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The dynamics on Teichmüller spaces induced by holomorphic self-coverings

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1 Introduction

The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of equivalence classes $[f]$ of quasiconformal homeomorphisms f on R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 on R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are consider to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete distance on $T(R)$ which is called the Teichmüller distance.

We assume that a Riemann surface R is of hyperbolic type. Namely, it is represented by a quotient space $\mathbb{H}^+/\Gamma$ of the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by a torsion free Fuchsian group Γ. Let $R' = \mathbb{H}^-/\Gamma$ be the complex conjugate of R where $\mathbb{H}^- = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$, and $B(R')$ the complex Banach space of all bounded holomorphic quadratic differentials on R' with the hyperbolic supremum norm. Then the Teichmüller space $T(R)$ is a complex Banach manifold modeled on $B(R')$. In fact, $T(R)$ is embedded in $B(R')$ as a bounded contractible domain. Hence it is equipped with the Kobayashi distance. If R is a Riemann surface whose fundamental group is infinitely generated, then the Teichmüller space is infinite dimensional. For details, see [4] and [8]. It was proved in [3] that the Teichmüller distance and the Kobayashi distance are coincident for all Riemann surfaces.

We consider a holomorphic map of $T(R)$ into $T(R)$. Every quasiconformal automorphism of a Riemann surface R induces a biholomorphic automorphism of $T(R)$. Then this is an isometry with respect to the Teichmüller-Kobayashi distance. Furthermore, the converse is also true, namely every biholomorphic automorphism of $T(R)$ is induced by a quasiconformal automorphism of the

Riemann surface. This is a combination of results of [1] and [5]. In [2], we have considered the dynamics of isometric automorphisms in general metric spaces as well as that of biholomorphic automorphisms of the Teichmüller space.

In this paper, we consider a Riemann surface R in which there exists a non-injective unramified holomorphic self-covering $f : R \to R$. Then the fundamental group of R is infinitely generated. For example, we can obtain such a surface by a Fatou component of the complex dynamics on the Riemann sphere. The holomorphic self-covering f is locally isometric with respect to the hyperbolic metric on R, and it induces a holomorphic self-map

$$f^* : T(R) \to T(R).$$

Then f^* is non-expanding with respect to the Teichmüller-Kobayashi distance d and not surjective. We investigate the dynamics of f^* on $T(R)$.

2 Dynamics of holomorphic self-maps

Definition 1 We define the full cluster set of f^* by

$$C(f^*) = \lim_{k\to\infty} \overline{\bigcup_{n=k}^{\infty} (f^*)^n(T(R))} = \bigcap_{n=1}^{\infty} (f^*)^n(T(R)).$$

The full cluster set $C(f^*)$ is the maximal closed and completely invariant set under the action of f^*.

Definition 2 For a point $x \in T(R)$, it is said that $y \in T(R)$ is a *ω-limit point* of x for f^* if there exists a sequence $\{n_i\} \subset \mathbf{Z}_+$ of positive integers such that $\lim_{i\to\infty} d((f^*)^{n_i}(x), y) = 0$. The set of all ω-limit points of x for f^* is called the ω-limit set of x for f^* and is denoted by $\Lambda(f^*, x)$. It is said that $x \in T(R)$ is a *recurrent point* for f^* if $x \in \Lambda(f^*, x)$. The set of all recurrent points for f^* is called the recurrent set for f^* and is denoted by $\mathrm{Rec}(f^*)$. The *ω-limit set* for f^* is defined by $\Lambda(f^*) = \bigcup_{x\in T(R)} \Lambda(f^*, x)$. The set of all periodic points for f^* is denoted by $\mathrm{Per}(f^*)$.

The following properties make the definitions for a non-expanding map simple.

Proposition 3 *The recurrent set* $\mathrm{Rec}(f^*)$ *is a subset of the full cluster set* $C(f^*)$*, and the recurrent set* $\mathrm{Rec}(f^*)$ *is coincident with the limit set* $\Lambda(f^*)$*. Moreover* $\mathrm{Rec}(f^*)$ *is closed, and so is* $\Lambda(f^*)$*.*

However $\mathrm{Rec}(f^*)$ is not coincident with $C(f^*)$. In fact, we have the following.

Theorem 4 (i) *For every point $x \in C(f^*)$, the orbit $O(x) = \{(f^*)^n(x) \mid n \in \mathbf{Z}_+\}$ is not dense in $C(f^*)$.* (ii) *The following inclusion relations are proper;*

$$C(f^*) \supset \mathrm{Rec}(f^*) \supset \overline{\mathrm{Per}(f^*)} \supset \mathrm{Per}(f^*) \supset \mathrm{Fix}(f^*).$$

(iii) *The recurrent set* $\mathrm{Rec}(f^*)$ *is nowhere dense in* $C(f^*)$*.*

3 Geometry of holomorphic self-map

Next, we consider the non-expanding property of f^* more closely. The injective holomorphic map f^* induces an injective holomorphic map

$$\hat{f^*} : T(T(R)) \to T(T(R))$$

of the holomorphic tangent bundle $T(T(R))$ of $T(R)$ such that f^* sends (p, v) to $(f^*(p), (df^*)_p(v))$. Then we define the magnification of a tangent vector v at p by

$$r(p,v) := \frac{||(df^*)_p(v)||_{T_{f^*(p)}(T(R))}}{||v||_{T_p(T(R))}}.$$

If a covering $f : R \to R$ is amenable, then $r(p, v) = 1$ for every $(p, v) \in T(T(R))$ (see [6]). Namely, f^* is an isometry on $T(R)$. Thus hereafter we assume that f is a non-amenable cover. In this case, we see that there are a lot of tangent vectors in $T(T(R))$ that are actually contracted by f^*.

Theorem 5 *The set* $\{(p, v) \in T(T(R)) \mid r(p, v) < 1\}$ *is dense in* $T(T(R))$.

This theorem is also followed by [6] combined with the fact that the Reich-Strebel functionals (tangent vectors) are dense in each tangent space $T_p(T(R))$.

However, we know that the magnification $r(p, v)$ is not uniformly bounded, for otherwise, the fixed point theorem says that the full cluster set $C(f^*)$ should be a unique fixed point of f^*.

Theorem 6 *For every point* $(p, v) \in T(T(R))$*, we have*

$$\lim_{n\to\infty} r((\hat{f^*})^n(p,v)) = 1.$$

Actually, there exists some tangent vector (p, v) such that $r(p, v) = 1$.

Theorem 7 (i) *For every point* $p \in \mathrm{Per}(f^*)$*, there exists a tangent vector* $v \in T_p(T(R))$ *such that* $r(p, v) = 1$. (ii) *For every point* $p \in \mathrm{Rec}(f^*)$*, we have* $\sup_{v\in T_p(T(R))} r(p, v) = 1$.

4 Dynamics on the base surface

We prove these theorems by the following structure theorem on the dynamics of a holomorphic self-covering on a Riemann surface. A similar result was proved also by McMullen and Sullivan [7].

Theorem 8 (Structure theorem I) *Suppose that there exist a Riemann surface* R *and a non-injective unramified holomorphic self-covering* $f : R \to R$. *Then there exist a Riemann surface* S*, a holomorphic covering* $\pi : R \to S$ *and a* *biholomorphic automorphism $g : S \to S$ of infinite order such that the following diagram commutes:*

$$\begin{array}{ccc} R & \xrightarrow{f} & R \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow[g]{} & S \end{array}$$

This theorem insists that the action of f^* is very similar to the isometry g^*.

Remark 9 The grand orbit of $x \in R$ under f is the set of points $y \in R$ such that $f^n(x) = f^m(y)$ for some $n, m \geq 0$. Furthermore, the small orbit of $x \in R$ under f is the set of points $y \in R$ such that $f^n(x) = f^n(y)$ for some $n \geq 0$. We define R/f as the quotient space by the grand orbit relation, and $R/(f)$ as the quotient space by the small orbit relation. The Riemann surface S as in Theorem 8 is coincident with $R/(f)$ and the quotient surface $S/\langle g \rangle$ is coincident with R/f.

Finally we consider another application obtained by the structure theorem.

Definition 10 For a holomorphic self-covering $f : R \to R$, we say that a subset $U \subset R$ is an absorbing domain if $f(\overline{U}) \subset U$ and if, for every point $x \in R$, there exists $n \in \mathbf{N}$ such that $f^n(x) \in U$. If f is injective in the absorbing domain U, then we call U simple. Furthermore we say that the absorbing domain U is escaping if, for every compact subset $K \subset R$, the number of integers n satisfying $f^n(U) \cap K \neq \emptyset$ is finite.

Theorem 11 *For every non-injective holomorphic self-covering $f : R \to R$, there exists a simple, escaping, absorbing domain.*

Corollary 12 (Denjoy-Wolff type theorem) *For a non-injective holomorphic self-covering $f : R \to R$, there exists a unique topological end e of R such that $f^n(x) \to e$ for every $x \in R$.*

In fact, there exists a unique analytical end which is determined by a fixed point of a lift of g to $\mathbb{H}$.

On the last of this section, we mention the existence of holomorphic self-coverings.

Theorem 13 (Structure theorem II) *For every Riemann surface S and for every biholomorphic automorphism $g : S \to S$ of infinite order, there exist a holomorphic covering $\pi : R \to S$ and a holomorphic self-covering $f : R \to R$ such that $\pi \circ f = g \circ \pi$.*

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