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燃焼過程を伴う
一次元粘性流体星モデル方程式の時間大域解

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1 Introduction.

We consider the one-dimensional motion of a compressible, viscous and heat conductive gas driven by the self-gravitation in the free-boundary case. In addition to this situation, we take into account the energy producing process inside the medium, that is, the gas consists of a reacting mixture and the combustion process is current at the high temperature stage.

The motion mentioned above is described by the following four equations in the Eulerian coordinate system corresponding to the conservation laws of mass, momentum and energy, and an equation of reaction-diffusion type:

$$\begin{cases} \rho_t + v\rho_y = -\rho v_y, \\ \rho(v_t + vv_y) = (-p + \mu v_y)_y + \rho f, \\ \rho(e_t + ve_y) = (\kappa\theta_y)_y + (-p + \mu v_y)v_y + \lambda\rho\phi z, \\ \rho(z_t + vz_y) = (d\rho z_y)_y - \rho\phi z \end{cases} \tag{1.1}$$

in $\bigcup_{t>0} (\Omega_t \times \{t\})$, where $\Omega_t := \{y \in \mathbb{R} \,|\, y_1(t) < y < y_2(t)\}$ and $y_i(\cdot)$ for $i = 1, 2$ are fluctuating boundary functions. Here the density $\rho = \rho(y,t)$, the velocity $v = v(y,t)$, the absolute temperature $\theta = \theta(y,t)$ and the mass fraction of the reactant $z = z(y,t)$ are the unknown functions, and positive constants μ, d and λ are the coefficients of viscosity, the species diffusion and the difference in heat between the reactant and the product.

The external force per unit mass $f = f(y,t)$ is given by the potential U due to the self-gravitation, $f = -U_y$. It is well known that U satisfies the boundary value problem

$$\begin{cases} U_{yy} = G\rho & \text{in } \bigcup_{t>0} (\Omega_t \times \{t\}), \\ U|_{y=y_1(t)} = U|_{y=y_2(t)} = 0 & \text{for } t > 0. \end{cases} \tag{1.2}$$

Here G is the Newtonian gravitational constant. The rate function $\phi = \phi(\theta)$ is defined by the Arrhenius law

$$\phi(\theta) = \theta^\beta e^{-\frac{A}{\theta}}, \tag{1.3}$$

where A is the activation energy (a positive constant) and β is a non-negative number. At high tempereture regimes, pressure $p = p(\rho, \theta)$ and internal energy $e = e(\rho, \theta)$ are given by $p = p_G + p_R$ and

$$e = C_v\theta + a\frac{\theta^4}{\rho}$$

with the specific heat at constant volume (positive constant) C_v, the Stefan-Boltzmann constant $a > 0$, respectively. Here $p_G = p_G(\rho, \theta)$ is the *gaseous* (elastic and thermal) pressure and $p_R = p_R(\rho, \theta)$ is the *radiative* pressure given by Stefan law

$$p_R = \frac{a}{3}\theta^4.$$

For technical reason, we assume the gas is ideal, that is, $p_G = R\rho\theta$ with the perfect gas constant R. We also assume the conductivity $\kappa = \kappa(\rho, \theta)$ has the following form (see for example, [1,6]):

$$\kappa = \kappa_1 + \kappa_2\frac{\theta^q}{\rho}$$

where κ_1, κ_2 and q are positive constants.

We impose the dynamical and kinematic boundary conditions for $i = 1, 2$

$$\begin{cases} (-p + \mu v_x)|_{y=y_i(t)} = -p_e & \text{for } t > 0, \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = v(y_i(t), t) & \text{for } t > 0, \end{cases}$$

where the positive constant p_e is the external pressure, and the thermal and chemical boundary conditions for $i = 1, 2$

$$\begin{cases} \kappa\theta_y|_{y=y_i(t)} = 0 & \text{for } t > 0, \\ d\rho z_y|_{y=y_i(t)} = 0 & \text{for } t > 0, \end{cases}$$

and the initial conditon

$$(\rho, v, \theta, z)|_{t=0} = \big(\rho_0(y), v_0(y), \theta_0(y), z_0(y)\big) \quad \text{for } y \in \overline{\Omega_0}.$$

We introduce the Lagrangian transformation. For arbitrary fixed point $(y, t) \in \bigcup_{t>0} \big(\overline{\Omega_t} \times \{t\}\big)$, we consider the solution curve $Y_{y,t}(\tau)$ of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}Y_{y,t}(\tau)}{\mathrm{d}\tau} = v\big(Y_{y,t}(\tau), \tau\big) & \text{for } 0 < \tau < t, \\ Y_{y,t}(t) = y. \end{cases}$$

The unique existence of such a solution curve is guaranteed from the fundamental existence theorem of an ordinary differential equation as long as v is suitably smooth. Let $Y_{y,t}(0) = \xi$. Then this is uniquely solvable in y,

$$y = Y_{\xi,0}(t) = \xi + \int_0^t v\big(Y_{\xi,0}(\tau), \tau\big)\,\mathrm{d}\tau.$$

It is well known that the kinematic boundary condition implies that for each $t \geq 0$ this mapping $(y,t) \mapsto (\xi,t)$ is one-to-one from $\overline{\Omega_t} \times \{t\}$ onto $\overline{\Omega_0} \times \{t\}$. We put $y_1(0) = 0$ and $y_2(0) = L$. Futhermore, we introduce the mass transformation

$$\xi \mapsto x = \int_0^\xi \rho_0(s)\,\mathrm{d}s.$$

Consequently, by putting $v(x,t) := 1/\tilde{\rho}(x,t)$, $u(x,t) := \tilde{v}(x,t)$ (the tilde "$\tilde{\ }$" means transformed functions) and normalizing $M := \int_0^L \rho_0(\xi)\,\mathrm{d}\xi = 1$ our problem becomes

$$\begin{cases} v_t &= u_x, \\ u_t &= \left(-p + \dfrac{\mu}{v}u_x\right)_x - G\left(x - \dfrac{\int_0^1 \eta v(\eta,t)\,\mathrm{d}\eta}{\int_0^1 v(\eta,t)\,\mathrm{d}\eta}\right), \\ e_t &= \left(\dfrac{\kappa}{v}\theta_x\right)_x + \left(-p + \dfrac{\mu}{v}u_x\right)u_x + \lambda\phi z, \\ z_t &= \left(\dfrac{d}{v^2}z_x\right)_x - \phi z \end{cases} \tag{1.4}$$

in $(0,1) \times (0,\infty)$ with the boundary conditions

$$\left(-p + \frac{\mu}{v}u_x,\ \frac{\kappa}{v}\theta_x,\ \frac{d}{v^2}z_x\right)\Bigg|_{x=0,1} = (-p_e,\ 0,\ 0) \quad \text{for } t > 0, \tag{1.5}$$

and the initial condition

$$(v,u,\theta,z)|_{t=0} = \big(v_0(x), u_0(x), \theta_0(x), z_0(x)\big) \quad \text{for } x \in [0,1]. \tag{1.6}$$

One-dimensional problems have been studied under various conditions. For the viscous polytropic ideal gas a pioneering work of global in time existence with large initial data was due to Kazhikhov and Shelukhin [7] under Dirichlet boundary condition with respect to the velocity. In the free-boundary case, Nagasawa [9] discussed the global existence problem and the asymptotic behavior for the polytropic ideal gas with the external pressure depending on time. Also Chen [2] studied a model equations for a reacting mixture. All works mentioned above were not taken into account the influence of an external force.

Ducomet [3–5] treated a one-dimensional self-gravitating gaseous model as some large-scale structure of the universe, called "pancakes" in the astrophysical literature (see [11]). Following the spirit of [11], he adopted as the self-gravitational term

$$\tilde{f}(x,t) = -G\left(x - \frac{1}{2}M\right)$$

not the exact form in $(1.4)^2$, and also assumed that the initial data and the solution are symmetric.

Now, by integration of $(1.4)^2$ with respect to x over $[0,1]$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u\,\mathrm{d}x = -G\left(\frac{1}{2} - \frac{\int_0^1 \eta v(\eta,t)\,\mathrm{d}\eta}{\int_0^1 v(\eta,t)\,\mathrm{d}\eta}\right). \tag{1.7}$$

Denoting $u - \int_0^1 u\,\mathrm{d}x$ by u again, we obtain the final form:

$$\begin{cases} v_t &= u_x, \\ u_t &= \left(-p + \dfrac{\mu}{v}u_x\right)_x - G\left(x - \dfrac{1}{2}\right), \\ e_t &= \left(\dfrac{\kappa}{v}\theta_x\right)_x + \left(-p + \dfrac{\mu}{v}u_x\right)u_x + \lambda\phi z, \\ z_t &= \left(\dfrac{d}{v^2}z_x\right)_x - \phi z \end{cases} \tag{1.8}$$

in $(0,1)\times(0,\infty)$ with the same initial-boundary conditions (1.5) and (1.6). For this system it is natural that initial function u_0 (which corresponds to $u_0 - \int_0^1 u_0\,\mathrm{d}x$ for the original system (1.4)) satisfies

$$\int_0^1 u_0\,\mathrm{d}x = 0. \tag{1.9}$$

In this paper we construct the unique global classical solution of system (1.8), (1.5), (1.6) with the equations of state

$$p = R\frac{\theta}{v} + \frac{a}{3}\theta^4, \qquad e = C_v\theta + av\theta^4 \tag{1.10}$$

and the conductivity

$$\kappa = \kappa_1 + \kappa_2 v\theta^q, \tag{1.11}$$

without the symmetric assumption to the initial data and the solution. From (1.7) it is easily seen that this solution leads to the one for the original problem (1.4)-(1.6) describing the exact one-dimensional self-gravitating fluid model, not the approximated one, "pancakes" which has been considered by Ducomet. The difficulty of our problem is mainly caused by radiative components of equations of state and θ-dependency of the conductivity. We can solve the problem only for the case of some $q \geq 4$, which is physically valid [14]. Similar result obtained in [5], but the proof in it is not clear for the authors.

Let $\Omega := (0,1)$, m a nonnegative integer, $0 < \sigma < 1$, T a positive constant and $Q_T := \Omega \times (0,T)$. We denote

$$|u|^{(0)} := \sup_{(x,t)\in Q_T} |u(x,t)|$$

and use the familiar notations $C^{m+\sigma}(\Omega)$, $C^{\sigma,\sigma/2}_{x,t}(Q_T)$, $C^{2+\sigma,1+\sigma/2}_{x,t}(Q_T)$ for the Hölder spaces (see for example, [8]).

Our main result is

Theorem 1 (Global Solution) *Let* $\alpha \in (0,1)$, $4 \le q \le 16$ *and* $0 \le \beta \le 13/2$. *Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^3 \tag{1.12}$$

satisfies the compatibility conditions, (1.9) *and*

$$v_0(x),\ \theta_0(x) > 0, \quad 0 \le z_0(x) \le 1 \quad \text{for } x \in \overline{\Omega}. \tag{1.13}$$

Then there exists a unique solution (v, u, θ, z) *of the initial-boundary value problem* (1.8), (1.5), (1.6) *with* (1.3), (1.10), (1.11) *such that for any* $T > 0$

$$(v,\ v_x, v_t) \in \left(C^{\alpha,\alpha/2}_{x,t}(Q_T)\right)^3, \quad (u, \theta, z) \in \left(C^{2+\alpha,\,1+\alpha/2}_{x,t}(Q_T)\right)^3, \tag{1.14}$$

$$v(x,t),\ \theta(x,t) > 0, \quad 0 \le z(x,t) \le 1 \quad \text{for } (x,t) \in \overline{Q_T}. \tag{1.15}$$

Proof of Theorem 1 is based on the local existence theorem and a priori estimates. The fundamental theorem about the existence and the uniqueness of the local in time solution in three-dimensional case was firstly established by Tani [12] under sufficiently general initial-boundary conditions. For a radiative fluid, Secchi [10] obtained the corresponding result. We can easily obtain suitable unique local solution to our problem in the same manner as these works. Therefore, to prove Theorem 1 it is sufficient to establish the following a priori boundedness.

Proposition 1 (A Priori Estimates) *Let* T *be an arbitrary positive constant,* $4 \le q \le 16$ *and* $0 \le \beta \le 13/2$. *Assume that the initial data satisfy the hypotheses of Theorem* 1 *and problem* (1.8), (1.5), (1.6) *with* (1.3), (1.10), (1.11) *has a solution* (v, u, θ, z) *such that*

$$(v,\ v_x, v_t) \in \left(C^{\alpha,\alpha/2}_{x,t}(Q_T)\right)^3, \quad (u, \theta, z) \in \left(C^{2+\alpha,\,1+\alpha/2}_{x,t}(Q_T)\right)^3. \tag{1.16}$$

Then there exists a positive constant M *depending on the initial data and* T *such that*

$$|v,\ v_x, v_t|_{\alpha,\alpha/2},\ |u, \theta, z|_{2+\alpha,\,1+\alpha/2} \le M, \tag{1.17}$$

$$v(x,t),\ \theta(x,t) \ge 1/M, \quad 0 \le z(x,t) \le 1 \quad \text{for } (x,t) \in \overline{Q_T}. \tag{1.18}$$

2 Key Lemmas for Proving Proposition 1.

In proving Proposition 1, we need several lemmas concerning the estimates of the solution and its derivatives (see [13] for the details). We use C as positive constants, and $\|\cdot\|$ denotes usual L^2 norm. At first, we easily obtain the following lemma by the standard energy method.

Lemma 1 *For any* $t \in [0,T]$

$$\int_0^1 \left(\frac{1}{2}u^2 + e + \lambda z + f(x)v\right) \mathrm{d}x$$
$$= \int_0^1 \left(\frac{1}{2}{u_0}^2 + e_0 + \lambda z_0 + f(x)v_0\right) \mathrm{d}x := E_0, \tag{2.1}$$

$$U(t) + \int_0^t V(\tau)\,\mathrm{d}\tau \le C, \tag{2.2}$$

$$\int_0^1 \frac{1}{2}z^2\,\mathrm{d}x + \int_0^t \int_0^1 \left(\frac{d}{v^2}{z_x}^2 + \phi z^2\right) \mathrm{d}x\,\mathrm{d}\tau = \int_0^1 \frac{1}{2}{z_0}^2\,\mathrm{d}x. \tag{2.3}$$

Here $e_0 := C_v\theta_0 + av_0{\theta_0}^4$, $f(x) := p_e + \dfrac{1}{2}Gx(1-x)$ *and*

$$\begin{cases} U(t) & := \displaystyle\int_0^1 \Big[C_v(\theta - 1 - \log\theta) + R(v - 1 - \log v)\Big]\,\mathrm{d}x, \\ V(t) & := \displaystyle\int_0^1 \left(\frac{\mu {u_x}^2}{v\theta} + \frac{\kappa{\theta_x}^2}{v\theta^2} + \lambda\frac{\phi}{\theta}z\right) \mathrm{d}x. \end{cases}$$

Kazhikhov and Shelukhin firstly derived the useful representation formula for v. In the present case, we can obtain the following similar form (see [7]).

Lemma 2 *The identity*

$$v(x,t) = \frac{1}{B(x,t)Y(x,t)D(x,t)}$$
$$\times \left(v_0 + \int_0^t \frac{R}{\mu}\theta(x,\tau)B(x,\tau)Y(x,\tau)D(x,\tau)\,\mathrm{d}\tau\right) \tag{2.4}$$

holds, where

$$B(x,t) := \exp\left[\frac{1}{\mu}\int_0^x \Big(u_0(\xi) - u(\xi,t)\Big)\,\mathrm{d}\xi\right], \quad Y(x,t) := \exp\left(\frac{1}{\mu}f(x)t\right),$$
$$D(x,t) := \exp\left(-\frac{a}{3\mu}\int_0^t \theta(x,\tau)^4\,\mathrm{d}\tau\right).$$

From this representation, we can obtain a priori bounds of v.

Lemma 3 *For any* $(x,t) \in \overline{Q_T}$

$$C^{-1} \le v(x,t) \le C. \tag{2.5}$$

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