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Self-similar scaling in power-law fluid models

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Abstract. Navier-Stokes equations represents a key reference model both in the analysis of nonlinear partial differential systems, and in non-Newtonian fluid mechanics. The power-law fluid, one of the most popular non-Newtonian models, generalizes the Navier-Stokes fluid in the fact that its non-constant viscosity is a power of the shear rate. A modification of the power-law fluids that consists in multiplying the power of the shear rate by the pressure is also considered. Such a model fits to the class of fluids with the viscosity depending on the shear rate and on the the pressure. We look at all these models from the point of view of invariance with respect to suitable scaling, and present some relevant results and open problems.

Dedicated to Professor Tai-Ping Liu on the occasion of his sixtieth birthday.

1. Power-law fluids

Denoting $\mathbf{v} = (v_1, v_2, v_3)$ the velocity and p the mean normal stress (the pressure), we consider an incompressible homogeneous fluid whose (generalized) viscosity ν_g depends on the pressure and on the shear rate $|\mathbf{D}(\mathbf{v})|^2$, $\mathbf{D}(\mathbf{v})$ representing the symmetric part of the velocity gradient $\nabla\mathbf{v}$. Unsteady flows of such fluids, that take place in a three-dimensional container $\Omega \subset \mathbb{R}^3$ at a constant temperature, are described by the system of partial differential equations of the form

$$\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) = -\nabla p + \operatorname{div}(2\nu_g(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})). \tag{1.1}$$

This system includes as special subsystems

- *Schaeffer's model*, derived in [19] to capture flows of certain granular materials, with

$$\nu_g(p, |\mathbf{D}(\mathbf{v})|^2) = \frac{\alpha p}{|\mathbf{D}(\mathbf{v})|} \quad \text{valid if } |\mathbf{D}(\mathbf{v})| \neq 0 \qquad (\alpha > 0).$$

- *Fluids with pressure dependent viscosity*, where ν_g is independent of the shear rate, but depends on the pressure, i.e., $\nu_g = \nu_g(p)$. Such models, frequently considered with an exponential viscosity-pressure relationship, are very important in elastohydrodynamics and in processes taking place under high pressures, see for example [2] or [22].

- *Fluids with shear rate dependent viscosity*, where ν_g is independent of the pressure, i.e., $\nu_g = \nu_g(|\mathbf{D}(\mathbf{v})|^2)$, including
  - *Ladyzhenskaya's fluids* with $\nu_g(|\mathbf{D}(\mathbf{v})|^2) = \nu_0 + \nu_1|\mathbf{D}(\mathbf{v})|^{r-2}$, where $r > 2$ is fixed, ν_0 and ν_1 are positive numbers. For $r = 3$ this system of PDEs is frequently called Smagorinski's model of turbulence, see [21]; then ν_0 is the molecular viscosity and ν_1 is the turbulent viscosity.
  - *Power-law fluids* with $\nu_g(|\mathbf{D}(\mathbf{v})|^2) = \nu_1|\mathbf{D}(\mathbf{v})|^{r-2}$ where $r \in (1,\infty)$ is fixed and ν_1 is a positive number.
  - *Navier-Stokes fluids* with $\nu_g(p, |\mathbf{D}(\mathbf{v})|^2) = \nu_0$ (ν_0 being a positive number).

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A Ladyzhenskaya's fluid reduces to the Navier-Stokes fluid by putting $\nu_1 = 0$, and to power-law fluids by setting $\nu_0 = 0$. Note also that taking $r = 2$ in the constitutive equation for the Ladyzhenskaya's fluid leads again to the Navier-Stokes fluid with the constant viscosity $2(\nu_0 + \nu_1)$. We refer to [16] and [17] for more details and views on these models.

As said above, we will primarily deal with three models. Time-dependent flows of fluids with the viscosity depending linearly on the pressure and polynomially on the shear rate are described by

$$\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - 2\nu_1 \operatorname{div}(p|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v})) = -\nabla p\,. \tag{1.2}$$

Note that setting $r = 1$ and $\alpha = 2\nu_1$ in (1.2) leads to Schaeffer's model.

Also, if ν_g in (1.2) does not depend on p one obtains the system describing unsteady flows of the power-law fluids

$$\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - 2\nu_1 \operatorname{div}(|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v})) = -\nabla p\,. \tag{1.3}$$

Taking $r = 2$ in (1.3), this system finally reduces to the evolutionary Navier-Stokes system

$$\operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nu_1 \Delta \mathbf{v} = -\nabla p\,. \tag{1.4}$$

2. Self-similar scaling

It is known that if $(\mathbf{v}, p)$ solves the Navier-Stokes system (1.4) then $(\mathbf{v}^\lambda, p^\lambda)$ defined through

$$\mathbf{v}^\lambda(t, x) = \lambda \mathbf{v}(\lambda^2 t, \lambda x), \quad p^\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x) \qquad (\lambda > 0) \tag{2.1}$$

solve (1.4) too.

Similarly, solutions of the equations for power-law fluids (1.3), considered for $r \in (1, 3)$, are invariant with respect to the scaling

$$\mathbf{v}^\lambda(t, x) := \lambda^{\frac{r-1}{3-r}} \mathbf{v}(\lambda^{\frac{2}{3-r}} t, \lambda x), \quad p^\lambda(t, x) := \lambda^{2\frac{r-1}{3-r}} p(\lambda^{\frac{2}{3-r}} t, \lambda x). \tag{2.2}$$

It means that if $(\mathbf{v}, p)$ solves (1.3), then $(\mathbf{v}^\lambda, p^\lambda)$ solves (1.3) as well. Note that the scaling (2.1) of the Navier-Stokes system is also included by setting $r = 2$ in (2.2).

Interestingly, looking for such values of α, β and γ that provide solutions of (1.2) invariant with respect to the scaling of the form

$$\mathbf{v}^\lambda(t, x) = \lambda^\alpha \mathbf{v}(\lambda^\beta t, \lambda x), \quad p^\lambda(t, x) = \lambda^\gamma p(\lambda^\beta t, \lambda x)\,, \tag{2.3}$$

we come to the conclusion that this is possible only if

either	$r = 1$ (Schaeffer's model)	and $\gamma = 2\alpha\,, \beta = \alpha + 1\,,$	(2.4)
or	$\alpha = -1$	$\Longrightarrow\ \beta = 0$ and $\gamma = -2\,.$	(2.5)

If (2.4) happens then we obtain the whole family of selfsimilar solutions of (1.2) of the form

$$\mathbf{v}^\lambda(t, x) = \lambda^\alpha \mathbf{v}(\lambda^{\alpha+1} t, \lambda x), \quad p^\lambda(t, x) = \lambda^{2\alpha} p(\lambda^{\alpha+1} t, \lambda x)\,, \tag{2.6}$$

while in the case (2.5) the self-similar solutions are of the form

$$\mathbf{v}^\lambda(t, x) = \frac{1}{\lambda} \mathbf{v}(t, \lambda x), \quad p^\lambda(t, x) = \frac{1}{\lambda^2} p(t, \lambda x)\,. \tag{2.7}$$

We use these scaling in two ways. First, in Section 3, studying the behavior of the rate of dissipation under these scalings, we classify the difficulty of the problems, and we make a relation between such behavior and the results achieved in dependence on the value of the power-law index r. Then, in Section 4, we come to the task of a construction of singular solutions to the governing systems of PDEs in a self-similar form.

3. Classification of the difficulty of the problems

We use above scaling to magnify the flow near the point of interest located inside the fluid domain. Studying the behavior of the averaged rate of dissipation $d(p,\mathbf{v})$ defined through

$$d(p,\mathbf{v}) := \int_{-1}^{0}\int_{\mathcal{B}_1(0)} \nu_g(p,|\mathbf{D}(\mathbf{v})|^2)|\mathbf{D}(\mathbf{v})|^2\,dx\,dt \tag{3.1}$$

for $d(p^\lambda,\mathbf{v}^\lambda)$ as $\lambda\to\infty$, we can give the following classification of the problem:

$$\text{if } d(p^\lambda,\mathbf{v}^\lambda)\begin{cases}\to 0\\ \to A\in(0,\infty) \text{ as } \lambda\to\infty\\ \to\infty\end{cases} \quad\text{then the problem is}\quad \begin{cases}\text{supercritical,}\\ \text{critical,}\\ \text{subcritical.}\end{cases}$$

Roughly speaking, we may say that for a subcritical problem the zooming (near a possible singularity) is penalized by $d(p^\lambda,\mathbf{v}^\lambda)$ as $\lambda\to\infty$, while for supercritical case the energy dissipated out of the system is an insensitive measure of this magnification.

Since for the power-law fluids

$$d(\mathbf{v}) := d(p,\mathbf{v}) = 2\nu_1\int_{-1}^{0}\int_{\mathcal{B}_1(0)} |\mathbf{D}(\mathbf{v})|^r\,dx\,dt\,, \tag{3.2}$$

and consequently

$$d(\mathbf{v}^\lambda) = 2\nu_1\lambda^{5r-11}\int_{-\lambda^{\frac{2}{3-r}}}^{0}\int_{\mathcal{B}_\lambda(0)} |\mathbf{D}_y(\mathbf{v})|^r\,dy\,d\tau\,, \tag{3.3}$$

we observe that

$$d(\mathbf{v}^\lambda)\begin{cases}\to 0\\ \to A\in(0,\infty)\\ \to\infty\end{cases} \quad \text{as } \lambda\to\infty \quad \text{if } \begin{cases} r<\frac{11}{5},\\ r=\frac{11}{5},\\ r>\frac{11}{5},\end{cases} \quad \text{implying that problem (1.3) is} \begin{cases}\text{supercritical.}\\ \text{critical.}\\ \text{subcritical.}\end{cases}$$

Here, we discuss what is known on mathematical consistency of the system (1.3). We shall see that the parameter $r=\frac{11}{5}$ plays an significant bound when formulating available results. Note that the Navier-Stokes equations in three spatial dimensions falls into the class of supercritical problems. Let us also recall (see [16]) that by *mathematical consistency* we mean long-time and large-data existence of (suitable) weak solution†, its uniqueness, regularity and further qualitative properties.

Evolutionary power-law fluid model (1.3) is for $r\geq\frac{11}{5}$ mathematically self-consistent; long-time existence of weak solution has been proved by Ladyzhenskaya (see [7], [8]), using the monotone operator theory and Minty's trick to identify the "correct" limit of the viscosity term, and the compact embedding to obtain the correct limit in quadratic term $\operatorname{div}(\mathbf{v}\otimes\mathbf{v})$. Regularity and consequently uniqueness for a smooth initial velocity is addressed in [15]. Note that the non-degenerate case (i.e. the Ladyzhenskaya's fluids) has been treated before and for $r\geq\frac{11}{5}$ the results on regularity and uniqueness for $r\geq\frac{11}{5}$ and on existence for $r>\frac{9}{5}$ were put into place by Bellout, Bloom and Nečas in [1], Málek, Nečas and Růžička in [13] and by Málek, Nečas, Rokyta and Růžička in [12], see also [16] for more details.

The long-time and large-data existence of weak solution for $r\leq\frac{11}{5}$ (under more general assumptions on smoothness of data than in [12] for example) and for $r>\frac{8}{5}$ is achieved by Bulíček et al. in [3] for the spatially periodic and Navier's slip boundary value problems‡. The same task to establish long-time and large-data existence result for the no-slip boundary conditions is non-trivial and it is addressed in the recent work by Wolf [24].

It is worth noting that the result in [3] holds also for a certain class of incompressible fluids with pressure and shear rate dependent viscosity for Navier's slip boundary conditions, extending thus the

† Since the balance equations of continuum physics are formulated over any measurable sets and such formulations are tantamount to the weak forms of the balance equations that are formulated pointwise, the notion of (suitable) weak solutions seems to primary, while the notion of smooth solution is rather secondary. This point of view has been obvious to Oseen, and consequently to Leray [9].

‡ This solution is suitable weak solution in the sense of Caffarelli, Kohn and Nirenberg, see [4], if $r>\frac{9}{5}$.

theory developed in [11] for spatially periodic problem. On the other hand, it is not at all clear if the result by Wolf can be extended to time-dependent flows of fluids with shear rate and pressure dependent viscosity that are subject to no-slip boundary conditions.

Another question of interest is the long-time and large-data existence for $r < \frac{8}{5}$. Supported by the analysis (see [6] and [5]) of steady flows one may conjecture that it may be true for $r \in (\frac{6}{5}, \frac{8}{5}]$. The main new ingredience incorporated by Frehse et al. in [6], see also [5], is based on an improvement of properties regarding Lipschitz truncations of Sobolev functions.

We complete this section by providing a classification of the difficulties of the problem (1.2) according to the behavior of $d(p^\lambda, \mathbf{v}^\lambda)$ as $\lambda \to \infty$ for $d(p, \mathbf{v})$ defined through (3.1), and $(\mathbf{v}^\lambda, p^\lambda)$ solving (1.2) and fulfilling either (2.6) or (2.7).

Considering first $(\mathbf{v}^\lambda, p^\lambda)$ of the form (2.6), we compute ($r = 1$)

$$d(p^\lambda, \mathbf{v}^\lambda) = 2\nu_1 \int_{-1}^{0} \int_{B_1(0)} p^\lambda |\mathbf{D}_x(\mathbf{v}^\lambda)| \, dx \, dt = \lambda^{2\alpha-3} 2\nu_1 \int_{-\lambda^{1+\alpha}}^{0} \int_{B_\lambda(0)} p |\mathbf{D}_y(\mathbf{v})| \, dy \, d\tau \,. \tag{3.4}$$

It implies that Schaeffer's model can be classified as supercritical (for $\alpha < \frac{3}{2}$), subcritical (for $\alpha > \frac{3}{2}$), or critical (for $\alpha = \frac{3}{2}$) in dependence of the considered scaling. Saying differently, despite the fact that solutions to the time-dependent Schaeffer's model ((2.1) with $r = 1$) are invariant with respect to various changes of scale, this invariance cannot be used to assign to the problem one of the labels: critical, subcritical, supercritical.

On the other hand, solutions of the system (1.2) for $r \geq 1$ are invariant with respect to the scaling (2.7) and then

$$d(p^\lambda, \mathbf{v}^\lambda) = 2\nu_1 \int_{-1}^{0} \int_{B_1(0)} p^\lambda |\mathbf{D}_x(\mathbf{v}^\lambda)|^r \, dx \, dt = \frac{2\nu_1}{\lambda^5} \int_{-1}^{0} \int_{B_\lambda(0)} p |\mathbf{D}_y(\mathbf{v})|^r \, dy \, d\tau \,. \tag{3.5}$$

Since $d(p^\lambda, \mathbf{v}^\lambda) \to 0$ as $\lambda \to \infty$, the problem is supercritical for all $r > 1$.

4. Singular self-similar solutions

In his article [9], Leray used the property (2.1) and proposed a construction of a self-simlar weak solution exhibiting a singularity at a critical time $T > 0$. Leray's program can be easily generalized to power-law fluids (1.3), which we will do next.

Assume that $(\mathbf{U}, Q)$ represents a nontrivial (weak†) solution of the system

$$\operatorname{div} \mathbf{U} = 0, \qquad \frac{3-r}{2} y_k \frac{\partial \mathbf{U}}{\partial y_k} + \frac{r-1}{2} \mathbf{U} + U_k \frac{\partial \mathbf{U}}{\partial y_k} - \operatorname{div}(|\mathbf{D}(\mathbf{U})|^{r-2} \mathbf{D}(\mathbf{U})) = -\nabla Q \,. \tag{4.1}$$

Then

$$\mathbf{v}_s(t,x) = \frac{1}{\sqrt{(T-t)^{r-1}}} \mathbf{U} \left(\frac{x}{\sqrt{(T-t)^{3-r}}} \right), \qquad p_s(t,x) = \frac{1}{(T-t)^{r-1}} Q \left(\frac{x}{\sqrt{(T-t)^{3-r}}} \right) \tag{4.2}$$

form a weak solution to (1.3) that develops the singularity at T as $t \to T-$.

Thus, in this appraoch the construction of singular weak solutions to (1.3) "reduces" to the question of existence of nontrivial solutions to (4.1).

For $r = 2$, Nečas et al. in [18] completed the original Leray's program by showing that if $\mathbf{U} \in L^3(\mathbb{R}^3)^3$ is a weak solution of (4.1) (with $r = 2$) then $\mathbf{U} \equiv \mathbf{0}$. The proof relies on the fact that the scalar quantity $\frac{1}{2}|\mathbf{U}|^2 + Q + y \cdot \mathbf{U}$ satisfies the maximum principle. This result was generalized in [23]; a simple proof under stronger assumption is given in [10].

† We may for example assume that $\mathbf{U} \in L^2(\mathbb{R}^3)^3 \cap W^{1,r}(\mathbb{R}^3)^3$.

If $r \neq 2$, one observes from how weak solutions to (4.1) are introduced that if $r > \frac{11}{5}$ then $\mathbf{U} \equiv 0$. For $r < \frac{11}{5}$ we have

$$\|\mathbf{v}_s(t)\|_2^2 = (T-t)^{11-5r}\|\mathbf{U}\|_2^2 \to 0 \text{ as } t \to T-,$$
$$\|\nabla \mathbf{v}_s(t)\|_r^r = \frac{\|\nabla \mathbf{U}\|_r^r}{(T-t)^{(5r-9)/2}} \to +\infty \text{ as } t \to T- \quad \text{for } r > \frac{9}{5},$$
$$\|\nabla \mathbf{v}_s(t)\|_s^s = \frac{\|\nabla \mathbf{U}\|_2^2}{(T-t)^{(3r+2s-9)/2}} \to +\infty \text{ as } t \to T- \quad \text{for } s > \frac{9-3r}{2}.$$

The last line just indicates that for $r \to 1+$, the higher norms of $\nabla \mathbf{v}$ blow-up.

We end up this section by preliminary results dealing with explicit constructions of nontrivial solutions to (4.1). We refer to [14] for details.

Set $\Psi(y) := (y_3 - y_2)^2 + (y_1 - y_3)^2 + (y_2 - y_1)^2$ and consider solutions to (4.1) of the form

$$\mathbf{U}(y) = \frac{A}{[\Psi(y)]^\alpha}(y_3 - y_2, y_1 - y_3, y_2 - y_1), \qquad P(y) = Q(\Psi(y)) \text{ for } Q \text{ smooth.} \tag{4.3}$$

As shown in [14], solutions of (4.1) having the structure (4.3) exist if

$$1 < r < \frac{3}{2} \text{ and } \alpha = \frac{1}{2-r}.$$

Note that if we consider time-independent flows, singular solutions of the form (4.3) exist for all $r > 1$ and $\alpha = \frac{1}{r-1}$. This suggests to look for counterexamples to regularity (as $C^{1,\alpha}$ for example) for the generalized Stokes-like systems, whereas "generalized"' means that ν_g is non-constant, and depends on p or $|\mathbf{D}(\mathbf{v})|$, etc.

5. Conclusion

The main objectives of this contribution are twofold.

First, we wished to point it out that there are interesting models of non-Newtonian mechanics for incompressible fluids that deserve the attention of mathematical analysts. Since the pressure is mostly eliminated from the analysis of the Navier-Stokes equations (despite the fact that the pressure can play a significant role in understanding of the whole system, see [20] for example), most of the studies of models for incompressible fluids with the viscosity depending on the pressure (and the shear rate) requires to deal with the pressure since the beginning. Here, we restrict ourselves to a few special models (power-law fluid, Schaeffer's fluid and their combinations) of one class of non-Newtonian fluids, namely fluids with non-constant viscosity, that share the property that their solutions are invariant with respect to a suitable scaling.

Second, using this scale invariance we show that power-law fluids can be classified in dependence on the power-law index r. Interestingly, this standard classification is not applicable to Schaeffer's model. We also documented that power-law models are well understood if the power-law index r satisfies the condition $r \geq \frac{11}{5}$ and the problem is subcritical or critical, while there is a lot of open problems if $r < \frac{11}{5}$ (the Navier-Stokes system falls to this range).

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