The space of postcritically bounded 2-generator polynomial semigroups with hyperbolicity

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Abstract

We consider dynamics of semigroups generated by polynomial maps on the Riemann sphere. We investigate the space of semigroups G such that G is generated by two polynomials, such that the planar postcritical set of G in the complex plane is bounded, and such that G is hyperbolic. We show that for a semigroup in the closure of the disconnectedness locus, the Julia set of the semigroup has Hausdorff dimension strictly less than two. Moreover, we show that the interior of the connectedness locus is dense in the connectedness locus. Furthermore, we investigate the function of probability of tending to infinity, with respect to the random dynamical systems. We show that, for a semigroup in the closure of the disconnectedness locus, the function above behaves like the devil's staircase. Moreover, for a semigroup in the closure of the disconnectedness locus, we find a kind of singular function defined on the complex plane, which is like the Takagi function.

1   Introduction

A **rational semigroup** is a semigroup generated by non-constant rational maps on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation being

functional composition([HM]). A **polynomial semigroup** is a semigroup generated by non-constant polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G.J. Martin ([HM]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group([ZR], [GR]), who studied such semigroups from the perspective of random dynamical systems. For other researches of rational semigroups, see [St1]-[St3], [SY], [StSu], [S1]-[S8].

The research of rational semigroups is directly related to that of random dynamics of holomorphic maps. For the study of random dynamics of holomorphic maps, see [FS], [BBR].

Definition 1.1. Let G be a rational semigroup.

- The **Fatou set** of G is defined to be
  $$F(G) := \{z \in \overline{\mathbb{C}} \mid \exists \text{ a nbd } U \text{ of } z \text{ s.t. } \{g|_U : U \to \overline{\mathbb{C}}\}_{g \in G} \text{ is normal on } U\}.$$

- The **Julia set** of G is defined to be $J(G) := \overline{\mathbb{C}} \setminus F(G)$.

- If G is generated by $\{g_i\}_i$, then we write $G = \langle g_1, g_2, \ldots \rangle$. More generally, if G is generated by $\{h_\lambda : \lambda \in \Lambda\}$, then we write $G = \langle h_\lambda : \lambda \in \Lambda \rangle$.

- For a polynomial g, we set $J(g) := J(\langle g \rangle)$.

Fact: If $G = \langle h_1, h_2, \ldots, h_m \rangle$, then $J(G) = h_1^{-1}(J(G)) \cup \cdots \cup h_m^{-1}(J(G))$.

Definition 1.2. Let G be a polynomial semigroup.

- The **postcritical set** of G is defined to be
  $$P(G) := \overline{\bigcup_{g \in G} \{\text{all critical values of } g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\}} \quad (\subset \overline{\mathbb{C}}.)$$

- G is said to be **hyperbolic** if $P(G) \subset F(G)$.

Definition 1.3. Let G be a polynomial semigroup.

- We set $P^*(G) := P(G) \setminus \{\infty\}$. This is called the **planar postcritical set** (or **finite postcritical set**) of G.

- G is said to be **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$.

It is well-known that for a polynomial g with $\deg(g) \geq 2$, $J(g)$ is connected if and only if $P^*(\langle g \rangle)$ is bounded in $\mathbb{C}$. It is natural for us to discuss the relationship between the planar postcritical set and the figure of the Julia set, in order to investigate the dynamics of polynomial semigroups. The first question in this direction is:

Question 1. Let G be a polynomial semigroup such that each element $g \in G$ is of degree at least two. If $P^*(G)$ is bounded in $\mathbb{C}$, then is $J(G)$ connected?

The answer is **NO**.

Example 1.4 ([SY]). Let $G = \langle z^3, \frac{z^2}{4} \rangle$. Then $P^*(G) = \{0\}$ (which is bounded in $\mathbb{C}$) and $J(G)$ is disconnected($J(G)$ is a Cantor set of round circles). Furthermore, by [S6], it can be shown that small perturbation H of G still satisfies that $P^*(H)$ is bounded in $\mathbb{C}$ and that $J(H)$ is disconnected. ($J(H)$ is a Cantor set of quasi-circles with uniform dilatation.)

Question 2. What happens if $P^*(G)$ is bounded in $\mathbb{C}$ and $J(G)$ is disconnected?

2 Main results

In this section, we present the main results of this paper.

2.1 Space of connected components of Julia sets, surrounding order

We present some results on connected components of the Julia set of a post-critically bounded polynomial semigroup.

Definition 2.1. We set Rat $:= \{h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid h$ is a non-constant rational map$\}$ endowed with topology induced by uniform convergence on $\overline{\mathbb{C}}$. We set Poly $:= \{h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid h$ is a non-constant polynomial $\}$ endowed with the relative topology from Rat. Moreover, we set $\text{Poly}_{\deg \geq 2} := \{g \in \text{Poly} \mid \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Definition 2.2. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- each element of G is of degree at least two, and

- $P^*(G)$ is bounded in $\mathbb{C}$.

Furthermore, we set $\mathcal{G}_{con} = \{G \in \mathcal{G} \mid J(G)$ is connected$\}$ and $\mathcal{G}_{dis} = \{G \in \mathcal{G} \mid J(G)$ is disconnected$\}$.

Notation: For a polynomial semigroup G, we denote by $\mathcal{J} = \mathcal{J}_G$ the set of all connected components J of $J(G)$ such that $J \subset \mathbb{C}$. Moreover, we denote by $\hat{\mathcal{J}} = \hat{\mathcal{J}}_G$ the set of all connected components of $J(G)$.

Definition 2.3. For any connected sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" indicates that $K_1 = K_2$, or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Furthermore, "$K_1 < K_2$" indicates $K_1 \leq K_2$ and $K_1 \neq K_2$. Note that "$\leq$" is a partial order in the space of all non-empty compact connected set in $\mathbb{C}$. This "$\leq$" is called the **surrounding order.**

Theorem 2.4. *Let $G \in \mathcal{G}$ (possibly infinitely generated). Then*

1. $(\mathcal{J}, \leq)$ *is totally ordered.*

2. *Each connected component of $F(G)$ is either simply or doubly connected.*

3. *Let $\mathcal{A}$ be the set of all doubly connected components of $F(G)$. Then, $(\mathcal{A}, \leq)$ is totally ordered.*

4. *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

Notation: For a polynomial semigroup G with $\infty \in F(G)$, we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. Moreover, for a polynomial g with $\deg(g) \geq 2$, we set $F_\infty(g) := F_\infty(\langle g \rangle)$.

Theorem 2.5. *Let $G \in \mathcal{G}_{dis}$. Under the above notation, we have the following.*

1. *We have that $\infty \in F(G)$. The connected component $F_\infty(G)$ of $F(G)$ containing ∞ is simply connected. Furthermore, the element $J_{\max} = J_{\max}(G) \in \mathcal{J}$ containing $\partial F_\infty(G)$ is the unique element of $\mathcal{J}$ satisfying that $J \leq J_{\max}$ for each $J \in \mathcal{J}$.*

2. *There exists a unique element $J_{\min} = J_{\min}(G) \in \mathcal{J}$ such that $J_{\min} \leq J$ for each element $J \in \mathcal{J}$. Furthermore, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Then $(P^*(G)) \cap D = \emptyset$ and $\partial \hat{K}(G) \subset J_{\min}$.*

3. *If G is generated by a family $\{h_\lambda\}_{\lambda \in \Lambda}$, then there exist two elements λ_1 and λ_2 of Λ satisfying:*

   (a) *there exist two elements J_1 and J_2 of $\mathcal{J}$ such that $J_1 \neq J_2$ and $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$,*

   (b) $J(h_{\lambda_1}) \cap J_{\min} = \emptyset$,

(c) for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ and $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$, and

(d) h_{λ_1} has an attracting fixed point z_1 in $\mathbb{C}$, $\text{int}(K(h_{\lambda_1}))$ consists of only one immediate attracting basin for z_1, and $K(h_{\lambda_2}) \subset \text{int}(K(h_{\lambda_1}))$. Furthermore, $z_1 \in \text{int}(K(h_{\lambda_2}))$.

Moreover, for each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have that g has an attracting fixed point z_g in $\mathbb{C}$, that $\text{int}(K(g))$ consists of only one immediate attracting basin for z_g, and $J_{\min} \subset \text{int}(K(g))$. Note that in general, $z_g \neq z_h$ for some g and h in G.

4. We have that $\text{int}(\hat{K}(G)) \neq \emptyset$. Moreover,

(a) $\mathbb{C} \setminus J_{\min}$ is disconnected, $\sharp J \geq 2$ for each $J \in \hat{\mathcal{J}}$, and

(b) for each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have $J_{\min} < g^*(J_{\min})$, $g^{-1}(J(G)) \cap J_{\min} = \emptyset$, $g(\hat{K}(G) \cup J_{\min}) \subset \text{int}(\hat{K}(G))$, and that the unique attracting fixed point z_g of g in $\mathbb{C}$ belongs to $\text{int}(\hat{K}(G))$.

Definition 2.6. A compact set K in $\overline{\mathbb{C}}$ is said to be uniformly perfect if $\sharp K \geq 2$ and there exists a constant $C > 0$ such that each annulus A that separates K satisfies that mod $A < C$, where mod A denotes the modulus of A (See the definition in [LV]).

Theorem 2.7. 1. Let G be a polynomial semigroup in $\mathcal{G}$. Then, $J(G)$ is uniformly perfect. Moreover, if $z_0 \in J(G)$ is a superattracting fixed point of an element of G, then $z_0 \in \text{int}(J(G))$.

2. If $G \in \mathcal{G}$ and $\infty \in J(G)$, then $G \in \mathcal{G}_{con}$ and $\infty \in \text{int}(J(G))$.

3. Suppose that $G \in \mathcal{G}_{dis}$. Let $g \in G$ and let $z_1 \in J(G) \cap \mathbb{C}$. If $g(z_1) = z_1$ and $g'(z_1) = 0$, then $z_1 \in \text{int}(J_{\min})$ and $J(g) \subset J_{\min}$.

2.2 Upper estimates of $\sharp(\hat{\mathcal{J}})$

We present some results on the space $\hat{\mathcal{J}}$ and some results on upper estimates of $\sharp(\hat{\mathcal{J}})$.

Definition 2.8. 1. For a polynomial g, we denote by $a(g) \in \mathbb{C}$ the coefficient of the highest degree term of g.

2. We set $RA := \{ax + b \in \mathbb{R}[x] \mid a, b \in \mathbb{R}, a \neq 0\}$ endowed with topology such that, $a_n x + b_n \to ax + b$ if and only if $a_n \to a$ and $b_n \to b$. The

space RA is a semigroup with the semigroup operation being functional composition. Any subsemigroup of RA will be called a *real affine semigroup*. We define a map Ψ : Poly $\to$ RA as follows. For a polynomial $g \in$ Poly, we set $\Psi(g)(x) := \deg(g)x + \log|a(g)|$.

Moreover, for a polynomial semigroup G, we set $\Psi(G) := \{\Psi(g) \mid g \in G\}$.

3. We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ endowed with topology such that $\{(r, +\infty]\}_{r\in\mathbb{R}}$ makes a fundamental neighborhood system of $+\infty$ and $\{[-\infty, r)\}_{r\in\mathbb{R}}$ makes a fundamental neighborhood system of $-\infty$. For a real affine semigroup H, we set

$$M(H) := \overline{\{x \in \mathbb{R} \mid \exists h \in H, h(x) = x, |h'(x)| > 1\}},$$

where the closure is taken in the space $\hat{\mathbb{R}}$. Moreover, we denote by $\mathcal{M}_H$ the set of all connected components of $M(H)$.

4. We denote by η : RA $\to$ Poly the natural embedding defined by $\eta(x \mapsto ax + b) = (z \mapsto az + b)$, where $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

5. We define a map Θ : Poly $\to$ Poly as follows. For a polynomial g, we set $\Theta(g)(z) = a(g)z^{\deg(g)}$. Moreover, for a polynomial semigroup G, we set $\Theta(G) := \{\Theta(g) \mid g \in G\}$.

Remark 1. 1. The map Ψ : Poly $\to$ RA is a semigroup homomorphism. That is, we have $\Psi(g \circ h) = \Psi(g) \circ \Psi(h)$. Hence, for a polynomial semigroup G, the image $\Psi(G)$ is a real affine semigroup. Similarly, the map Θ : Poly $\to$ Poly is a semigroup homomorphism. Hence, for a polynomial semigroup G, the image $\Theta(G)$ is a polynomial semigroup.

2. The maps Ψ : Poly $\to$ RA and η : RA $\to$ Poly are continuous.

Theorem 2.9. *(Theorem A)*

1. Let G be a polynomial semigroup in $\mathcal{G}$. Then, we have $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\mathcal{M}_{\Psi(G)})$.

2. If $G \in \mathcal{G}_{dis}$, then we have that $M(\Psi(G)) \subset \mathbb{R}$ and $M(\Psi(G)) = J(\eta(\Psi(G)))$.

3. Let G be a polynomial semigroup in $\mathcal{G}$. Then, $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\hat{\mathcal{J}}_{\eta(\Psi(G))})$.

Corollary 2.10. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, we have $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\hat{\mathcal{J}}_{\Theta(G)})$.*

Theorem 2.11. *Let* $G = \langle h_1, \ldots, h_m \rangle$ *be a finitely generated polynomial semigroup in $\mathcal{G}$. For each $j = 1, \ldots, m$, let a_j be the coefficient of the highest degree term of polynomial h_j. Let $\alpha := \min_{j=1,\ldots,m}\{\frac{1}{\deg(h_j)-1} \log |a_j|\}$ and $\beta := \max_{j=1,\ldots,m}\{\frac{1}{\deg(h_j)-1} \log |a_j|\}$. We set $[\alpha, \beta] := \{x \in \mathbb{R} \mid \alpha \leq x \leq \beta\}$. If $[\alpha, \beta] \subset \cup_{j=1}^{m} \Psi(h_j)^{-1}([\alpha, \beta])$, then $J(G)$ is connected.*

Theorem 2.12. *Let G be a (possibly infinitely generated) polynomial semigroup in $\mathcal{G}$ generated by polynomials of degree two. Then, $J(G)$ is connected.*

Theorem 2.13. *Let G be a (possibly infinitely generated) polynomial semigroup in $\mathcal{G}$ generated by a family $\{h_\lambda\}_\Lambda$ of polynomials. Let a_λ be the coefficient of the highest degree term of the polynomial h_λ. Suppose that for any $\lambda, \xi \in \Lambda$, we have $(\deg(h_\xi) - 1) \log(|a_\lambda|) = (\deg(h_\lambda) - 1) \log(|a_\xi|)$. Then, $J(G)$ is connected.*

2.3 Random dynamics of polynomials

In this section, we present some results on random dynamics of polynomials on $\overline{\mathbb{C}}$. The (outline of) proofs are given in section 4.2.

Let τ be a Borel probability measure on $\mathrm{Poly}_{\deg \geq 2}$. We consider the i.i.d. random dynamics on $\overline{\mathbb{C}}$ such that at every step we choose a polynomial map $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ according to the distribution τ. (This is a kind of Markov process on $\overline{\mathbb{C}}$.)

Notation: We use the following notation.

1. We set $\mathrm{supp}\,\tau :=$ support of τ ($\subset \mathrm{Poly}_{\deg \geq 2}$)

2. We set $X_\tau := (\mathrm{supp}\,\tau)^{\mathbb{N}}$ endowed with the product topology.

3. We set $\tilde{\tau} := \otimes_{j=1}^{\infty}\tau$. This is a Borel probability measure on X_τ.

4. Let G_τ be the polynomial semigroup generated by $\mathrm{supp}\,\tau$.

5. For any $z \in \overline{\mathbb{C}}$, we set

$$T_{\infty,\tau}(z) := \tilde{\tau}(\{\rho = (\rho_1, \rho_2, \ldots) \in X_\tau \mid \rho_n \cdots \rho_1(z) \to \infty \text{ as } n \to \infty\}).$$

This is the **probability of tending to ∞ starting from the initial value z.**

Remark 2. If τ is a Borel probability measure on $\mathrm{Poly}_{\deg \geq 2}$ and $\infty \in F(G_\tau)$ (for example, if $\mathrm{supp}\,\tau$ is compact), then for each connected component U of $F(G_\tau)$, $T_{\infty,\tau}|_U$ is constant.

Theorem 2.14. *(Theorem B) Suppose that supp τ is compact in* $\text{Poly}_{\deg \geq 2}$ *and* $G_\tau \in \mathcal{G}_{dis}$. *Then, we have all of the following.*

1. *For any component U of $F(G_\tau)$, there exists a constant $C_U \in [0,1]$ such that $T_{\infty,\tau}|_U \equiv C_U$.*

2. *$T_{\infty,\tau} : \overline{\mathbb{C}} \to [0,1]$ is a continuous function on $\overline{\mathbb{C}}$.*

3. *(Monotonicity)*

   (a) *Let $\mathcal{A}$ be the set of all doubly connected components of $F(G_\tau)$. If $A_1, A_2 \in \mathcal{A}$ and $A_1 < A_2$, then $C_{A_1} < C_{A_2}$. Hence $\{C_A \mid A \in \mathcal{A}\}$ are mutually distinct.*

   (b) *If $J_1, J_2 \in \mathcal{J}_{G_\tau}$ and $J_1 < J_2$, then $\max_{z \in J_1} T_{\infty,\tau}(z) \leq \min_{z \in J_2} T_{\infty,\tau}(z)$.*

4. *For any $A \in \mathcal{A}$, we have $T_{\infty,\tau}|_{\hat{K}(G_\tau)} \equiv 0 < C_A < 1 = C_{F_\infty(G_\tau)}$, where $F_\infty(G_\tau)$ is the component of $F(G_\tau)$ containing ∞.*

5. *Let Q be an open set in $\overline{\mathbb{C}}$ with*

   $$Q \cap \left(\partial(F_\infty(G_\tau)) \cup \partial(\hat{K}(G_\tau)) \cup \bigcup_{A \in \mathcal{A}} \partial A \right) \neq \emptyset.$$

   Then $T_{\infty,\tau}|_Q$ is not constant.

   *(The above 1-5 tells us that $T_{\infty,\tau} : \overline{\mathbb{C}} \to [0,1]$ is like the devil's staircase. We call such a function a "**devil's coliseum**".)*

6. *(**No Julia set for** $(M_\tau)_*$) Let M_τ be an operator on the Banach space $C(\overline{\mathbb{C}}) := \{\varphi : \overline{\mathbb{C}} \to \mathbb{R} \mid \varphi \text{ is continuous}\}$ endowed with the supremum norm defined by $M_\tau(\varphi)(z) := \int_{supp\ \tau} \varphi(g(z))\, d\tau(g)$. Then, there exists a unique Borel probability measure μ on $\hat{K}(G_\tau)$ such that for all $\varphi \in C(\overline{\mathbb{C}})$,*

   $$M_\tau^n(\varphi)(z) \to T_{\infty,\tau}(z) \cdot \varphi(\infty) + (1 - T_{\infty,\tau}(z)) \cdot \int_{\overline{\mathbb{C}}} \varphi\, d\mu$$

   as $n \to \infty$ uniformly on $\overline{\mathbb{C}}$. Hence

   $$(M_\tau)_*^n(\nu) \to \left(\int_{\overline{\mathbb{C}}} T_{\infty,\tau}\, d\nu \right) \cdot \delta_\infty + \left(\int_{\overline{\mathbb{C}}} (1 - T_{\infty,\tau})\, d\nu \right) \cdot \mu$$

   as $n \to \infty$ uniformly on the space $\mathcal{M}_1(\overline{\mathbb{C}})$ of all Borel probability measures ν on $\overline{\mathbb{C}}$.

2.4 Differentiability of $T_{\infty,\tau}$

In this section, we present some results on differentiability of $T_{\infty,\tau}$.

Theorem 2.15. *Let* $\tau = \sum_{j=1}^{m} p_j \delta_{g_j}$, *where* $\sum_{j=1}^{m} p_j = 1, p_j > 0$, *and* $g_j \in$ $\mathrm{Poly}_{\deg \geq 2}$ $(\forall j = 1, \ldots, m)$. *Suppose that* $\bigcup_{(i,j),\ i \neq j} g_i^{-1}(J(G_\tau)) \cap g_j^{-1}(J(G_\tau))$ *is either empty or totally disconnected. Moreover, suppose that* $T_{\infty,\tau} : \overline{\mathbb{C}} \to [0,1]$ *is not constant. Then,* $\mathrm{int}(J(G_\tau)) = \emptyset$ *and*
$$J(G_\tau) = \{z \in \overline{\mathbb{C}} \mid \text{for any neighborhood } U \text{ of } z,\ T_{\infty,\tau}|_U \text{ is not constant}\}.$$

In particular, if $m = 2$ *and* $G_\tau \in \mathcal{G}_{dis}$, *then* $\mathrm{int}(J(G_\tau)) = \emptyset$ *and* $J(G_\tau) =$ $\{z \in \overline{\mathbb{C}} \mid \text{for any neighborhood } U \text{ of } z,\ T_{\infty,\tau}|_U \text{ is not constant}\}$.

Definition 2.16. Let Γ be a subset of Rat. Let $\Gamma^{\mathbb{N}} := \{\rho = (\rho_1, \rho_2, \ldots) \mid \forall j, \rho_j \in \Gamma\}$ endowed with product topology. We define a map $f : \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}}$ as follows. For a point $(\rho, y) \in \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}}$, we set $f(\rho, y) := (\sigma(\rho), \rho_1(y))$, where $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map, that is, $\sigma(\rho_1, \rho_2, \ldots) = (\rho_2, \rho_3, \ldots)$. The map $f : \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}}$ is called the **skew product associated with the generator system** Γ. Let $\pi : \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_{\overline{\mathbb{C}}} : \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the natural projections. For each $\rho \in \Gamma^{\mathbb{N}}$ and $n \in \mathbb{N}$, we set $f_\rho^n := f^n|_{\pi^{-1}\{\rho\}} : \pi^{-1}\{\rho\} \to \pi^{-1}\{\sigma^n(\rho)\}$. Moreover, we set $f_{\rho,n} := \rho_n \circ \cdots \circ \rho_1$. For each $\rho = (\rho_1, \rho_2, \ldots) \in \Gamma^{\mathbb{N}}$, we denote by $F_\rho'(f)$ the set of $z \in \overline{\mathbb{C}}$ satisfying that there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that the sequence $\{\rho_n \circ \cdots \circ \rho_1\}_{n \in \mathbb{N}}$ of maps from U to $\overline{\mathbb{C}}$ is equicontinuous on U. Moreover, we set $J_\rho'(f) := \overline{\mathbb{C}} \setminus F_\rho'(f)$. Furthermore, we set $J_\rho(f) := \{\rho\} \times J_\rho'(f)$ $(\subset \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}})$ and $\tilde{J}(f) := \overline{\bigcup_{\rho \in \Gamma^{\mathbb{N}}} J_\rho(f)}$, where the closure is taken in the product space $\Gamma^{\mathbb{N}} \times \overline{\mathbb{C}}$. For each $\rho \in \Gamma^{\mathbb{N}}$, we set $\hat{J}_\rho(f) := \pi^{-1}\{\rho\} \cap \tilde{J}(f)$. Furthermore, for each $z = (\rho, y) \in \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}}$, we set $f'(z) := (f_{\rho,1})'(y)$. More generally, for each $n \in \mathbb{N}$ and $z = (\rho, y) \in \Gamma^{\mathbb{N}} \times \overline{\mathbb{C}}$, we set $(f^n)'(z) := (f_{\rho,n})'(y)$.

Theorem 2.17. *Let* $\tau = \sum_{j=1}^{m} p_j \delta_{g_j}$, *where* $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ *for each* $j = 1, \ldots, m$, *and* $\sum_{j=1}^{m} p_j = 1$. *Let* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *be the skew product associated with* $\mathrm{supp}\ \tau$. *For each* $z = (\rho, y) \in X_\tau \times \overline{\mathbb{C}}$, *we set* $p(z) := p_j$ *if* $\rho_1 = g_j$. *Suppose that* $G_\tau \in \mathcal{G}$ *and that* $\{g_j^{-1}(J(G_\tau))\}_{j=1}^{m}$ *are mutually disjoint. Then, the following statements hold.*

1. *Let* $z_0 = (\rho, y_0) \in \tilde{J}(f)$ *and* $t \geq 0$. *Suppose that there exists a sequence* $\{n_j\}_{j \in \mathbb{N}}$ *of positive integers and a point* $y_1 \in J(G_\tau) \setminus P(G_\tau)$ *such that* $f_{\rho,n_j}(y_0) \to y_1$ *and* $p(f^{n_j}(z_0)) \cdots p(z_0) \cdot |(f_{\rho,n_j})'(y_0)|^t \to \infty$ *as* $j \to \infty$. *Then,* $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|^t} = \infty$.

2. *Suppose that for each* $j = 1, \ldots, m$, $p_j \cdot \min_{y \in g_j^{-1}(J(G_\tau))} |(g_j)'(y)| > 1$. *Then, for each* $y_0 \in J(G_\tau)$, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|} = \infty$ *and* $T_{\infty,\tau}$ *is not differentiable at* y_0.

Theorem 2.18. *Let* $\tau = \sum_{j=1}^{m} p_j \delta_{g_j}$, *where* $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ *for each* $j = 1, \ldots, m$, *and* $\sum_{j=1}^{m} p_j = 1$. *Let* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *be the skew product associated with* supp τ. *For each* $z = (\rho, y) \in X_\tau \times \overline{\mathbb{C}}$, *we set* $p(z) := p_j$ *if* $\rho_1 = g_j$. *Suppose that* $G_\tau \in \mathcal{G}$ *and that* $\{g_j^{-1}(J(G_\tau))\}_{j=1}^{m}$ *are mutually disjoint. Moreover, suppose that* G_τ *is hyperbolic. Let* $z_0 = (\rho, y_0) \in \tilde{J}(f)$ *and* $t \geq 0$. *Suppose that* $p(f^n(z_0)) \cdots p(z_0) \cdot |(f_{\rho,n})'(y_0)|^t \to 0$ *as* $n \to \infty$. *Then,* $\lim_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y-y_0|^t} = 0$.

Theorem 2.19. *Let* $\tau = \sum_{j=1}^{m} p_j \delta_{g_j}$, *where* $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ *for each* $j = 1, \ldots, m$, *and* $\sum_{j=1}^{m} p_j = 1$. *Suppose that* $G_\tau \in \mathcal{G}$ *and that* $\{g_j^{-1}(J(G_\tau))\}_{j=1}^{m}$ *are mutually disjoint. Moreover, suppose that* G_τ *is hyperbolic. Then,* $T_{\infty,\tau} : \overline{\mathbb{C}} \to [0,1]$ *is Hölder continuous with respect to the spherical distance.*

Theorem 2.20. *Let* $\tau = \sum_{j=1}^{m} p_j \delta_{g_j}$, *where* $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ *for each* $j = 1, \ldots, m$, *and* $\sum_{j=1}^{m} p_j = 1$. *Let* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *be the skew product associated with* supp τ. *Let* μ *be the maximal entropy measure of* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *with respect to* $(\sigma, \tilde{\tau})$ *on* X_τ; *that is, let* μ *be the unique* f-*invariant Borel probability measure on* $X_\tau \times \overline{\mathbb{C}}$ *such that* $h_\mu(f|\sigma) = \max\{h_\nu(f|\sigma) \mid f_*\nu = \nu, \pi_*(\nu) = \tilde{\tau}\}$, *where* $h_\nu(f|\sigma)$ *denotes the relative entropy of* (f, ν) *with respect to* $\sigma : X_\tau \to X_\tau$ *(Remark: For the existence and uniqueness of* μ, *see [S8]). Suppose that* $G_\tau \in \mathcal{G}$ *and that* $\{g_j^{-1}(J(G_\tau))\}_{j=1}^{m}$ *are mutually disjoint. Then, the following statements hold.*

1. *Suppose* $\int_{X_\tau \times \overline{\mathbb{C}}} \log |f'(z)| \, d\mu > -\sum_{j=1}^{m} p_j \log p_j$. *Then, for almost every* $y_0 \in J(G_\tau)$ *with respect to* $(\pi_{\overline{\mathbb{C}}})_*(\mu)$, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y-y_0|} = \infty$ *and* $T_{\infty,\tau}$ *is not differentiable at* y_0.

2. *Suppose that* $\sum_{j=1}^{m} p_j \log p_j + \frac{1}{2} \sum_{j=1}^{m} p_j \log(\deg(g_j)) > 0$. *Then, for almost every* $y_0 \in J(G_\tau)$ *with respect to* $(\pi_{\overline{\mathbb{C}}})_*(\mu)$, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y-y_0|} = \infty$ *and* $T_{\infty,\tau}$ *is not differentiable at* y_0.

3. *Suppose that for each* $j = 1, \ldots, m$, $\deg(g_j) \geq m^2$, *and that there exists an* i *such that* $\deg(g_i) > m^2$. *Then, for almost every* $y_0 \in J(G_\tau)$ *with respect to* $(\pi_{\overline{\mathbb{C}}})_*(\mu)$, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y-y_0|} = \infty$ *and* $T_{\infty,\tau}$ *is not differentiable at* y_0.

Theorem 2.21. *Let* $\tau = \sum_{j=1}^{m} p_j \delta_{g_j}$, *where* $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ *for each* $j = 1, \ldots, m$, *and* $\sum_{j=1}^{m} p_j = 1$. *Let* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *be the skew product associated with* supp τ. *For each* $z = (\rho, y) \in X_\tau \times \overline{\mathbb{C}}$, *we set* $p(z) := p_j$ *if* $\rho_1 = g_j$. *Suppose that* $G_\tau \in \mathcal{G}$ *and that* $\{g_j^{-1}(J(G_\tau))\}_{j=1}^{m}$ *are mutually*

disjoint. Moreover, suppose that G_τ is hyperbolic. Let ν be an f-invariant ergodic Borel probability measure on $\tilde{J}(f)$. Let

$$t_0 := -\frac{\int_{X_\tau \times \overline{\mathbb{C}}} \log p(z)\, d\nu}{\int_{X_\tau \times \overline{\mathbb{C}}} \log |f'(z)|\, d\nu}.$$

Then, the following statements hold.

1. *Let $0 \leq t < t_0$. Then, for almost every $y_0 \in J(G_\tau)$ with respect to $(\pi_{\overline{\mathbb{C}}})_*(\nu)$, $\lim_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|^t} = 0$.*

2. *Let $t > t_0$. Then, for almost every $y_0 \in J(G_\tau)$ with respect to $(\pi_{\overline{\mathbb{C}}})_*(\nu)$, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|^t} = \infty$.*

Theorem 2.22. *Let $\tau = \sum_{j=1}^m p_j \delta_{g_j}$, where $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ for each $j = 1, \ldots, m$, and $\sum_{j=1}^m p_j = 1$. Suppose that $G_\tau \in \mathcal{G}$ and that $\{g_j^{-1}(J(G_\tau))\}_{j=1}^m$ are mutually disjoint. Moreover, suppose that G_τ is hyperbolic. Let $\delta := \dim_H(J(G_\tau))$ and let H^δ be the δ-dimensional Hausdorff measure. (Remark: By [S3], $0 < H^\delta(J(G_\tau)) < \infty$.) Let $C(J(G_\tau)) := \{\psi : J(G_\tau) \to \mathbb{R} \mid \psi$ is continuous $\}$ endowed with the supremum norm. Let $L : C(J(G_\tau)) \to C(J(G_\tau))$ be the operator defined by: $L(\psi)(y) := \sum_{j=1}^m \sum_{g_j(w)=y} \frac{\psi(w)}{|g_j'(w)|^\delta}$. Let $\alpha := \lim_{n \to \infty} L^n(1) \in C(J(G_\tau))$, where 1 denotes the constant function taking the value 1. (Note that by [S3], the above α exists.) Let*

$$s_0 := \sum_{j=1}^m (\log p_j) \cdot \int_{g_j^{-1}(J(G_\tau))} \alpha(y)\, dH^\delta(y) + \sum_{j=1}^m \int_{g_j^{-1}(J(G_\tau))} \alpha(y) \log |g_j'(y)|\, dH^\delta(y).$$

Then, the following statements hold.

1. *Suppose $s_0 < 0$. Then, for almost every $y_0 \in J(G_\tau)$ with respect to H^δ, $\lim_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|} = 0$ and $T_{\infty,\tau}$ is differentiable at y_0.*

2. *Suppose $s_0 > 0$. Then, for almost every $y_0 \in J(G_\tau)$ with respect to H^δ, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|} = \infty$ and $T_{\infty,\tau}$ is not differentiable at y_0.*

Remark 3. Combining Theorem 2.20 and Theorem 2.22, it follows that there exists a $\tau = \sum_{j=1}^m p_j \delta_{g_j}$, where $p_j > 0$, $g_j \in \mathrm{Poly}_{\deg \geq 2}$ for each $j = 1, \ldots, m$, and $\sum_{j=1}^m p_j = 1$, such that all of the following statements 1,2, and 3 hold.

1. G_τ is hyperbolic, $G_\tau \in \mathcal{G}$, and $\{g_j^{-1}(J(G_\tau))\}_{j=1}^m$ are mutually disjoint.

2. Let $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ be the skew product associated with supp τ. Let μ be the maximal entropy measure of $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$. Then, for almost every $y_0 \in J(G_\tau)$ with respect to $(\pi_{\overline{\mathbb{C}}})_*(\mu)$, $\limsup_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|} = \infty$ and $T_{\infty,\tau}$ is not differentiable at y_0.

3. Let $\delta := \dim_H(J(G_\tau))$ and let H^δ be the δ-dimensional Hausdorff measure. Then, $0 < H^\delta(J(G_\tau)) < \infty$ and for almost every $y_0 \in J(G_\tau)$ with respect to H^δ, $\lim_{y \to y_0} \frac{|T_{\infty,\tau}(y) - T_{\infty,\tau}(y_0)|}{|y - y_0|} = 0$ and $T_{\infty,\tau}$ is differentiable at y_0.

2.5 The space of 2-generator polynomial semigroups

In this section, we present some results on the space of 2-generator polynomial semigroups.

Definition 2.23. We use the following notation.

- $\mathcal{Y} := \{g : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with topology induced by uniform convergence on $\overline{\mathbb{C}}$. Moreover, for any $m \in \mathbb{N}$, we set $\mathcal{Y}^m := \mathcal{Y} \times \cdots \times \mathcal{Y}$ (m factors) endowed with product topology.

- $\mathcal{B} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid P^*(\langle h_1, h_2 \rangle) \text{ is bounded in } \mathbb{C}\}$.

- $\mathcal{C} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(\langle h_1, h_2 \rangle) \text{ is connected}\}$.

- $\mathcal{D} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(\langle h_1, h_2 \rangle) \text{ is disconnected}\}$.

- $\mathcal{H} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid \langle h_1, h_2 \rangle \text{ is hyperbolic}\}$.

- $\mathcal{I} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(h_1) \cap J(h_2) \neq \emptyset\}$.

- $\mathcal{Q} := \{(h_1, h_2 \in \mathcal{Y}^2 \mid J(h_1) = J(h_2), \text{ and } J(h_1) \text{ and } J(h_2) \text{ are quasicircles}\}$.

Lemma 2.24. *The sets* $\mathcal{H}, \mathcal{H} \cap \mathcal{B}, \mathcal{H} \cap \mathcal{B} \cap \mathcal{D}$ *are non-empty and open in* $\mathcal{Y}^2$.

Definition 2.25. Let $m \in \mathbb{N}$, $(h_1, \ldots, h_m) \in \mathcal{Y}^m$ and $z \in \overline{\mathbb{C}}$.

- We set $S(h_1, \ldots, h_m, z) :=$

$$\inf \left\{ t \geq 0 \mid \sum_{n \in \mathbb{N}} \sum_{(w_1, \ldots, w_n) \in \{1, \ldots, m\}^n} \sum_{h_{w_n} \cdots h_{w_1}(y) = z} \| (h_{w_n} \cdots h_{w_1})'(y) \|^{-t} < \infty \right\}$$

$\in [0, \infty]$, where $\| \cdot \|$ denotes the norm of the derivative with respect to the spherical metric.

- For any $p \in (0,1)$, let $T(h_1, h_2, p, z)$ be the probability of tending to infinity starting from the initial value z, with respect to the i.i.d. random dynamical system on $\overline{\mathbb{C}}$ such that at every step we choose h_1 with probability p and choose h_2 with probability $1 - p$. More precisely, setting $\tau_{h_1, h_2, p} := p\delta_{h_1} + (1-p)\delta_{h_2}$, where δ_h denotes the Dirac measure concentrated at h, we set $T(h_1, h_2, p, z) := T_{\infty, \tau_{h_1, h_2, p}}(z)$. (Note that $z \mapsto T(h_1, h_2, p, z)$ is locally constant on $F(\langle h_1, h_2 \rangle)$.)

- For any subset A of $\overline{\mathbb{C}}$, $\dim_H(A)$ denotes the Hausdorff dimension of the set A with respect to the spherical distance.

Theorem 2.26. *(Theorem C) We have the following.*

1. *Let $(h_1, h_2) \in \mathcal{B} \cap \mathcal{D}$ and $G = \langle h_1, h_2 \rangle$. Then, $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$.*

2. *$\mathcal{H} \cap \overline{\mathrm{int}(\mathcal{H} \cap \mathcal{B} \cap \mathcal{C})} = \mathcal{H} \cap \mathcal{B} \cap \mathcal{C}$.*

3. *For any $(h_1, h_2) \in \mathcal{H} \cap \overline{\mathcal{B} \cap \mathcal{D}}$, $\dim_H(J(\langle h_1, h_2 \rangle)) < 2$ and $J(\langle h_1, h_2 \rangle)$ is porous.*

4. *Let $(h_1, h_2) \in (\mathcal{H} \cap \overline{\mathcal{B} \cap \mathcal{D}}) \setminus \mathcal{Q}$. Then, $\dim_H(J(\langle h_1, h_2 \rangle)) = S(h_1, h_2, z)$ for each $z \in \overline{\mathbb{C}} \setminus P(\langle h_1, h_2 \rangle)$. Moreover, there exist an $\epsilon > 0$ and an open neighborhood V of (h_1, h_2) in $\mathcal{H} \cap \mathcal{B}$ such that for any $(g_1, g_2) \in V$, $\dim_H(J(\langle g_1, g_2 \rangle)) \leq S(g_1, g_2, z) \leq 2 - \epsilon$, for any $z \in \overline{\mathbb{C}} \setminus P(\langle g_1, g_2 \rangle)$.*

5. *$\mathcal{D} \cap \mathcal{Q} = \emptyset$.*

6. *For each connected component V of $\mathcal{Y}^2$, $\mathcal{Q} \cap V$ is included in a proper holomorphic subvariety of V.*

7. *$(\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})) \setminus \mathcal{Q}$ is dense in $\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})$.*

8. *For any $(h_1, h_2) \in (\mathcal{B} \cap \mathcal{D}) \cup (\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C}))$ and any $0 < p < 1$,*
   $J(\langle h_1, h_2 \rangle) =$
   $\{z_0 \in \overline{\mathbb{C}} \mid \forall \mathrm{nbd}\ U\ \text{of}\ z_0,\ z \mapsto T(h_1, h_2, p, z)\ \text{is not constant on}\ U\}.$

9. *Let $(h_1, h_2) \in \mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})$ and $0 < p < 1$. Then, $z \mapsto T(h_1, h_2, p, z)$ is continuous on $\overline{\mathbb{C}}$ if and only if $J(h_1) \cap J(h_2) = \emptyset$.*

10. *Let $(h_1, h_2) \in (\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})) \setminus \mathcal{I}$. Then, there exists a neighborhood V of (h_1, h_2) in $\mathcal{H} \cap \mathcal{B}$ such that for any $(g_1, g_2) \in V$ and any $0 < p < 1$, $z \mapsto T(g_1, g_2, p, z)$ is continuous on $\overline{\mathbb{C}}$.*

11. *Let $(h_1, h_2) \in (\mathcal{B} \cap \mathcal{D}) \cup ((\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})) \setminus \mathcal{I})$. Then, for any $z \in \overline{\mathbb{C}}$, the function $p \mapsto T(h_1, h_2, p, z)$ is real analytic on $(0,1)$. Moreover, for any $n \in \mathbb{N} \cup \{0\}$, the function $(p, z) \mapsto \frac{\partial^n T}{\partial p^n}(h_1, h_2, p, z)$ is continuous on $(0,1) \times \overline{\mathbb{C}}$.*

Remark 4.   1. Let $\alpha_1, \alpha_2 \in$ RA. For each $x \in \mathbb{R}$ and each $0 < p < 1$, let $T_{+\infty}(\alpha_1, \alpha_2, p, x)$ be the probability of tending to $+\infty$ starting from the initial value x, with respect to the i.i.d. random dynamical system on $\hat{\mathbb{R}}$ such that at every step we choose α_1 with probability p and choose α_2 with probability $1 - p$. Then, setting $\beta_1(x) := 3x$ and $\beta_2(x) := 3(x - 1) + 1$, the function $x \mapsto T_{+\infty}(\beta_1, \beta_2, \frac{1}{2}, x)$ on $[0, 1]$ is the devil's staircase (or the Cantor function). Moreover, setting $\rho_1(x) := 2x$ and $\rho_2(x) := 2(x-1)+1$, for each $0 < p < 1$ with $p \neq \frac{1}{2}$, the function $x \mapsto T_{+\infty}(\rho_1, \rho_2, p, x)$ on $[0, 1]$ is Lebesgue's singular function. Furthermore, the function $x \mapsto \frac{\partial T_{+\infty}}{\partial p}(\rho_1, \rho_2, \frac{1}{2}, x)$ on $[0, 1]$ is two times the Takagi function.

2. For researches of Lebesgue's singular functions and the Takagi function and its generalization, see [AK], [D], [HY], [SS], and [T].

3. The function $z \mapsto T(h_1, h_2, p, z)$ on $\overline{\mathbb{C}}$ is a complex analogue of Lebesgue's singular function.

4. The function $z \mapsto \frac{\partial T}{\partial p}(h_1, h_2, p, z)$ on $\overline{\mathbb{C}}$ is a complex analogue of the Takagi function.

Figure 1: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1 := z^2 - 1, g_2 := \frac{z^2}{4}, h_1 := g_1^2, h_2 := g_2^2$. The semigroup G belongs to $\mathcal{G}_{dis}$. Moreover, G is hyperbolic.
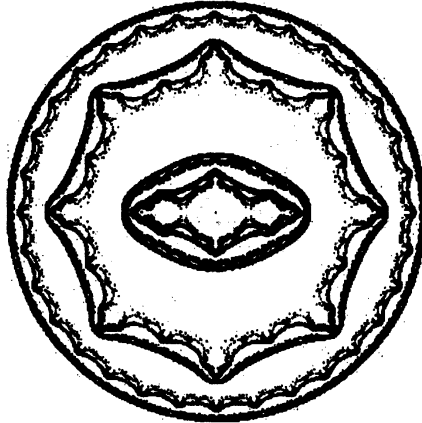
Figure 2: The graph of $z \mapsto T(h_1, h_2, \frac{1}{2}, z)$ (a devil's coliseum).
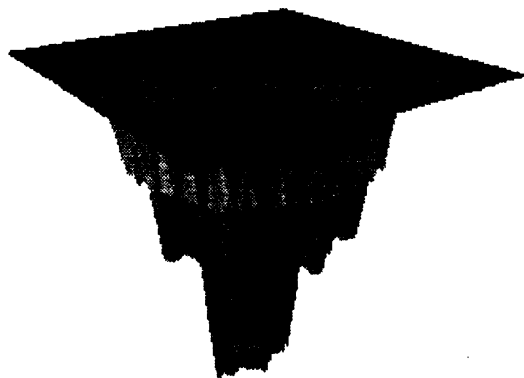

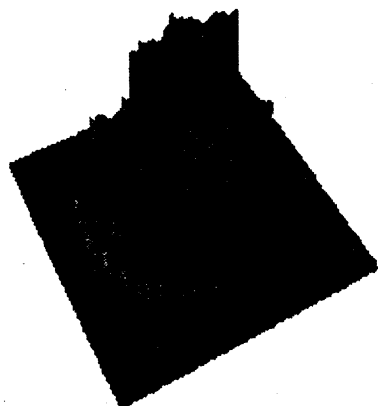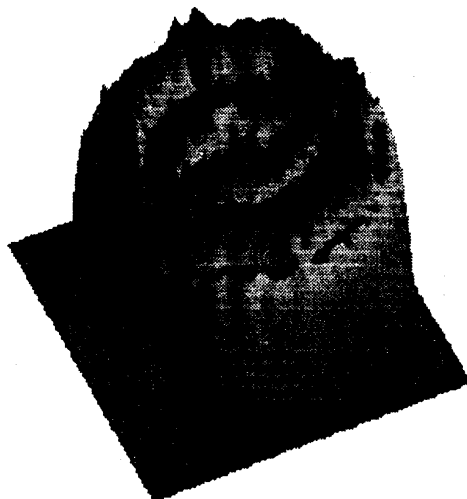Figure 3: The graph of $z \mapsto 1 - T(h_1, h_2, \frac{1}{2}, z)$.

Figure 4: The graph of $z \mapsto \frac{\partial T}{\partial p}(h_1, h_2, \frac{1}{2}, z)$ (a complex analogue of the Takagi function).

3 Example

Proposition 3.1. *Let h_1 be a hyperbolic polynomial such that $P^*(\langle h_1 \rangle)$ is bounded in $\mathbb{C}$ and $\deg(h_1) \geq 2$. Let $d \in \mathbb{N}$ with $(\deg(h_1), d) \neq (2,2)$. Then, there exists a holomorphic family $\{h_{2,a}\}_{a \in W}$ of polynomials such that all of the following hold.*

1. *W is a subdomain of $\mathbb{C}$. For any $a \in W$, $(h_1, h_{2,a}) \in \mathcal{H} \cap \mathcal{B}$ and $\deg(h_{2,a}) = d$.*

2. *There exists an $a_0 \in W$ such that $(h_1, h_{2,a_0}) \in \mathcal{H} \cap \mathcal{B} \cap \mathcal{D}$.*

3. *There exists an $a_1 \in W$ satisfying the following:*

   - *$(h_1, h_{2,a_1}) \in (\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})) \setminus \mathcal{I} \subset (\mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})) \setminus \mathcal{Q}$ and*

   - *for any neighborhood V of a_1 in W, there exists an $a_2 \in V$ such that $(h_1, h_{2,a_2}) \in \mathrm{int}(\mathcal{H} \cap \mathcal{B} \cap \mathcal{C})$.*

4 Tools and Proofs

To show the main results, we need some tools in this section.

4.1 Fundamental properties of rational semigroups

Lemma 4.1 ([HM],[GR],[S1]). *Let G be a rational semigroup.*

1. *For each $f \in G$, we have $f(F(G)) \subset F(G)$ and $f^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*

2. *If G is generated by a compact subset Γ of Rat, then $J(G) = \cup_{h \in \Gamma} h^{-1}(J(G))$. (This is called the **backward self-similarity** of $J(G)$.*

3. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set.*

4. *If $\sharp(J(G)) \geq 3$, then $\sharp E(G) \leq 2$.*

5. *If a point z is not in $E(G)$, then $J(G) \subset \overline{\cup_{g \in G} g^{-1}(z)}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then $\overline{\cup_{g \in G} g^{-1}(z)} = J(G)$.*

6. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

Theorem 4.2 ([HM],[GR]). *Let G be a rational semigroup. If $\sharp(J(G)) \geq 3$, then $J(G) = \{z \in \overline{\mathbb{C}} \mid \exists g \in G, \ g(z) = z, \ |g'(z)| > 1\}$. In particular, $J(G) = \bigcup_{g \in G} J(g)$.*

Lemma 4.3 ([N]). *Let X be a compact metric space and let $f : X \to X$ be a continuous open map. Let A be a compact connected subset of X. Then for each connected component B of $f^{-1}(A)$, we have $f(B) = A$.*

4.2 Tools for random dynamics of holomorphic maps

To show the results in section 2.3, we need some tools in this section.

Definition 4.4. Let τ be a Borel probability measure on Rat.

1. We denote by supp τ the support of τ. Moreover, we set $X_\tau := (\text{supp } \tau)^{\mathbb{N}}$ $(= \{\rho = (\rho_1, \rho_2, \dots) \mid \rho_j \in \text{supp } \tau\})$ endowed with the product topology. Furthermore, we set $\tilde{\tau} := \otimes_{j=1}^{\infty}\tau$. This is a Borel probability measure on X_τ. We denote by G_τ the rational semigroup generated by supp τ.

2. We denote by $\mathcal{M}_1(\overline{\mathbb{C}})$ the space of all Borel probability measures on $\overline{\mathbb{C}}$, endowed with the weak topology. Note that $\mathcal{M}_1(\overline{\mathbb{C}})$ is a compact metric space. Let M_τ be an operator on $C(\overline{\mathbb{C}})$ defined by $M_\tau(\phi)(z) := \int_{\text{supp}\tau} \phi(g(z)) \, d\tau(g)$. Moreover, let $(M_\tau)_* : \mathcal{M}_1(\overline{\mathbb{C}}) \to \mathcal{M}_1(\overline{\mathbb{C}})$ be the dual of M_τ.

3. We denote by $F_{meas}(\tau)$ the set of $\mu \in \mathcal{M}_1(\overline{\mathbb{C}})$ satisfying that there exists a neighborhood B of μ in $\mathcal{M}_1(\overline{\mathbb{C}})$ such that the sequence $\{(M_\tau)_*^n|_B : B \to \mathcal{M}_1(\overline{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous on B.

4. We set $J_{meas}(\tau) := \mathcal{M}_1(\overline{\mathbb{C}}) \setminus F_{meas}(\tau)$.

5. We denote by $F^0_{meas}(\tau)$ the set of $\mu \in \mathcal{M}_1(\overline{\mathbb{C}})$ satisfying that the sequence $\{(M_\tau)_*^n : \mathcal{M}_1(\overline{\mathbb{C}}) \to \mathcal{M}_1(\overline{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous at the one point μ. Note that $F_{meas}(\tau) \subset F^0_{meas}(\tau)$.

6. Using the embedding $z \in \overline{\mathbb{C}} \mapsto \delta_z \in \mathcal{M}_1(\overline{\mathbb{C}})$, where δ_z denotes the Dirac measure at z, we denote by $F_{pt}(\tau)$ the set of $z \in \overline{\mathbb{C}}$ satisfying that there exists a neighborhood B of z in $\overline{\mathbb{C}}$ such that the sequence $\{(M_\tau)_*^n|_B : B \to \mathcal{M}_1(\overline{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous on B.

7. Similarly, we denote by $F^0_{pt}(\tau)$ the set of $z \in \overline{\mathbb{C}}$ such that the sequence $\{(M_\tau)_*^n|_{\overline{\mathbb{C}}} : \overline{\mathbb{C}} \to \mathcal{M}_1(\overline{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous at the one point $z \in \overline{\mathbb{C}}$. Note that $F_{pt}(\tau) \subset F^0_{pt}(\tau)$.

Lemma 4.5. *Let τ be a Borel probability measure on Rat. Then, we have the following.*

1. $(M_\tau)_*^{-1}(F_{meas}(\tau)) \subset F_{meas}(\tau)$, *and* $(M_\tau)_*^{-1}(F^0_{meas}(\tau)) \subset F^0_{meas}(\tau)$.

2. *Let* $y \in \overline{\mathbb{C}}$ *be a point. Then,* $y \in F_{pt}(\tau)$ *if and only if for any* $\phi \in C(\overline{\mathbb{C}})$, *there exists a neighborhood U of y in $\overline{\mathbb{C}}$ such that the sequence $\{z \mapsto M_\tau^n(\phi)(z)\}_{n \in \mathbb{N}}$ of functions on U is equicontinuous on U. Similarly,* $y \in F^0_{pt}(\tau)$ *if and only if for any* $\phi \in C(\overline{\mathbb{C}})$, *the sequence* $\{z \mapsto M_\tau^n(\phi)(z)\}_{\in \mathbb{N}}$ *of functions on $\overline{\mathbb{C}}$ is equicontinuous at the one point y.*

3. $F_{meas}(\tau) \cap \overline{\mathbb{C}} \subset F_{pt}(\tau)$.

4. $F^0_{meas}(\tau) \cap \overline{\mathbb{C}} = F^0_{pt}(\tau)$.

5. $F(G_\tau) \subset F_{pt}(\tau)$.

6. *Let* $y \in \overline{\mathbb{C}}$ *be a point. Suppose that supp τ is compact, and that*
$$\tilde{\tau}\left(\{\rho = (\rho_1, \rho_2, \rho_3, \dots) \in X_\tau \mid y \in \cap_{j=1}^\infty \rho_1^{-1} \cdots \rho_j^{-1}(J(G_\tau))\}\right) = 0. \text{ Then,}$$
we have that $y \in F^0_{pt}(\tau) = F^0_{meas}(\tau) \cap \overline{\mathbb{C}}$.

7. $F^0_{pt}(\tau) = \overline{\mathbb{C}}$ *if and only if* $F_{meas}(\tau) = \mathcal{M}_1(\overline{\mathbb{C}})$.

Lemma 4.6. *Suppose that supp τ is compact, and that $\sharp(J(G_\tau)) \geq 3$. Let $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ be the skew product associated with supp τ. Then, we have* $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G_\tau)$.

Lemma 4.7. *Suppose that supp τ is compact, and that $\sharp(J(G_\tau)) \geq 3$. Let $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ be the skew product associated with supp τ. Then, for each $\rho = (\rho_1, \rho_2, \dots) \in X_\tau$, we have* $\pi_{\overline{\mathbb{C}}}(\hat{J}_\rho(f)) = \cap_{j=1}^\infty \rho_1^{-1} \cdots \rho_j^{-1}(J(G_\tau))$.

Definition 4.8. Let G be a rational semigroup. We set $J_{ker}(G) := \cap_{g \in G} g^{-1}(J(G))$. This is called the **kernel Julia set** of a rational semigroup G.

Lemma 4.9. *Suppose that supp τ is compact, and that $\sharp(J(G_\tau)) \geq 3$. Let $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ be the skew product associated with supp τ. Let $y \in J(G_\tau)$ be a point. Then,*

$$\tilde{\tau}(\{\rho \in X_\tau \mid (\rho, y) \in \tilde{J}(f), \liminf_{n \to \infty} d(f_{\rho,n}(y), J_{ker}(G_\tau)) > 0\}) = 0.$$

By these arguments, we obtain the following result.

Theorem 4.10. *Suppose that* supp τ *is compact,* $\sharp(J(G_\tau)) \geq 3$, *and that* $J_{ker}(G_\tau) = \emptyset$. *Let* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *be the skew product associated with* supp τ. *Then,* $F_{meas}(\tau) = \mathcal{M}_1(\overline{\mathbb{C}})$, *and for almost sure* $\rho \in X_\tau$ *with respect to* $\tilde{\tau}$, *the two-dimensional Lebesgue measure of* $\pi_{\overline{\mathbb{C}}}(\hat{J}_\rho(f))$ *is zero.*

Lemma 4.11. *Let* τ *be a Borel probability measure on* Poly$_{deg \geq 2}$. *Suppose that* $\infty \in F(G_\tau)$. *Let* $\phi \in C(\overline{\mathbb{C}})$ *be a function satisfying that there exists a neighborhood* U *of* ∞ *in* $\overline{\mathbb{C}}$ *such that* $\phi|_U \equiv 1$. *Moreover, suppose that* supp $\phi \subset F_\infty(G_\tau)$. *Then, we have*
$$T_{\infty,\tau}(y) = \tilde{\tau}(\{\rho \in X_\tau \mid \phi(\rho_n \circ \cdots \circ \rho_1(y)) \to \infty, n \to \infty\})$$
$$= \tilde{\tau}(\{\rho \in X_\tau \mid \exists n, \ \phi(\rho_n \circ \cdots \circ \rho_1(y)) = 1\}) = \lim_{n \to \infty} M_\tau^n(\phi)(y).$$

Lemma 4.12. *Let* τ *be a Borel probability measure on* Poly$_{deg \geq 2}$. *Suppose that* $\infty \in F(G_\tau)$. *Then, for each connected component* U *of* $F(G_\tau)$, *there exists a constant* $C_U \in [0,1]$ *such that* $T_{\infty,\tau}|_U \equiv C_U$.

Proposition 4.13. *Let* τ *be a Borel probability measure on* Poly$_{deg \geq 2}$ *such that* supp τ *is compact. Suppose that* $J_{ker}(G_\tau) = \emptyset$. *Then, the function* $T_{\infty,\tau} : \overline{\mathbb{C}} \to [0,1]$ *is continuous on the whole* $\overline{\mathbb{C}}$.

Lemma 4.14. *Let* τ *be a Borel probability measure on* Rat *such that* supp τ *is compact. Let* $f : X_\tau \times \overline{\mathbb{C}} \to X_\tau \times \overline{\mathbb{C}}$ *be the skew product associated with* supp τ. *Let* V *be a non-empty open subset of* $\overline{\mathbb{C}}$ *such that for each* $g \in G$, $g(V) \subset V$. *For each* $\rho = (\rho_1, \rho_2, \dots) \in X_\tau$, *we set* $L_\rho := \cap_{j=1}^\infty \rho_1^{-1} \cdots \rho_j^{-1}(\overline{\mathbb{C}} \setminus V)$. *Moreover, we set* $L_{ker} := \cap_{g \in G} g^{-1}(\overline{\mathbb{C}} \setminus V)$. *Let* $y \in \overline{\mathbb{C}}$ *be a point. Then, we have that* $\tilde{\tau}(\{\rho \in X_\tau \mid y \in L_\rho, \ \liminf_{n \to \infty} d(f_{\rho,n}(y), L_{ker}) > 0\}) = 0$.

Lemma 4.15. *Let* $G \in \mathcal{G}_{dis}$ *be a polynomial semigroup generated by a compact set of* Poly$_{deg \geq 2}$. *Then, for each* $y \in \overline{\mathbb{C}}$, *there exists an element* $g \in G$ *such that* $g(y) \in F_\infty(G) \cup$ int$(\hat{K}(G))$. *In particular, we have* $J_{ker}(G) = \emptyset$.

Corollary 4.16. *Let* τ *be a Borel probability measure on* Poly$_{deg \geq 2}$ *such that* supp τ *is compact. Suppose that* $G_\tau \in \mathcal{G}_{dis}$. *Then,* $J_{ker}(G_\tau) = \emptyset$ *and* $F_{meas}(\tau) = \mathcal{M}_1(\overline{\mathbb{C}})$.

Lemma 4.17. *Let* τ *be a Borel probability measure on* Poly$_{deg \geq 2}$ *such that* supp τ *is compact. Suppose that* $G_\tau \in \mathcal{G}_{dis}$. *Let* $y \in \overline{\mathbb{C}}$ *be a point. Then, we have*

$$\tilde{\tau}(\{\rho = (\rho_1, \rho_2, \dots) \in X_\tau \mid \exists n, \ \rho_n \circ \cdots \circ \rho_1(y) \in F_\infty(G_\tau) \cup \text{int}(\hat{K}(G))\}) = 1.$$

Lemma 4.18. *Under the assumption of Lemma 4.17, there exists an element* $\alpha \in$ supp τ, *a positive integer* n, *a neighborhood* U *of polynomial hull*

of $J_{\min}(G_\tau)$ in $\mathbb{C}$, an attracting fixed point $a \in \mathbb{C}$ of α, a positive number δ, and a neighborhood W of $(\alpha, \ldots, \alpha) \in$ (supp $\tau)^n$ in (supp $\tau)^n$, such that for each $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in W$, $(\gamma_n \circ \cdots \circ \gamma_1)(U) \subset B(a, \delta) \subset B(a, 2\delta) \subset \mathrm{int}(\hat{K}(G))$.

Lemma 4.19. *Suppose that we have the assumption of Lemma 4.17. Under the notation of Lemma 4.18, we denote by $\tilde{W}$ the following set*

$$\{\rho = (\rho_1, \rho_2, \ldots) \in X_\tau \mid \exists(k_j) \to \infty, \text{ such that } (\rho_{k_j}, \rho_{k_j+1}, \ldots, \rho_{k_j+n-1}) \in W\}.$$

Then, we have that $\tilde{\tau}(\tilde{W}) = 1$, and that for each $\rho \in \tilde{W}$ and each $y \in \hat{K}(G_\tau)$, $d(\rho_l \cdots \rho_1(a), \rho_l \cdots \rho_1(y)) \to 0$ as $l \to \infty$.

Lemma 4.20. *Suppose that we have the assumption of Lemma 4.17. Under the notation of Lemma 4.18, for any $y \in \hat{K}(G)$ and any $\phi \in C(\overline{\mathbb{C}})$,*

$$|M_\tau^p(\phi)(a) - M_\tau^p(\phi)(y)| \to 0, \quad as \ p \to \infty.$$

Lemma 4.21. *Suppose that we have the assumption of Lemma 4.17. Then, for any $\phi \in C(\overline{\mathbb{C}})$ there exists a number $\mu_\phi \in \mathbb{R}$ such that*

$$\|M_\tau^n(\phi) - \mu_\phi \cdot \mathbf{1}\|_{\infty, \hat{K}(G_\tau)} \to 0, \quad as \ n \to \infty,$$

where $\mathbf{1}$ denotes the constant function taking the value 1 and $\| \cdot \|_{\infty, \hat{K}(G_\tau)}$ denotes the supremum norm on $\hat{K}(G_\tau)$. Moreover, the map $\mu(\phi) = \mu_\phi$ from $C(\overline{\mathbb{C}})$ to $\mathbb{R}$ is continuous on $C(\overline{\mathbb{C}})$. Furthermore, μ defines a Borel probability measure on $\overline{\mathbb{C}}$ such that supp μ is contained in $\hat{K}(G_\tau)$.

By these arguments, we obtain Theorem 2.14.

4.3 Tools to prove Theorem C

We give some tools to prove Theorem C.

Lemma 4.22 ([S2]). *Let $(h_1, h_2) \in \mathcal{B}$. Then, $J(\langle h_1, h_2 \rangle)$ is connected if and only if $h_1^{-1}(J(\langle h_1, h_2 \rangle)) \cap h_2^{-1}(J(\langle h_1, h_2 \rangle)) \neq \emptyset$.*

Lemma 4.23. *Let $(h_1, h_2) \in (\mathcal{H} \cap \overline{\mathcal{B}} \cap \overline{\mathcal{D}}) \setminus \mathcal{Q}$. Then, the following holds.*

1. *We have either $K(h_1) \subset K(h_2)$ or $K(h_2) \subset K(h_1)$.*

2. *If $K(h_1) \subset K(h_2)$, let $U := \mathrm{int}(K(h_2)) \setminus K(h_1)$. Then, U is a non-empty open set, $h_1^{-1}(U) \cup h_2^{-1}(U) \subset U$ and $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Moreover, $\overline{U} \neq J(\langle h_1, h_2 \rangle)$.*

3. If $K(h_2) \subset K(h_1)$, let $U := \operatorname{int}(K(h_1))\backslash K(h_2)$. Then, U is a non-empty open set, $h_1^{-1}(U) \cup h_2^{-1}(U) \subset U$ and $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Moreover, $\overline{U} \neq J(\langle h_1, h_2 \rangle)$.

Lemma 4.24. *Let $(h_1, h_2) \in \mathcal{H} \cap \partial(\mathcal{B} \cap \mathcal{C})$. Then, $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) \neq \emptyset$.*

Theorem 4.25 ([S3]). *Let $(h_1, \dots, h_m) \in \mathcal{Y}^m$ and $z \in \overline{\mathbb{C}} \backslash P(\langle h_1, \dots, h_m \rangle)$. Suppose that $\langle h_1, \dots, h_m \rangle$ is hyperbolic. Then, all of the following statements hold.*

1. $\dim_H(J(\langle h_1, \cdots, h_m \rangle)) \leq S(h_1, \dots, h_m, z)$.

2. If there exists a non-empty open set U in $\overline{\mathbb{C}}$ such that $\cup_{j=1}^m h_j^{-1}(U) \subset U$ and such that $\{h_j^{-1}(U)\}_{j=1}^m$ are mutually disjoint, then

$$\dim_H(J(\langle h_1, \dots, h_m \rangle)) = S(h_1, \dots, h_m, z).$$

Theorem 4.26 ([S5]). *Let $(h_1, \dots, h_m) \in \mathcal{Y}^m$ and suppose that $\langle h_1, \dots, h_m \rangle$ is hyperbolic. Moreover, suppose that there exists a non-empty open set U in $\overline{\mathbb{C}}$ such that $\cup_{j=1}^m h_j^{-1}(U) \subset U$ and such that $\{h_j^{-1}(U)\}_{j=1}^m$ are mutually disjoint. Furthermore, suppose that $\overline{U} \neq J(\langle h_1, \dots, h_m \rangle)$. Then, $J(\langle h_1, \dots, h_m \rangle)$ is porous and $\dim_H(J(\langle h_1, \dots, h_m \rangle)) < 2$.*

Lemma 4.27. *Let $(h_1, \dots, h_m) \in \mathcal{Y}^m$ and $z \in \overline{\mathbb{C}} \backslash P(\langle h_1, \dots, h_m \rangle)$. Suppose that $\langle h_1, \dots, h_m \rangle$ is hyperbolic. Then, the function $(g_1, \dots g_m) \mapsto S(g_1, \dots g_m, z)$ defined on $\mathcal{Y}^m$ is continuous around $(h_1, \dots, h_m)$.*

Lemma 4.28. *Let $(h_1, h_2) \in (\mathcal{H} \cap \overline{\mathcal{B} \cap \mathcal{D}}) \backslash \mathcal{I}$. Then there exists an open neighborhood V of (h_1, h_2) in $\mathcal{Y}^2$ such that for each $(g_1, g_2) \in V$, $J_{ker}(\langle g_1, g_2 \rangle) = \emptyset$.*

Lemma 4.29 (cf. [B]). *Let $\mathcal{L} := \{(h_1, h_2) \in \mathcal{Y}^2 \mid J(h_1) = J(h_2)\}$. Then, for each connected component $\mathcal{V}$ of $\mathcal{Y}^2$, $\mathcal{L} \cap \mathcal{V}$ is included in a proper subvariety of $\mathcal{V}$.*

Theorem 4.30. *Let $(h_1, h_2) \in \mathcal{B}$. Suppose that $J(h_1) < J(h_2)$. Let $U := int(K(h_2))\backslash K(h_1)$. Suppose that $h_1^{-1}(U) \cup h_2^{-1}(U) \subset U$ and $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Then, for any $z \in \overline{\mathbb{C}}$, the function $p \mapsto T(h_1, h_2, p, z)$ is real analytic on $(0, 1)$. Moreover, for any $n \in \mathbb{N} \cup \{0\}$, the function $(p, z) \mapsto \frac{\partial^n T}{\partial p^n}(h_1, h_2, p, z)$ is continuous on $(0, 1) \times \overline{\mathbb{C}}$.*

Combining these results above with results in section 4.2, we can prove Theorem C.

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