数理解析研究所講究録
1494巻 2006年 106-116

Bijectivity of straightenings for families of renormalizable cubic polynomials

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May 23, 2006

Abstract

We give a description of combinatorics of renormalizable polynomials in terms of rational lamination, and study the set $\mathcal{R}_d(f_0)$ of renormalizable polynomials of degree d with a given center f_0, and prove that if f_0 is non-contiguous, then $\mathcal{R}_d(f_0)$ is compact.

Moreover, in the case of capture (or disjoint) renormalizations of cubic polynomials, if $\mathcal{R}_3(f_0)$ is compact, then the straightening map on it is bijective (or homeomorphism). We also prove that we can always characterize the combinatorics of a renormalizable cubic polynomial of capture (or disjoint) type in such a way.

1 Introduction

1.1 The case of quadratic polynomials

It is well-known that the Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C};\ J(z^2+c) \text{ is connected}\}$ contains infinitely many homeomorphic copies of itself, called *baby Mandelbrot sets*, proved by Douady and Hubbard [Ha]. A baby Mandelbrot set corresponds to a set of renormalizable parameters of given combinatorics. More precisely, consider a quadratic polynomial z^2+c_0 such that the critical point 0 is periodic. Then there exists a homeomorphism $T_{c_0} : \mathcal{M} \to \mathcal{M}(c_0) \subset \mathcal{M}$ such that $T_{c_0}(0) = c_0$ and for $c' = T_{c_0}(c)$, z^2+c' has a renormalization hybrid equivalent to z^2+c except when $c = \frac{1}{4}$ and $\mathcal{M}(c_0)$ is of *satellite type*, that is, $T_{c_0}(c)$ is on the boundary of some hyperbolic component other than the one containing c_0.

Definition. Let z^2+c_0, $\mathcal{M}(c_0)$ and T_{c_0} be as above. We call c_0 the *center* and $T_{c_0}(\frac{1}{4})$ the *root* of $\mathcal{M}(c_0)$. A homeomorphism T_{c_0} is the *tuning map* and its inverse $S_{c_0} = T_{c_0}^{-1}$ is the *straightening map* for $\mathcal{M}(c_0)$.

If $\mathcal{M}(c_0)$ is not of satellite type, then it is of *primitive type*.

Here we consider another description of a baby Mandelbrot set, which is based on the notion of rational laminations introduced by Thurston [Th] (see also [Ki]).

Definition. Let f be a polynomial of degree $d \geq 2$ such that $J(f)$ is connected. Consider a equivalence relation λ_f on $\mathbb{Q}/\mathbb{Z}$ such that $\theta \sim_{\lambda_{c_0}} \theta'$ if and only if the external rays $R_f(\theta)$ and $R_f(\theta')$ for f land at the same point. It is called the *rational lamination* of f.

We can characterize $\mathcal{M}(c_0)$ combinatorially as follows:

$$\mathcal{M}(c_0) = \{c \in \mathbb{C};\ \lambda_{z^2+c} \supset \lambda_{z^2+c_0}\}.$$

Here rational laminations are considered as subsets of $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$. In other words, $\lambda \subset \lambda'$ if λ' is stronger relation than λ;

$$\theta \sim_\lambda \theta' \Rightarrow \theta \sim'_\lambda \theta'.$$

Let s be the period of 0 for $z^2 + c_0$. Define

$$\mathcal{R}(c_0) = \{c \in \mathcal{M}(c_0);\ z^2 + c \text{ has a renormalization of period } s\}.$$

Then

$$\mathcal{M}(c_0) \setminus \mathcal{R}(c_0) = \begin{cases} \{\text{root}\} & \text{for satellite case,} \\ \emptyset & \text{for primitive case.} \end{cases}$$

In particular, $\mathcal{M}(c_0) = \mathcal{R}(c_0)$ if and only if it is of primitive case and it is equivalent to $\mathcal{R}(c_0)$ is compact.

1.2 The case of cubic polynomials

For $d \geq 2$, let Poly_d be the family of monic centered polynomials of degree d. Let $\mathcal{C}_d = \{f \in \mathrm{Poly}_d;\ J(f) \text{ is connected}\}$ be the *connectedness locus* of Poly_d. We notice that Poly_d can be considered as the parameter space of affine conjugacy class of polynomial of degree d with marked invariant external ray (which determines that of "angle 0"). In this paper, we always consider an polynomials (or a polynomial-like map) has marked invariant external ray.

Now we consider the case $d = 3$. Milnor [Mi] classified hyperbolic cubic polynomials into four cases:

Adjacent case One periodic Fatou component contains two critical points (with multiplicity).

Bitransitive case One periodic Fatou component U contains a critical point and its forward image $f^n(U)(\neq U)$ contains the other.

Capture case One periodic Fatou component U contains a critical point and the other critical point is contained in a strictly preperiodic Fatou component U', hence $f^N(U') = U$ for some $N \geq 1$.

Disjoint case There exist two disjoint periodic cycle of Fatou components. Each contains one critical point.

We apply this classification also to renormalizations of cubic polynomials.

Definition. For $f \in \mathcal{C}_3$, a *renormalization* of f is a polynomial-like restriction $f^s : U' \to U$ of some iterate of f with connected filled Julia set $K(f^s; U', U) = \bigcap_{n \geq 0}(f^s|_{U'})^{-n}(U')$ such that every critical point ω of f is contained in at most one of $U', f(U'), \dots, f^{s-1}(U')$. We identify renormalizations of f if they have the same filled Julia set.

Adjacent case We say f has an *adjacent renormalization* if f has a renormalization of degree 3.

Bitransitive case We say f has an *bitransitive renormalization* if f has a renormalization of degree 4.

Capture case We say f has an *capture renormalization* if f has a renormalization $f^s : U' \to U$ of degree two such that a critical point $\omega_0 \in U'$ and the other critical point ω (called the *captured critical point*) satisfied that $f^N(\omega) \in K(f^s; U', U)$ for some $N > 0$. The smallest such N is called the *capture time.*

Disjoint case We say f has *disjoint renormalizations* if f has two renormalizations $f^{s_i} : U_i' \to U_i$ of degree two such that critical points ω_1, ω_2 of f satisfies $\omega_i \in U_i'$.

When $d \geq 4$, we can similarly classify renormalizable polynomials whose renormalizations contain or capture all the critical points.

As in the case of quadratic polynomials, we would like to consider combinatorial characterization of renormalizations.

Definition. We say $f_0 \in \mathcal{C}_d$ is a *center* if f_0 is hyperbolic and postcritically finite (every critical point is eventually periodic).

Note that even if a critical point of a center f_0 is preperiodic, its periodic orbit is superattracting.

For a center $f_0 \in \mathcal{C}_d$, define

$$\mathcal{C}_d(f_0) = \{f \in \mathcal{C}_d;\ \lambda_f \supset \lambda_{f_0}\},$$
$$\mathcal{R}_d(f_0) = \left\{ f \in \mathcal{C}_d(f_0);\ \begin{array}{c} f \text{ has a renormalization (renormalizations)} \\ \text{of the same type and period(s) as } f_0 \end{array} \right\}$$

By definition, $\mathcal{C}_2(z^2 + c_0) = \mathcal{M}(c_0)$ and $\mathcal{R}_2(z^2 + c_0) = \mathcal{R}(c_0)$ when $d = 2$.

Definition. Let $f_0 \in \mathcal{C}_d$ be a center. We say f_0 is *contiguous* if there exist Fatou components U_1, U_2 such that $\overline{U_1}$ intersects $\overline{U_2}$.

It is equivalent that there exist *periodic* Fatou components U_1, U_2 such that $\overline{U_1}$ intersects $\overline{U_2}$.

The first theorem is valid for any degree:

Main Theorem 1. *Let $f_0 \in \mathcal{C}_d$ be a center. If f_0 is contiguous, then $\mathcal{R}_d(f_0) = \mathcal{C}_d(f_0)$*

and it is compact.

This theorem is a generalization of the results in [In2].

Remark 1. When $d \geq 3$, $\mathcal{C}_d(f_0)$ is not compact in general. In fact, let $f_0(z) = z^3 - \frac{3}{2}z$. The critical points of f_0 are $\omega_\pm = \pm\frac{1}{\sqrt{2}}i$ and $f_0(\omega_\pm) = -2\omega_\pm^3 = \omega_\pm$. Therefore, f_0 has two invariant Fatou components $U_\pm$. It is easy to see that $\overline{U_+} \cap \overline{U_-} = \{0\}$. Since f_0 is real, the external rays of angle 0 and $\frac{1}{2}$ land at 0. In other words, 0 and $\frac{1}{2}$ are λ_{f_0}-equivalent.

Now consider $f(z) = z^3 + \frac{11}{12}z + \frac{1}{108}$. It is affinely conjugate to $g(z) = z^3 + \frac{1}{2}z^2 + z$. It is easy to see that f and g are real and g (hence f) has a real parabolic fixed point of multiplier one and a real repelling fixed point. Therefore, $f \notin \mathcal{C}_3(f_0)$. However, for small $\varepsilon > 0$, $f + \varepsilon \in \mathcal{C}_3(f_0)$. Indeed, since f is parabolic-attracting (following Adam Epstein), $f + \varepsilon$ must has attracting fixed point. However, $f + \varepsilon$ has only one real fixed point, which is repelling. Hence the other fixed points x and $\bar{x}$ are both attracting. This implies that their attractive basins both contains the real fixed point. Thus $\lambda_{f+\varepsilon} = \lambda_{f_0}$ and $f + \varepsilon \in \mathcal{C}_3(f_0)$.

Therefore, $\mathcal{C}_3(f_0) \ni f + \varepsilon \to f \notin \mathcal{C}_3(f_0)$ and $\mathcal{C}_3(f_0)$ is not compact.

In the following, we assume $d = 3$ and consider only quadratic renormalizations (i.e., capture or disjoint renormalizations).

Main Theorem 2. *Consider a capture renormalization (or disjoint renormalizations) of $f \in \mathcal{C}_3$. Then there exists a center f_0 such that $f \in \mathcal{R}_d(f_0)$ and the given renormalization(s) of f are the renormalization(s) in the definition of $\mathcal{R}_d(f_0)$.*

The Julia set of f can be non-locally connected (such as the case f has a Cremer periodic point). However, we can still find enough landing relations which characterize the renormalization(s).

Now we define the straightening map, which is based on the straightening theorem by Douady and Hubbard [DH2].

Theorem 1.1 (Straightening theorem). *Every polynomial-like map $f : U' \to U$ of degree $d \geq 2$ is hybrid equivalent to a polynomial g of the same degree.*

Furthermore, if $K(f; U', U)$ is connected, then g is unique up to affine conjugacy.

We say a polynomial-like map $f : U' \to U$ is hybrid equivalent to another polynomial-like map $g : V' \to V$ (or polynomial g) if there exists a quasiconformal map ψ defined near $K(f; U', U)$ such that $\bar{\partial}\psi = 0$ a.e. on $K(f; U', U)$ and $g \circ \psi = \psi \circ f$. Such ψ is called a *hybrid conjugacy* between f and g.

As noticed before, we always consider polynomials and polynomial-like maps with marked invariant external rays. Hence if $K(f; U', U)$ is connected, $g \in \mathrm{Poly}_d$ is uniquely determined. Furthermore, although ψ is not uniquely determined, $\psi|_{K(f;U',U)}$ is uniquely determined.

Definition. Let $f_0 \in \mathcal{C}_3$ be a center of capture or disjoint type. Define the *straightening map S_{f_0}* as follows:

Capture case For $f \in \mathcal{R}_3(f_0)$, let $f^s : U' \to U$ be the corresponding renormalization and ω be the captured critical point and N be the capture time. By the straightening theorem, $f^s : U' \to U$ is hybrid equivalent to a unique quadratic polynomial of the form $g(z) = z^2 + c$ by a hybrid conjugacy ψ. Denote $z = \psi(f^N(\omega))$ (it is well-defined).
Define $S_{f_0}(f) = (c, z)$. Then S_{f_0} is defined as a map $S_{f_0} : \mathcal{R}_3(f_0) \to \mathcal{MK} = \bigcup_{c \in \mathcal{M}} \{c\} \times K(z^2 + c)$.

Disjoint case For $f \in \mathcal{R}_3(f_0)$, let $f^{s_i} : U_i' \to U_i$ $(i = 1, 2)$ be the corresponding renormalizations. (We fix the order of renormalizations.) By the straightening theorem, there exists a polynomial $z^2 + c_i$ hybrid equivalent to $f^{s_i} : U_i' \to U_i$.
Define $S_{f_0}(f) = (c_1, c_2) : \mathcal{R}_3(f_0) \to \mathcal{M}^2$.

Remark 2. By the upper semi-continuity of $K(z^2 + c)$ on c, $\mathcal{MK}$ is a compact set in $\mathbb{C}^2$.

Remark 3. For a quadratic-like family $(f_\lambda : U_\lambda' \to U_\lambda)_{\lambda \in \Lambda}$, we can take a straightening map $S : \Lambda \to \mathrm{Poly}_2$ $(f_\lambda \sim_{hb} S(f_\lambda))$ to be continuous [DH2].

Hence if $f_0 \in \mathcal{C}_3$ is of capture type, $f \mapsto c$ is continuous. It is not known whether $f \mapsto z$ is continuous or not.

If f_0 is of disjoint type, S_{f_0} is continuous.

Main Theorem 3. *Let $f_0 \in \mathcal{C}_3$ be a center of capture or disjoint type. If $\mathcal{R}_d(f_0)$ is compact, then the straightening map S_{f_0} is bijective.*

In particular, if f_0 is of disjoint type, then $S_{f_0} : \mathcal{R}_d(f_0) \to \mathcal{M}^2$ is a homeomorphism.

Remark 4. Our proofs of Main Theorem 2 and Main Theorem 3 depends on the continuous dependence of straightening maps of quadratic-like families (Remark 3). Hence our proofs can be generalized only to quadratic renormalizations (of higher degree polynomials), and we cannot apply them to renormalizations of higher degree.

2 Proof of Main Theorem 1

The proof of Main Theorem 1 depends on [In2, Theorem 1.1], which is the following in our context.

Theorem 2.1. *Let $f_0 \in \mathcal{C}_d$ be a center and $f_l \to f$ be a convergent sequence with $f_l \in \mathcal{C}_d(f_0)$.*

If there exist more than $d-1$ disjoint sets O_k $(k = 1, \ldots N \geq d-1)$ which consist of periodic orbits and separate periodic Fatou components of f_0, then $f \in \mathcal{C}_d(f_0)$.

Here, we say a union O of (repelling or parabolic) periodic orbits of f_0 *separates periodic Fatou components* if there exists $n \geq 0$ such that different periodic Fatou components lie in different components of

$$\mathbb{C} \setminus \overline{\bigcup_{\theta \in \Theta} R_{f_0}(\theta)}$$

where Θ be the set of landing angles of $f^{-n}(O)$.

Hence it is sufficient to show the following two lemmas to prove Main Theorem 1:

Lemma 2.2. *If $f_0 \in \mathcal{C}_d$ is a center of non-contiguous type, then there exist infinitely many sets consisting of periodic orbits which separate periodic Fatou components of f_0.*

Lemma 2.3. *If $f_0 \in \mathcal{C}_d$ is a center of non-contiguous type, then $\mathcal{R}_d(f_0) = \mathcal{C}_d(f_0)$.*

The proof of Lemma 2.3 is a standard argument using Yoccoz puzzles. Compare [In2, Proposition 5.6] (see also [Mc, §8.2] and [In1, Lemma 4.1]).

To prove Lemma 2.2, we need the result of Poirier [Po]. First we introduce the notion of Hubbard trees [DH1].

Definition. Let f_0 be a postcritically finite polynomial. A Jordan arc in the closure of a bounded Fatou component of f_0 is *regulated* if it consists of (at most two) segments of internal rays. More generally, a Jordan arc in $K(f_0)$ is *regulated* if its intersection with the closure of any Fatou component is regulated.

Any two points $z_1, z_2 \in K(f_0)$ can be joined uniquely by a regulated arc, which we denote by $[z_1, z_2]_{f_0}$.

Definition. A set $X \subset K(f_0)$ is *regulated connected* if for any $z_1, z_2 \in X$, we have $[z_1, z_2]_{f_0} \subset X$. A *regulated hull* $[X]_{f_0}$ of $X \subset K(f_0)$ is the minimal closed regulated connected subset of $K(f_0)$ containing X.

The *Hubbard tree* T_{f_0} of f_0 is the regulated hull of the postcritical set of f_0.

The following result is proved by Poirier [Po, Theorem B].

Theorem 2.4. *Let f_0 be a postcritically finite polynomial. Let $v \in T_{f_0} \cap J(f_0)$ be a periodic point. Then The number of rays land at v is equal to the number of incident edges of T_{f_0} at v. Furthermore, there is exactly one ray landing between each pair of consecutive edges.*

Proof of Lemma 2.2. Let f_0 is a center of non-contiguous type. Consider an edge e of T_{f_0}. Then there exist some $n \geq 0$, and $p > 0$ such that $f_0^{n+p}(e) \supset f_0^n(e)$.

Since f_0 is non-contiguous, e is not contained in a union of the closure of periodic Fatou components. This implies that $f_0^n(e)$ contains infinitely many repelling periodic points.

Therefore we can take O by choosing one periodic point for each edge e and taking the union of their orbits. Clearly, there exist infinitely many such choices. $\square$

3 Proofs of Main Theorem 2 and Main Theorem 3

For simplicity, we only treat capture renormalizations in this section. The case of disjoint renormalizations is similar.

3.1 Proof of Main Theorem 3, part 1: injectivity

Here we prove the injectivity of the straightening map. Let f_0 be a center of capture type and let $f_1, f_2 \in \mathcal{C}_d$. Assume $S(f_1) = S(f_2) = (c_0, z_0)$. Take a set O consists of periodic orbits of f_0 which separates periodic Fatou components of f_0 and let Θ be the set of landing angles of O.

Consider a sequence $\Lambda = (\lambda_k)_{k\geq 0}$ of equivalence relations on $\mathbb{Q}/\mathbb{Z}$ by $\lambda_k = \lambda_{f_0}|_{f_0^{-n}(O)}$. For each $k \geq 0$, we say $\theta, \theta' \in \mathbb{R}/\mathbb{Z}$ are λ_k*-unlinked* if for any $\theta_1 \sim_{\lambda_k} \theta_2$, θ and θ' lie in the same component of $\mathbb{R}/\mathbb{Z} \setminus \{\theta_1, \theta_2\}$. It is an equivalence relation and let $\mathcal{P}_k(\Lambda)$ be the set of λ_k-unlinked classes. For each $L \in \mathcal{P}_k(\Lambda)$, we can define the corresponding *Yoccoz puzzle* $P_{f_i}(L)$ *of depth* k *for* f_i (see [In2] for more details). Denote $\mathcal{P}_k(f_i, \Lambda)$ the set of Yoccoz puzzles of depth k for f_i.

Then $\mathcal{P}_k(f_i, \Lambda)$ has the following properties:

(i) $\mathcal{P}_k(f_i, \Lambda)$ is a partition of $K(f_i)$; the interiors of each puzzle piece of depth k are disjoint and $\bigcup_{P\in\mathcal{P}_k(f_i,\Lambda)} P \supset K(f_i)$.
(ii) For any $P \in \mathcal{P}_{k+1}(f_i, \Lambda)$, there exists some $P' \in \mathcal{P}_k(f_i, \Lambda)$ such that $P' \supset P$.
(iii) For any $P \in \mathcal{P}_{k+1}(f_i, \Lambda)$, $f(P) \in \mathcal{P}_k(f_i, \Lambda)$.
(iv) $\theta \in L \in \mathcal{P}_k(\Lambda)$ is equivalent that $R_{f_i}(\theta)$ intersects the interior of $P_{f_i}(L)$.

Define a quasiconformal homeomorphism $\varphi_0 : \mathbb{C} \to \mathbb{C}$ such that for each $P_{f_1}(L) \in \mathcal{P}_0(f_1, \Lambda)$, $\varphi_0(P_{f_1}(L)) = P_{f_2}(L)$ and $\varphi_0 \circ f_1 = f_2 \circ \varphi_0$ on ∂P (for any $P \in \mathcal{P}_0(f_1, \Lambda)$), $\partial U'$ and $\bigcup_{n=0}^{s-1} f_1^n(K(f_1^s; U_1', U_1))$. Since $f_1^s : U_1' \to U_1$ and $f_2^s : U_2' \to U_2$ are hybrid equivalent, such a homeomorphism φ_0 exists.

For each $k > 0$, define $\varphi_k : U_1 \to U_2$ inductively as follows. Let $\varphi_k = \varphi_{k-1}$ outside $\bigcup_{P\in\mathcal{P}_k(f_1,\Lambda)} P$. For $P \in \mathcal{P}_k(f_1, \Lambda)$, take the corresponding puzzle piece $P' \in \mathcal{P}_k(f_2, \Lambda)$. We can define a quasiconformal homeomorphism $\varphi_k : P \to P'$ such that $\varphi_{k-1} \circ f_1 = f_2 \circ \varphi_k$. Indeed, it is trivial if $f_1 : P \to f_1(P)$ is conformal. Assume P contains a critical point. Then it is $\omega_0 \in K(f_1^s; U_1', U_1)$ or the captured critical point ω. If $\omega_0 \in P$, then since $\omega_0 \in K(f_1^s; U_1', U_1)$, define $\varphi_k = \varphi_{k-1}$ on $K(f_1^s; U_1', U_1)$ and it can be extend quasiconformally on P. If $\omega \in P$, then since $\varphi_0(f_1^N(\omega)) = f_2^N(\omega')$, where $\omega' \in P'$ is the captured critical point of f_2. Therefore, we can lift φ_{k-1} by f_1 and f_2 and define φ_k on P.

By construction, φ_k is a quasiconformal homeomorphism and the maximal dilatation of φ_k does not depend on k. Therefore, passing to a subsequence, φ_k converges to a quasiconformal homeomorphism φ. It is a conjugacy between f_1 and f_2 and since $\bigcup_{k\geq 0} f^{-k}(K(f_i^s; U_i', U_i)) = K(f_i)$ a.e., it is a hybrid conjugacy. Since f_1 and f_2 are cubic polynomials with connected Julia set, they are affinely conjugate.

Therefore, the straightening map $S : \mathcal{C}_3(f_0) \to \mathcal{MK}$ is injective.

3.2 Proof of Main Theorem 2

A basic tool for the proofs of Main Theorem 2 and the surjective part of Main Theorem 3 is the following (cf. [DH2, Proposition 12]):

Lemma 3.1. *Let $f_n : U'_n \to U_n$ be a sequence of quadratic-like maps and let $z_n \in K(f_n; U'_n, U_n)$. Assume*

(i) *f_n converges to a quadratic-like map $f : U' \to U$, i.e., $f_n \to f$ uniformly on some neighborhood of $K(f; U', U)$.*
(ii) *$z_n \to z \in K(f; U', U)$.*
(iii) *f_n is hybrid equivalent to $g_n(z) = z^2 + c_n$ and f is hybrid equivalent to $g(z) = z^2 + c$.*
(iv) *$c_n \to c$.*
(v) *$K(f_n) \to K(f)$ in the Hausdorff topology.*

Then g_n converges to g and passing to a subsequence, ψ_n converges to a hybrid conjugacy ψ between f and g.
In particular, $\psi_n(z_n)$ converges to $\psi(z)$.

Proof of Main Theorem 2. Assume $f \in \mathcal{C}_3$ has a renormalization $f^s : U' \to U$ of capture type hybrid equivalent to $Q(z) = z^2 + c$. Let ω be the captured critical point and let $N > 0$ be the capture time.

The case when f is locally connected (in particular, when f is hyperbolic) is easy because $J(f)$ is homeomorphic to S^1/λ_f.

Consider the case $c \in \partial\mathcal{M}$. There exists an open set $\mathcal{U} \subset \mathrm{Poly}_3$ such that there exists a holomorphic family of quadratic-like restriction $(g^s : U'_g \to U_g)_{g\in\mathcal{U}}$ and $U'_f = U'$ and $U_f = U$. Then $E = \{g \in \mathcal{U};\ g^s : U'_g \to U_g$ is hybrid equivalent to $Q\}$ is an analytic set of $\mathcal{U}$ [DH2, §II.6, Corollary 2].

Lemma 3.2. *E has dimension one.*

Before we prove the lemma, we introduce the notion of active critical points for a family of polynomials with marked critical points.

Definition. For an analytic family of polynomial $(g_\lambda, \omega_\lambda)_{\lambda\in\Lambda}$ with marked critical points, we say ω_λ is *active* at λ_0 (or simply, ω_{λ_0} is *active*) if $(\lambda \mapsto g^n_\lambda(\omega_\lambda))$ does not form a normal family at λ_0.

Proof. If E has dimension two (E is an open set in $\mathcal{U}$), then $S : E \to \mathcal{MK}$ is continuous by Lemma 3.1 and $S(g) = (Q, z(g))$ for any $g \in E$. Let $g^s : U'_g \to U_g$ be the corresponding renormalization for g, $\omega(g)$ be the captured critical point and $\omega_0(g) \in U'_g$ be the other critical point. Then ω_0 is not active because $g^n(\omega_0(g)) \in K(g^s; U'_g, U_g)$. On the contrary, if ω is also not active, then f is structurally stable in E and this implies f carries an invariant line field on $J(f)$. But since $J(f) \setminus \bigcup_{n\geq 0} f^{-n}(J(f^s; U', U))$ has measure zero, it follows what Q also carries an invariant line field, which contradicts the assumption $c \in \partial M$. Therefore, ω is active. Hence there exists some g_1 arbitrarily close to f such that $\omega(g_1)$ is preperiodic. Take an analytic subset A of E containing g_1 where $\omega(g)$ is preperiodic. Then $\omega(g)$ is not active on A and repeating the above argument, Q carries an invariant line field and it is a contradiction. □

Similar argument shows that ω_0 is not active and ω is active on E. Take a sequence $x_n \to f^N(\omega)$ such that $x_n \in K(f^s; U', U)$ is periodic. For $g \in E$, let $x_n(g) \in K(g^s; U'_g, U_g)$ be the continuation of x_n. Since for any $g \in E$, $g^s : U'_g \to U_g$ is a quadratic-like map hybrid equivalent to Q, x_n is defined a neighborhood of E (independent of n). If $g^N(\omega(g)) - x_n(g)$ converges uniformly to zero as $n \to \infty$, then this implies that $g^n(\omega(g)) \in K(g^s; U'_g, U_g)$ for all $g \in E$ near f and $n \geq N$, which contradicts that ω is active on E. Hence $g^N(\omega(g)) - x_n(g)$ converges to a non-constant holomorphic function h with $h(f) = 0$. This implies that for sufficiently large n, there exists some g_2 such that $g_2^N(\omega(g_2)) - x_n(g_2) = 0$. In other words, there exists some g_2 arbitrarily close to f such that it has a capture renormalization of the same period and capture time as f such that the captured critical point is preperiodic.

Let F be an one-dimensional analytic set containing g_2 such that the marked critical point $\omega(g)$ is preperiodic. Then, the straightening map S satisfies that $\pi_1 \circ S$ is not constant, where $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ is the projection to the first coordinate. Hence there exists some g_3 arbitrarily close to g_2 such that $\pi_1 \circ S(g_3)$ is hyperbolic. Hence there exists some f_0 such that $g_3 \in \mathcal{C}_3(f_0)$. Since for a given $s, N > 0$, there exist only finitely many center f_0 of capture type of period s and capture time N, hence we may assume there exists a center f_0 such that we choose such g_3 arbitrarily close to f and $g_3 \in \mathcal{C}_3(f_0)$. Taking a limit $g_3 \to f$, we can verify that f also lies in $\mathcal{C}_3(f_0)$, this implies that $f \in \mathcal{R}_3(f_0)$.

Finally, consider the case c lies in a queer component (non-hyperbolic component of the interior of $\mathcal{M}$). Then f is infinitely renormalizable. In particular, all periodic points are repelling. Therefore, we can find many biaccessible points (e.g., there exists exactly d fixed points and $d-1$ invariant landing angles. Hence there exists a fixed point with non-invariant landing angles, which implies that it is biaccessible), and we can construct f_0 by using the result of Kiwi [Ki].

□

3.3 Proof of Main Theorem 3, part 2: surjectivity

Let $(c, z) \in \mathcal{MK}$. We want to find $f \in \mathcal{C}_3(f_0)$ such that $S(f) = (c, z)$.

If $z^2 + c$ is hyperbolic and z lies in the interior of $K(z^2 + c)$, then we can construct f by constructing a rational lamination ("combinatorial tuning") λ from λ_{f_0} and λ_{z^2+c}, realizing it by a cubic polynomial [Ki] and perturbing it by the standard quasiconformal deformation technique.

Taking a limit on z, we can also construct f for the case $z^2 + c$ hyperbolic and $z \in J(z^2 + c)$ by Lemma 3.1.

Consider the case $c \in \partial\mathcal{M}$. If $z^2 + c$ does not has a Siegel disk, then $K(z^2 + c) = J(z^2 + c)$ has no interior and $c \mapsto K(z^2 + c)$ is continuous at c. Since $z^2 + c$ can be approximated by hyperbolic polynomials, we can similarly find f by Lemma 3.1. If $z^2 + c$ has a Siegel disk, then taking a radial limit from the hyperbolic component H with $c \in \partial H$, we can obtain a sequence $c_n \to c$ with $K(z^2 + c_n) \to K(z^2 + c)$. Hence this case is also similar.

The last case is when c lies in a queer component A. First take $c' \in \partial A$ and (eventually) periodic $z' \in K(z^2+c')$. Then there exists $f' \in \mathcal{C}_3(f_0)$ such that $S(f') = (c', z')$. The captured critical point ω' for f' is preperiodic. Hence it is contained in a one-dimensional subspace E of Poly_3, where one marked critical point is preperiodic. The straightening map S is defined a neighborhood $\mathcal{U}$ of z in E. Since S is injective, S is non-constant. Hence S is an open map [DH2, §III.3]. This implies that there exists a map $f'' \in E$ such that $S(f'') = (c'', z'')$ with $c'' \in A$ and z'' is (eventually) periodic for $z^2 + c''$.

Since $z^2 + c''$ carries an invariant line field on its Julia set, we can deform quasi-conformally $z^2 + c''$ and f'' at the same time, and obtain an open set $\mathcal{U}' \subset E \cap \mathcal{C}_3(f_0)$ such that $\pi_1 \circ S(\mathcal{U}') = A$.

This proves that for any (c, z) with $c \in A$ and (eventually) periodic z, there exists some $f \in \mathcal{C}_3(f_0)$ such that $S(f) = (c, z)$. Since such points are dense in $\mathcal{MK} \cap (A \times \mathbb{C})$, this holds for any $(c, z) \in \mathcal{MK} \cap (A \times \mathbb{C})$ (note that the interiors of $K(f)$ and $K(z^2+c)$ are empty).

Acknowledgment. The author would express his gratitude to Jan Kiwi who told me the main idea of Main Theorem 3. He would also thank Tomoki Kawahira for valuable discussion.

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