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On cubic polynomials with a parabolic fixed point of a capture type

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Abstract

We consider the location of each critical point of a cubic polynomial map with a parabolic fixed point. We show that, for any given number of iterations, there exists a cubic polynomial map with a parabolic fixed point such that the immediate parabolic basin contains just one of the critical points and the image of another critical point under the specified number of iterations.

1 Introduction

Let f be any cubic polynomial. If f has a parabolic fixed point α, then a cycle of Fatou components of f is called the immediate parabolic basin for α if the cycle contains a parabolic petal for α.

Roughly speaking, in this note we consider the dynamically location of each critical points of f with a parabolic fixed point whose basin contains both the critical points. We denote by c_0 and c_1 the critical points of f. Using the Haissinsky pinching deformation, we prove the following result:

Theorem 1.1. For any positive integer n, there exists a cubic polynomial map f with a parabolic fixed point such that the immediate parabolic basin contains c_0 and $f^{\circ n}(c_1)$, and does not contain $f^{\circ k}(c_1)$ for any integer k with $0 \leq k < n$.

Now, suppose that f has a parabolic fixed point, and the parabolic basin contains c_0 and c_1. By analogy with Milnor [3], we shall define the types of this parabolic fixed point. For $j = 0, 1$, we denote by U_j the Fatou component which contains c_j. Without loss of generality, we may assume that U_0 is contained in the immediate basin of the parabolic fixed point. Following from [3], there exist four possibilities as follows.

Case 1: The Fatou component is adjacent, i.e., $U_0 = U_1$.

Case 2: The Fatou component is bitransitive. Namely, $U_0 \neq U_1$, and moreover there exist the smallest positive integers $p, q > 0$ such that $f^{\circ p}(U_0) = U_1$ and $f^{\circ q}(U_1) = U_0$.

Case 3: The immediate parabolic basin captures U_1. Namely, the immediate parabolic basin does not contain U_1, but $f^{\circ k}(U_1)$ for some integer $k \geq 1$.

Case 4: Each of U_0 and U_1 is contained in the disjoint cycle of the immediate parabolic basin. Namely, U_0 and U_1 is contained in the immediate parabolic basin, and it follows that $f^{\circ n}(U_0) \cap f^{\circ m}(U_1) = \emptyset$ for any integers $n, m \geq 0$.

We define the types of the parabolic fixed point α as follows:

Definition 1.2. In Case 1, 2, 3 or 4, we say that α is a parabolic fixed point of an *adjacent*, *bitransitive*, *capture*, or *disjoint* type, respectively.

We will consider the type of the parabolic fixed point a the cubic polynomial map obtained by the Haissinsky pinching deformation, which is illustrated in the next section.

2 The Haissinsky Pinching deformation

Suppose that f is any cubic polynomial map with an attracting fixed point α. Let $B_f(\alpha)$ be the attracting basin for α. We consider the Haissinsky pinching deformation of f defined by pinching curves in $B_f(\alpha)$.

Following from [1], for any integer $q \geq 1$, there exist a smooth open arc γ and a neighborhood $U \subset B_f(\alpha)$ of γ satisfying the following conditions.

- $\overline{\gamma} \setminus \gamma$ consists of the attracting fixed point α and a repelling periodic point β of period q.
- $f^{\circ q}(\gamma) = \gamma$, $f^{\circ q}(U) = U$, and $f^{\circ q}|_U$ is univalent.
- $f^{\circ n}(U) \cap f^{\circ m}(U) = \emptyset$ for any $0 \leq n < m < q$.
- There exist a number $\sigma > 0$ and a conformal map $\Phi_\sigma : U \to \{|z| < \pi\}$ such that $\Phi_\sigma \circ f^{\circ q}(z) = \Phi_\sigma(z) + \sigma$ for all $z \in U$.

We call the union $S := \bigcup_{k \geq 0} f^{\circ -k}(\overline{\gamma})$ the *support of pinching*, and define $S_0 := \bigcup_{k \geq 0} f^{\circ k}(\overline{\gamma})$. It follows from [1] that we have a sequence of quasiconformal maps $(h_t)_{t \geq 0}$ satisfying the following conditions.

- h_t converges uniformly on $\widehat{\mathbb{C}}$ to a local quasiconformal map h_∞ on $\widehat{\mathbb{C}} \setminus S$.
- $f_t := h_t \circ f \circ h_t^{-1}$ converges uniformly on $\widehat{\mathbb{C}}$ to a cubic polynomial f_∞.
- $h_\infty(\alpha)$ is a parabolic fixed point of f_∞.
- $h_\infty(S_0) = h_\infty(\alpha)$.

For further details, see [1] or [2].

3 Proof of Theorem 1.1

We first prove the following lemma needed later.

Lemma 3.1. Let n be any positive integer, and let λ be any complex number in $\mathbb{D} \setminus \{0\}$. Then there exists a cubic polynomial f, with $f^{\circ n}(c_1) = c_0$, such that f has an attracting fixed point of multiplier λ whose attracting basin is simply connected.

Proof. Consider a monic and centered cubic polynomial

$$P_{A,B}(z) = z^3 - 3Az + \sqrt{B},\ (A, B) \in \mathbb{C}^2.$$

Suppose that $P_{A,B}$ has a fixed point of multiplier λ. Then the fixed point is $\alpha_{A,\lambda} := \sqrt{A + \lambda/3}$, and hence, $P_{A,B}$ is affine conjugate to the cubic polynomial map

$$Q_{A,\lambda}(z) = z^3 + 3\alpha_{A,\lambda} z^2 + \lambda z$$

with critical points $c^{\pm}_{A,\lambda} := -\alpha_{A,\lambda} \pm \sqrt{A}$.

Suppose that $\lambda \in (-1, 0)$, and the parameter A is any real number $> -\lambda/3$ such that the attracting basin for zero is simply connected.

For each integer $k \geq 0$, we denote by $z_{A,\lambda}(k)$ the unique point on $\mathbb{R}_+$ such that $Q^{\circ k}_{A,\lambda}(z_{A,\lambda}(k)) = c^+_{A,\lambda}$. For any integer $k > 0$ and for any real number A' with $A' > A$, we have $z_{A,\lambda}(k) < z_{A,\lambda}(k+1)$ and $z_{A,\lambda}(k) > z_{A',\lambda}(k)$. Thus since $Q_{A,\lambda}(c^-_{A,\lambda}) \to +\infty$ as $A \to +\infty$, for any integer $n > 0$ there exists a real number A such that $Q^{\circ n}_{A,\lambda}(c^-_{A,\lambda}) = c^+_{A,\lambda}$.

Let λ' be any complex number in $\mathbb{D} \setminus \{0\}$. Then it follows from [5] that there exists a quasiconformal map h such that the cubic polynomial map $g := h \circ Q_{A,\lambda} \circ h^{-1}$ has an attracting fixed point with multiplier λ'. $\square$

We use the Haissinsky pinching deformation of f obtained from this lemma.

Proof of Theorem 1.1. Without loss of generality, we may assume that $f(z) = z^3 + 3\alpha_{A,\lambda} z^2 + \lambda z$, $c_0 = c^+_{A,\lambda}$ and $c_1 = c^-_{A,\lambda}$.

Suppose that λ is any real number with $-1 < \lambda < 0$, and A is a real number $> -\lambda/3$ such that the attracting basin for zero is simply connected. Recall that $B_f(0)$ is the attracting basin for zero. Let φ_f be the Kœnigs map such that $\varphi_f(0) = 0$, and $\varphi_f(z) = \lambda z$ for all $z \in B_f(0)$. We may assume that $\varphi_f(c_0) = 1$.

Define the half-line $\hat{\gamma} := i\mathbb{R}^+$, so that $\hat{\gamma}$ is periodic of period two under the iterates of the map $L(z) := \lambda z$. We denoted by γ the connected component of the preimage of $\hat{\gamma}$ under φ_f whose closure contains zero. Thus, we have the support of pinching $S := \bigcup_{k \geq 0} f^{\circ -k}(\overline{\gamma})$, and denote by f_∞ the limit of the Haissinsky pinching deformation of f defined by S.

Let n be any positive integer. From Lemma 3.1, we have a parameter A such that $f^{\circ n}(c_1) = c_0$. For each integer $k \geq 1$, we denote by $\alpha(k)$ the point on $\mathbb{R}_+$ such that $f^{\circ k}(\alpha(k)) = 0$, and by $S_{\alpha(k)}$ the connected component of S which contains $\alpha(k)$.

At first consider the case $n \geq 2$. Since for each integer $k \geq 1$ the component $S_{\alpha(k)}$ separates the origin and $f^{\circ k}(c_1)$, it follows that f_∞ has a parabolic fixed point of a capture type.

Next, consider the case $n = 1$. Since no connected component of S separates the origin and c_1, it follows that f_∞ has a parabolic fixed point of a bitransitive type.

In order to obtain a polynomial with a parabolic fixed point of a capture type, we will use the Branner-Hubbard deformation of f obtained by wringing the almost complex structure on the attracting basin for zero (cf. [5]). In particular, we consider the Branner-Hubbard deformation which does not change the multiplier of the origin.

Let $s = 1 + 2\pi i/\log\lambda$, and let l be the quasi-conformal map defined as $l(z) := z\,|z|^{s-1}$.

Recall that φ_f is the Kœnigs map defined on $B_f(0)$. We define the holomorphic map $\psi_f : \mathbb{D} \to \mathbb{C}$ as the inverse map of φ_f such that $\psi_f(0) = 0$.

Let σ_0 be the standard almost complex structure of $\widehat{\mathbb{C}}$, and let σ be the almost complex structure defined as follows:

$$\sigma = \begin{cases} \sigma_0 & \text{on } \widehat{\mathbb{C}} \setminus B_f(0) \\ (l \circ \varphi_f)^*(\sigma_0) & \text{on } \psi_f(\mathbb{D}) \\ (l \circ \varphi_f \circ f^{\circ k})^*(\sigma_0) & \text{on } f^{-k}(\psi_f(\mathbb{D})) \setminus f^{-k+1}(\psi_f(\mathbb{D})), \end{cases} \tag{1}$$

where k is an integer ≥ 1.

From the Measurable Riemann Mapping Theorem, we obtain the quasi-conformal map h such that $h^*\sigma_0 = \sigma$. Suppose that $h(0) = 0$, $h(1) = 1$

and $h(\infty) = \infty$. Then, we obtain a cubic polynomial map $g = h \circ f \circ h^{-1}$ with the attracting fixed point zero. It follows from [5] that the multiplier is $g'(0) = h(\lambda) = \lambda\,|\lambda|^{s-1} = \lambda$, and that the Kœnigs map $\varphi_g = l \circ \varphi_f \circ h^{-1}$.

Following from the argument similar to the above discussion, we define $S' \subset B_g(0)$ as the support of the pinching deformation, and denote by g_∞ the limit of the pinching deformation of g defined by the support S'.

There exists a cycle of connected components of $B_f(0) \setminus h^{-1}(S')$ under the iterates of f,

If c_1 is not contained in this cycle, then one of the critical points of g_∞ is not contained in the immediate parabolic basin of g_∞.

We consider the inverse image of $i\mathbb{R}$ under $\varphi \circ h^{-1}$. We introduce a preliminary definition as follows. For any point z of the backward orbit of the origin, we denote by $D_f(z; r)$ the connected component of the set $\{w : |\varphi_f(w)| < r\}$ which contains the point z.

Since f has no critical point in the open set $D_f(0; |\lambda|^{-1})$ except c_0, it follows that f maps $D_f(0; |\lambda|^{-1}) \setminus \{c_0\}$ to $D_f(0; 1) \setminus \{c_0\}$ in two-to-one correspondence. Thus f has the unique preimage α' of the origin such that $\alpha' \neq 0$ and $\alpha' \in D_f(0; |\lambda|^{-1}) \setminus \{c_0\}$.

We extend ψ_f to the conformal map $\psi_{f,0}$ defined on $\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1})$ to a subset of $D_f(0; |\lambda|^{-1})$. Moreover, we define $\psi_{f,1}$ as the conformal map defined on $\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1})$ such that $\varphi_f \circ \psi_{f,1} \equiv$ identity map and $\psi_{f,1}(0) = \alpha'$.

The end points of the image of the set $\{yi \mid -|\lambda|^{-1} < y < |\lambda|^{-1}\}$ under $\psi_{f,0} \circ h^{-1}$ is contained in the boundary of $\psi_{f,1}(\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1}))$. Hence, the connected component of the preimage of $i\mathbb{R}$ under $\varphi_f \circ h^{-1}$ which contains zero passes through the boundary of $\psi_{f,1}(\mathbb{D}(0; |\lambda|^{-1}) \setminus [1, |\lambda|^{-1}))$, and does not separate c_0 and c_1. On the other hand, the connected component of the preimage of $i\mathbb{R}$ under $\varphi_f \circ h^{-1}$ which contains α' separates c_0 and c_1. Therefore, the cycle of the Fatou components of g does not contain one of the critical points of g, and hence g_∞ has a parabolic fixed point of a capture type. $\square$

4 Notes

Consider the family of cubic polynomials $P_{A,B}(z) := z^3 - 3Az + \sqrt{B}$ with $P_{A,B}(-\sqrt{A}) = \sqrt{A}$. We have $B = A(1-2A)^2$. The connectedness locus of the family of $P_{A,A(1-2A)^2}(z) = z^3 - 3Az + \sqrt{A} - 2A\sqrt{A}$, $A \in \mathbb{C}$, is showed in Figure 2.

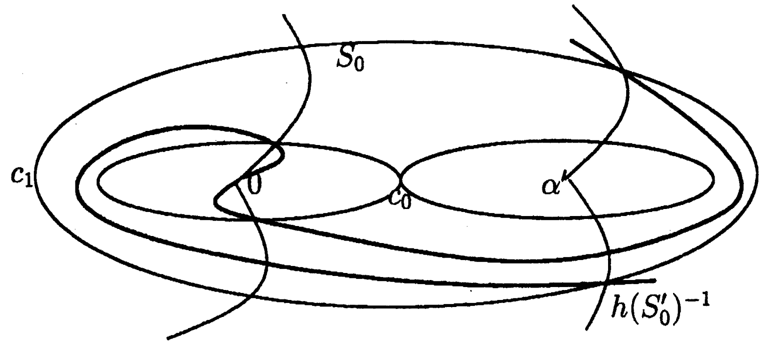

Figure 1: Sketch for the pinching curves.

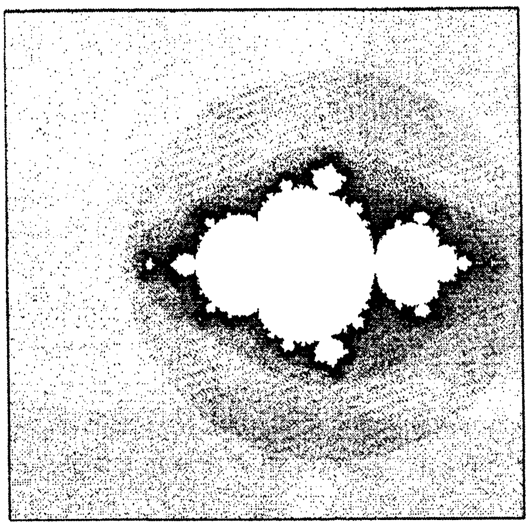

Figure 2: The connectedness locus of the family of cubic polynomials $P_{A,A(1-2A)^2}$, $A \in \mathbb{C}$.

$P_{A,A(1-2A)^2}$ is affine conjugate to the cubic polynomial map

$$F_A(z) := (P_{A,A(1-2A)^2}(\sqrt{A}z + \sqrt{A}) - \sqrt{A})/\sqrt{A} = Az^3 + 3Az^2 - 4A.$$

Suppose that $0 < |A| < 1/4$. Then the map F_A satisfies the inequality $|F_A(z) + 4A| < |4A|$, that is, F_A maps the disk of radius $|F_A(0)|$ centered at $F_A(0)$ into itself. Hence F_A has an attracting fixed point in the disk.

Let α_A be the attracting fixed point.

Proposition 4.1. If A turns around the origin once, then the multiplier of the attracting fixed point of F_A turns around the origin twice.

Proof. Let D be the disk of radius $|F_A(0)|$ centered at $F_A(0)$. If A turns around the origin once, then the center of D turns around the origin once.

Set $0 < r < 1/4$, $\theta \in [0,1]$, and $A = re^{2\pi i\theta}$. Since the radius of D is the constant $|F_A(0)|$, the attracting fixed point α_A also turns around the origin once. Thus the multiplier $F'_A(\alpha_A) = 3A\alpha_A(\alpha_A + 2)$ turns around the origin twice. □

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