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An example of J^+ for complex Hénon mappings which is locally connected nowhere

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Abstract

It is known that J^+ for complex Hénon mappings is connected. We give a sufficient condition so that J^+ is locally connected nowhere.

1 Introduction

In this paper we denote $z = (x, y) \in \mathbb{C}^2$. Let $p_j(y)$ be monic polynomials of $\deg g_j = d_j > 1$ for $j = 1, \ldots, m$. We call $g_j(x, y) = (y, p_j(y) - \delta_j x)$ generalized Hénon mappings, where $\delta_j \neq 0$. Moreover we define

$$f = f_m \circ \cdots \circ f_1, \quad \delta = \delta_1 \cdots \delta_m, \quad d = d_1 \cdots d_m.$$

Friedland and Milnor [5] classified polynomial automorphisms of $\mathbb{C}^2$ into three types: affine mapping, elementary mapping, composite of generalized Hénon mappings. The last one has complicated dynamical structures.

We define $K^{\pm} = \{z \in \mathbb{C}^2 \mid \{f^{\pm n}(z) \mid n \in \mathbb{N}\} \text{ is bounded}\}$, $J^{\pm} = \partial K^{\pm}$, $K = K^+ \cap K^-$ and $J = J^+ \cap J^-$. They are closed invariant sets.

Let $d(\ , \ )$ be the Euclidean distance in $\mathbb{C}^2$. For $X \subset \mathbb{C}^2$, define the stable set $W^s(X)$ and the unstable set $W^u(X)$ as follows: $W^s(X) = \{z \in \mathbb{C}^2 \mid d(f^n(z), f^n(X)) \to 0 \ (n \to \infty)\}$, $W^u(X) = \{z \in \mathbb{C}^2 \mid d(f^n(z), f^n(X)) \to 0 \ (n \to -\infty)\}$.

Let a be a periodic point with the period l such that the eigenvalues of $D(f^l)(a)$ are λ_s and λ_u ($|\lambda_s| < 1 < |\lambda_u|$). Such a periodic point is called a saddle point. Then we call $W^s(a)$ a stable manifold and $W^u(a)$ an unstable manifold since there are non-singular bijective entire mappings $H_s : \mathbb{C} \to W^s(a)$ and $H_u : \mathbb{C} \to W^u(a)$ with $f \circ H_s(t) = H_s(\lambda_s t)$ and $f \circ H_u(t) = H_u(\lambda_u t)$. See [9] for example. Bedford and Smillie [2] showed $\overline{W^s(a)} = J^+$ and $\overline{W^u(a)} = J^-$.

We call $\widetilde{K}^s = H_s^{-1}(K)$ a stable slice and $\widetilde{K}^u = H_u^{-1}(K)$ an unstable slice. We say $\widetilde{K}^s$ is *stably connected* if $\widetilde{K}^s$ has no compact connected components [4]. We say $\widetilde{K}^s$ is *bridged* if the connected component of $\widetilde{K}^s$ containing the origin is not a point [7]. An unstable connectivity and a bridgedness for $\widetilde{K}^u$ are defined similarly. Note that a stable (unstable) connectivity implies a bridgedness and that the following are equivalent [7]:

- $\widetilde{K}^s$ is bridged,
- the connected component of $\widetilde{K}^s$ containing the origin is unbounded,
- $\widetilde{K}^s$ has an unbounded connected component.

In particular $\widetilde{K}^s$ is not bridged if and only if each component of $\widetilde{K}^s$ is compact.

2 Main theorems

Theorem 2.1. *If $\widetilde{K}^u$ is not unstably connected and $\widetilde{K}^s$ is not bridged then J^+ is not locally connected anywhere.*

Theorem 2.2. *Assume $\widetilde{K}^u$ is not unstably connected. Then there are at most finitely many periodic points $p_1, \dots, p_n$ such that J^+ is locally connected only at the points.*

Note that $\overline{W^s(a)} = J^+$ and hence J^+ is connected. It implies that Theorem 2.1 gives an example of a connected set which is not locally connected anywhere.

It was shown [7] if $\widetilde{K}^u$ is bridged then the Yoccoz inequality holds. Therefore if $\widetilde{K}^u$ does not satisfy the inequality then it is not unstably connected, and if $\widetilde{K}^s$ does not then not bridged. Note that it is easy to give examples such that either $\widetilde{K}^u$ or $\widetilde{K}^s$ do not satisfy the inequality. It implies many Hénon mappings satisfy the assumptions of Theorem 2.1.

3 Proofs of the main theorems

In this section we assume the unstable slice $\widetilde{K}^u$ is not unstably connected. For $X \subset \mathbb{C}^2$ we define $B(X, r) = \{z \in \mathbb{C}^2 \mid d(z, X) < r\}$. Recall that the Green functions $G^\pm$ are defined [1] as:

$$G^\pm(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^{\pm n}(z)\|$$

and have the following properties:

- $G^\pm$ are nonnegative continuous plurisubharmonic functions,
- $G^\pm(z) = 0$ if and only if $z \in K^\pm$,
- $G^\pm|_{\mathbb{C}^2 \setminus K^\pm}$ are positive pluriharmonic functions,
- $G^\pm \circ f = d^{\pm 1} \cdot G^\pm$.

It is well-known [9] that in a neighborhood of saddle point a, f^l is conjugate to

$$\widetilde{f}(s,t) = (\lambda_s s + st\alpha(s,t), \lambda_u t + st\beta(s,t)), \tag{3.1}$$

where α, β are holomorphic functions defined in a bidisk $\widetilde{\Delta}$ centered at the origin. We denote by Φ the conjugation mapping whose domain is $\widetilde{\Delta}$. Define $\Delta = \Phi(\widetilde{\Delta})$.

Proposition 3.1. *Assume J^+ is locally connected at $z_0 \in J^+$. Then for any $r > 0$, $H_s^{-1}(B(z_0, r))$ has an unbounded connected component. Moreover we have $z_0 \notin W^s(a)$.*

Proof. The local connectivity implies there is an open neighborhood V of z_0 in $\mathbb{C}^2$ such that $V \cap J^+$ is connected and $V \Subset B(z_0, r)$. Let $\widetilde{V}^s$ be a component of $H_s^{-1}(V)$ and $\widetilde{B}^s$ the component of $H_s^{-1}(B(z_0, r))$ containing $\widetilde{V}^s$. We assume $\widetilde{B}^s$ is bounded and derive a contradiction.

We define $B^s = H_s(\widetilde{B}^s)$. Choose $n \geq 0$ so that $f^{ln}(B^s) \Subset \Phi(\{(s,0) \in \widetilde{\Delta}\})$ and define $B_1^s = f^{ln}(B^s)$, $\widetilde{B}_1^s = \Phi^{-1}(B_1^s)$. Let C be a simple closed curve in

$\widetilde{\Delta} \cap \{t=0\}$ which surrounds $\widetilde{B}_1^s$ and does not intersect with $\Phi^{-1}(f^{ln}(B(z_0,r)))$. Choose $\varepsilon > 0$ so small and decrease $r > 0$ slightly if necessary so that $\hat{C} = \{(s,t) \in \widetilde{\Delta} \mid (s,0) \in C, |t| < \varepsilon\}$ and $\Phi^{-1}(f^{ln}(B(z_0,r)))$ do not intersect.

On the other hand, take a compact component K_1^u of $H_u(\widetilde{K}^u)$ contained in $\Phi(\{(0,t) \in \widetilde{\Delta}\})$ and define $\widetilde{K}_1^u = \Phi^{-1}(K_1^u)$. Let Γ be a closed curve in $\widetilde{\Delta} \cap \{s=0\}$ which surrounds $\widetilde{K}_1^u$ and does not intersect with $\Phi^{-1}(H_u(\widetilde{K}^u))$ [7, section 6]. Choose $\delta > 0$ so that $\hat{\Gamma} = \{(s,t) \mid (0,t) \in \Gamma, |s| < \delta\}$ does not intersect with $\Phi^{-1}(\Delta \cap K^+)$. By properties of the Green function G^+, for any s_1 with $|s_1| < \delta$, $\Phi^{-1}(K^+) \cap \{s = s_1\}$ is not empty inside of $\hat{\Gamma}$ [4].

By (3.1), $\widetilde{f}^k(\hat{C})$ approaches $\{t=0\}$ uniformly and expand along $\{t=0\}$ uniformly. Therefore if we take k large, $\hat{\Gamma}$ goes through $\widetilde{f}^k(\hat{C})$.

Let us return to the starting point. Then $f^{-l(n+k)}(\Phi(\hat{\Gamma}))$ goes through $B(z_0,r)$ and V if we take k large if necessary. Since K^+ runs through inside of $f^{-l(n+k)}(\Phi(\hat{\Gamma}))$, we conclude that $V \cap J^+$ is not connected, which is a contradiction.

Let show the last statement of the theorem. Take $z_0 \in W^s(a)$. Since $W^s(a)$ is a 1-dimensional manifold, if we take $r > 0$ small, the connected component of $H_s^{-1}(B(z_0,r))$ containing $H_s^{-1}(z_0)$ is bounded. But an arbitrary open neighborhood V of z_0 intersects with the component, which is a contradiction. □

Proof of Theorem 2.1. By the assumption there is a closed curve γ surrounding the origin and not intersecting with $\widetilde{K}^s$ [7, section 6]. Since f^{-n} diverges in $\mathbb{C}^2 \setminus K^+$ locally uniformly as $n \to +\infty$, $f^{-n}(H_s(\gamma)) = H_s(\lambda_s^{-n}\gamma)$ diverges uniformly.

Assume J^+ is locally connected at $z_0 \in J^+$. Then some component of $H_s^{-1}(B(z_0,r))$ is unbounded. But if we choose n large, $f^{-n}(H_s(\gamma))$ is far from $B(z_0,r)$ and $\lambda_s^{-n}\gamma$ intersects $H_s^{-1}(B(z_0,r))$, which is a contradiction. □

Let us proceed to prove Theorem 2.2. For $z_0 \in J^+ \setminus W^s(a)$ and $n \in \mathbb{Z}$, we define

$$u(t) = \log d(H_s(t), z_0), \quad u_n(t) = \max\{0, u(t)+n\}.$$

For a nonnegative subharmonic function v on $\mathbb{C}$ we define the order of v as follows:

$$\operatorname{ord} v = \limsup_{r\to\infty} \frac{\log \max_{|t|=r} v(t)}{\log r}.$$

Lemma 3.2. *The functions u and u_n are continuous subharmonic functions and we have*

$$\rho = \operatorname{ord} u_n = \frac{l \log d}{-\log|\lambda_s|}.$$

Proof. Since $\log\|z\|$ is plurisubharmonic, u, u_n are subharmonic functions.

If we set $(h_1, h_2) = H_s$, the orders of h_1, h_2 are [7]:

$$\operatorname{ord} h_1 = \limsup_{r\to\infty} \frac{\log\log\max_{|t|=r}|h_1(t)|}{\log r} = \frac{l\log d}{-\log|\lambda_s|},$$
$$\operatorname{ord} h_2 = \limsup_{r\to\infty} \frac{\log\log\max_{|t|=r}|h_2(t)|}{\log r} = \frac{l\log d}{-\log|\lambda_s|},$$

since the period of a is l and the degree of f^l is d^l. It is easy to compute the order of u_n using the above equations. □

Lemma 3.3. *Let v be a nonnegative bounded subharmonic function in an unbounded open set $\Omega(\subset \mathbb{C})$ with an unbounded boundary. Let c be a bounded subset of $\partial\Omega$. If $v \equiv 0$ on $\partial\Omega \setminus c$, then $v(t)$ converges 0 uniformly as $|t| \to \infty$ with $t \in \Omega$.*

Proof. We define

$$w(\tau) = \begin{cases} u(1/\tau) & \text{if } 1/\tau \in \Omega, \\ 0 & \text{if } 1/\tau \notin \Omega \cup \bar{c}. \end{cases}$$

Then w is a nonnegative bounded subharmonic function for $1/\tau \notin \bar{c}$. Moreover since w is bounded in a neighborhood of $\tau = 0$, Removable Singularity Theorem [8, p. 53] implies w is subharmonic around the origin.

We may assume w is non-constant for any neighborhood of the origin. Therefore we can apply Tsuji inequality [6, p. 548] to w. In fact, for $e^{-1} < \kappa < 1$ and $0 < r \le \kappa^2 R$, we have

$$B(r) \le C_2(\kappa)B(R)\exp\left\{-\int_{r/\kappa}^{\kappa R} \frac{\alpha(\rho)d\rho}{\rho}\right\},$$

where $B(r) = \max\{w(t) \mid |t| = r\}$, $C_2(\kappa) = 6(1-\kappa)^{-3/2}$. In our case we can set $\alpha(\rho) = 1/2$ by the structure of Ω. We have

$$B(r) \le C_2(\kappa)B(R)\exp\left\{-\int_{r/\kappa}^{\kappa R} \frac{d\rho}{2\rho}\right\} \le C_2(\kappa)B(R)\sqrt{\frac{r}{\kappa^2 R}}.$$

Therefore $B(r) \to 0$ as $r \to 0$, i.e., $u(t) \to 0$ as $|t| \to \infty$. □

Proof of Theorem 2.2. Assume J^+ is locally connected at $z_0 \in J^+$. The above proposition implies $z_0 \notin W^s(a)$. In the following we will show that z_0 is an asymptotic point of H_s. Once we obtain the fact, since each holomorphic function of finite order [7] has at most finitely many asymptotic values, the proof is completed.

In general, let v be a nonnegative subharmonic function of complex one variable. Each connected component of $\{s \mid v(s) > 0\}$ is called *tract*. Then the number of tracts of v is at most $\max\{1, 2\,\mathrm{ord}\,v\}$ [6, Chapter 8].

Therefore the number of tracts of u_n is at most $\max\{1, 2\rho\}$. Take an appropriate $n_0 \in \mathbb{Z}$ such that the number of tracts of u_{n_0} attains its maximum q. For each tract of u_{n_0} choose an asymptotic path $\gamma_j : [0,\infty) \to \mathbb{C}$ $(0 \le j \le q)$ with $u_{n_0}(\gamma(\xi)) > 0$ and $u_{n_0}(\gamma(\xi)) \to \infty$ as $\xi \to \infty$. Take sufficiently large $R > 0$ and we may assume all paths γ_j intersect with $\{|t| = R\}$ only at their starting points. Then $\mathbb{C} \setminus (\overline{D_R} \cup \gamma_1 \cup \cdots \cup \gamma_q)$ consists of q-unbounded connected components, where $D_R = \{|t| < R\}$.

Choose U which is one of the components such that the infimum of u is $-\infty$ in the domain. Moreover choose large N so that

$$\min\{u_N(s) \mid t \in \overline{D_R} \cup \gamma_1 \cup \cdots \cup \gamma_q\} > 0.$$

For each $j = 1, 2, \ldots,$ the above proposition implies we can take a point $s_j \in U$ such that the component of $\{s \in U \mid u(s) < -N - j\}$ containing s_j is unbounded.

Let us show that we can draw a path joining s_1 and s_2 such that $u < -N$ on the path. By the construction, s_1 and s_2 are contained in the unbounded components U_1 and U_2 of $\{s \in U \mid u(s) < -N-1\}$, resp. Draw a smooth curve c_0 in U joining s_1 and s_2. We may assume $\overline{U_1} \cap \overline{U_2} \neq \emptyset$. Let us regard $U \setminus (\overline{U_2} \cup \overline{U_2} \cup c_0)$. Clearly the set is divided into two sides with respect to c_j: one can access ∂U, another cannot. We choose the open set which cannot and name it Ω. Then $\partial\Omega$ consists of a part of ∂U_1 and ∂U_2 and c_0. Note that $u_{N+1} \equiv 0$ on ∂U_1 and ∂U_2, and that Ω is unbounded and that u_{N+1} is bounded in Ω. At this point, we can apply the above lemma, and obtain that u_{N+1} decrease to 0 uniformly as $|s| \to \infty$ in Ω. Therefore we can draw a path $\Gamma_1 : [0,1] \to U$ joining s_1 and s_2 such that $u < -N$ on Γ_1.

Similarly we can draw paths $\Gamma_j : [0,1] \to U$ joining s_j and s_{j+1} such that $u < -N - j + 1$ on Γ_j for $j = 2, 3, \ldots$. If we define

$$\Gamma(\xi) = \Gamma_j(\xi - j + 1) \quad \text{for } j - 1 \leq \xi < j,$$

Γ is an asymptotic path such that $u(\Gamma(\xi)) \to -\infty$ as $\xi \to \infty$, i.e., $H_s(\Gamma(\xi)) \to z_0$ as $\xi \to \infty$, which implies z_0 is an asymptotic point of H_s. □

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