text{ in } \mathcal{B}_Q \cap g_{Q,\sigma}^{-n}(\mathbb{C} - K(Q)), \\ \mu_0 \text{ otherwise.} \end{cases}$$

Then, as before, there exists a qc-map $\chi_{s,\sigma}$ such that $\mu_{s,\sigma} = \chi_{s,\sigma}^* \mu_0$, $Q_{s,\sigma} := \chi_{s,\sigma} \circ Q \circ \chi_{s,\sigma}^{-1} \in Per_1(1)$.

Lemma 2.1. *The map $\chi_{s,\sigma} \circ g_{Q,\sigma} \circ \chi_{s,\sigma}^{-1}$ is a Lavaurs map of $Q_{s,\sigma}$ with some lifted phase $\sigma(s)$.*

We call $(Q_{s,\sigma}, \sigma(s))$ the *Branner-Hubbard-Lavaurs deformation* of (Q, σ). We also define the *BHL-ray* $L(Q,\sigma)$ through (Q,σ) by

$$L(Q,\sigma) = \{(Q_{s,\sigma}, \sigma(s)) \in Per_1(1) \times \mathbb{R}; s \in \mathbb{R}_+\}$$

and the *Böttcher-Lavaurs vector* by

$$\eta(Q,\sigma) := \frac{1}{\log 3} \log\log \varphi_Q(g_{Q,\sigma}(-\sqrt{A})) - \frac{1}{\log 3} \log\log \varphi_Q(g_{Q,\sigma}(\sqrt{A})).$$

Note that this is well defined because $\varphi_Q(g_{Q,\sigma}(\pm\sqrt{A})) > 1$. It satisfies $\eta(Q, \sigma+1) = \eta(Q,\sigma)$. By the same argument as in the proof of Lemma 1.1, we have the following.

Lemma 2.2. *The Böttcher-Lavaurs vector $\eta(Q,\sigma)$ is constant on each BHL-ray.*

For $Q \in Per_1(1)$, we define the *Fatou vector* by $\tau(Q) := \phi_{Q,-}(-\sqrt{A}) - \phi_{Q,-}(\sqrt{A})$.

Lemma 2.3. *The Fatou vector gives a real analytic parametrization of the locus $Per_1(1)$.*

It easily follows that $Q_{s,\sigma} \equiv Q$ if $\tau(Q) \in \mathbb{Z}$, that is, if Q has a critical orbit relation.

Theorem 2.1. *(Non-trivial BHL-deformation)*
If $\tau(Q) \notin \mathbb{Z}$, then the map $s \mapsto Q_{s,\sigma}$ is not constant for any σ.

Such a map is first obtained in Willumsen [W] in the region $A < 0$. See also Tan Lei [T]. Once we get such a non-trivial deformation, the following corollary is essentially due to [W].

Corollary 2.1. *(Discontinuity of wring operation)*
Suppose $\tau(Q) \notin \mathbb{Z}$. Then the map $(P, s) \mapsto W_P(s)$ is discontinuous at (Q, s) if $Q_{s,\sigma} \neq Q$ for some σ.

The region $\mathcal{R}_0 := \{B > 4(A + 1/3)^3\}$ is contained in the shift locus. Stretching rays in $\mathcal{R}_0$ are uniquely labelled by the Böttcher vector. Let $R(\eta)$ denote the ray with level η.

Theorem 2.2. *([KN], Non-landing of stretching rays)*
If $\eta \in \mathbb{Z}$, then $R(\eta)$ lands at $Q \in Per_1(1)$ with $\tau(Q) = \eta$. If $\eta \notin \mathbb{Z}$, then $R(\eta)$ has a non-trivial accumulation set on $Per_1(1)$.

The following theorem suggests that stretching rays are obtained from the rescaling of BHL-rays and seems to explain the regular oscillation of stretching rays.

Theorem 2.3. *Suppose $\tau(Q) \notin \mathbb{Z}$. Then the BHL-deformation set $\{Q_{s,\sigma}; s > 0\}$ of Q coincides with the accumulation set of the stretching ray $R(\eta)$, where $\eta = \eta(Q, \sigma)$.*

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