Reduced rescaled problem of some activator-inhibitor systems

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1   Introduction

We consider reaction-diffusion systems as follows:

$$u_\tau = \epsilon^2 \Delta u + f(u) - v, \qquad v_\tau = D \Delta v + u - \gamma v, \qquad (1.1)$$

where $u = u(y, \tau)$ and $v = v(y, \tau)$ denote activator and inhibitor respectively; $\gamma \geq 0$ is a nonnegative constant; and $\varepsilon, D > 0$ are positive constants. We assume that $f(s) = -W'(s)$ where $W \in C^2(\mathbb{R})$ is a double-equal-well potential satisfying

$$W(h^+) = W(h^-) = 0 < W(s) \quad {}^\forall s \in \mathbb{R} \backslash \{h^+, h^-\}, \quad W''(h^+)W''(h^-) > 0$$

with constants $h^- < 0 < h^+$, and there exists a unique value $h^0 \in (h^-, h^+)$ such that $f(h^0) = 0$ with $f'(h^0) > 0$. There hold $f(h^\pm) = 0$, $f'(h^\pm) < 0$ and $\int_{h^-}^{h^+} f(s)\, ds = 0$. A prototype is $f(u) = u - u^3$.

The system (1.1) describes the reaction and the diffusion phenomena of substances. When the ratio of the diffusion constants, ε^2/D, is extremely small, very interesting stationary patterns, such as stripes or spots, often appear. As a mathematical approach to understand this pattern formation, we consider the limit $\varepsilon \to 0$. Then usually the domain is divided into two regions and the remaining part becomes a thin layer. In some cases, the width of the internal transition layer approaches 0 in the limit,

and the discontinuity surface inside the domain, which is called sharp interface, appears. On the other hand, it is known that (1.1) can have very fine layered patterns. See [6, 12, 13]. We consider this fine pattern which has the space scale of $\epsilon^{1/3}$ order. This is the unique scale that the order of the two driving forces of the sharp interface, the inhibitor v and the curvature of the sharp interface, balances. See [10]. This scale also appears in [6]. After rescaling

$$x = \frac{y}{\epsilon^{1/3}}, t = \epsilon^{4/3}\tau, \varepsilon = \epsilon^{2/3},$$

we obtain

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - v), \\ \varepsilon^3 v_t = D\Delta v + \varepsilon(u - \gamma v). \end{cases} \tag{1.2}$$

We consider the stationary solutions of (1.2) subject to the homogeneous Neumann boundary condition:

$$\begin{cases} -\varepsilon^2\Delta u = f(u) - v, & \text{in } \Omega, \\ -D\Delta v = \varepsilon(u - \gamma v), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth bounday $\partial\Omega$. This is the elliptic system of FitzHugh–Nagumo type and the associated functional is
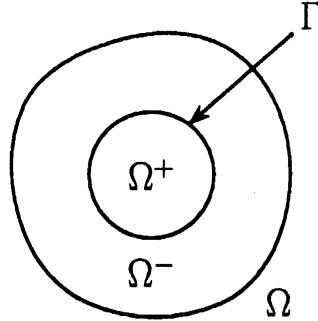
$$I(u, \varepsilon) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u) + \frac{1}{2\varepsilon^2}(D|\nabla v|^2 + \varepsilon\gamma v^2)\, dx,$$

where v solves

$$\begin{cases} -D\Delta v + \varepsilon\gamma v = \varepsilon u, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

In what follows, we deduce the reduced problem. If we assume $u \to u_0$ and $v \to v_0$ in the limit $\varepsilon \to 0$, we formally have

$$f(u_0) = v_0, \quad \Delta v_0 = 0, \quad \text{in } \Omega$$

図 1: sharp interface Γ and the domain Ω

$$\frac{\partial v_0}{\partial n} = 0 \quad \text{on } \partial\Omega$$

Hence v_0 is a constant. Now assume that v_0 is close to 0 and

$$u_0 = f^{-1}(v_0; h^+)1_{\Omega^+} + f^{-1}(v_0; h^-)1_{\Omega^-}.$$

Here Ω^+, Ω^- are open sets in Ω; $1_{\Omega^\pm}$ denotes the characteristic functions of $\Omega^\pm$; $u = f^{-1}(v; h^\pm)$ is the inverse function of $v = f(u)$ near $u = h^\pm$ respectively.

We assume that $\Gamma = \Omega \backslash (\Omega^+ \cup \Omega^-)$ is a curve embedded in Ω. We call Γ sharp interface. We shall identify the profile of layer near Γ.

It is known that there exists a constant $\tau > 0$, depending on f, such that for any $v \in (-\tau, \tau)$, the equation for u, $u_t = u_{xx} + f(u) - v$, has a traveling wave solution $u(x, t) = Q(x - ct; v)$ with the speed $c = c(v)$ and the profile $Q = Q(\xi; v)$. More precisely, $c(v)$ and $Q(\xi; v)$ for $v \in (-\tau, \tau), \xi \in \mathbb{R}$ satisfy

$$\begin{cases} \ddot{Q} + c(v)\dot{Q} + f(Q) - v = 0, & \text{in } \mathbb{R}, \\ \lim_{\xi \to -\infty} Q(\xi; v) = f^{-1}(v; h^+), \\ \lim_{\xi \to +\infty} Q(\xi; v) = f^{-1}(v; h^-), \\ c(0) = 0. \end{cases}$$

Here ˙ means $d/d\xi$. See, for example, [4]. Near the sharp interface Γ,

consider the function

$$u(x) = Q\left(\frac{d(x)}{\varepsilon}; v\right),$$

where $d = d(x)$ is the signed distance function from Γ such that $d(x) > 0$ if $x \in \Omega^-$ and $d(x) < 0$ if $x \in \Omega^+$. If the above function satisfy the first equation of (1.3) for each prescribed v, noting that $|\nabla d| = 1$, there holds

$$\ddot{Q} + \varepsilon(\Delta d)\dot{Q} + f(Q) - v = 0.$$

Since Δd is equal to the curvature κ of Γ on the interface Γ (here we choose the sign such that $\kappa > 0$ when Ω^+ is a disk), it follows that

$$c(v) = \varepsilon\kappa \qquad \text{on } \Gamma.$$

Since $c(0) = 0$ by the assumption, we may assume that

$$v_0 = 0$$

and

$$u_0 = h^+ \mathbf{1}_{\Omega^+} + h^- \mathbf{1}_{\Omega^-}.$$

Next we consider the higher order term. Assume

$$v = \varepsilon v_1 + O(\varepsilon^2).$$

Then we obtain the reduced problem

$$\begin{cases} -D\Delta v_1 = h^+ \mathbf{1}_{\Omega^+} + h^- \mathbf{1}_{\Omega^-}, & \text{in } \Omega, \\ \dfrac{\partial v_1}{\partial n} = 0, & \text{on } \partial\Omega, \\ c'(0)v_1 = \kappa, & \text{on } \Gamma. \end{cases} \qquad \text{(RP)}$$

It is known that

$$c'(0) = -\frac{h^+ - h^-}{\sigma} \qquad (1.4)$$

with

$$\sigma = \int_{h^-}^{h^+} \sqrt{2W(s)}\, ds.$$

The reduced functional becomes

$$I_0[\Gamma] = \sigma|\Gamma| + \frac{1}{2}\int_\Omega D|\nabla v_1|^2 \, dx,$$

where v_1 solves

$$\begin{cases} -D\Delta v_1 = h^+ 1_{\Omega^+} + h^- 1_{\Omega^-}, & \text{in } \Omega, \\ \dfrac{\partial v_1}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

See Lemma 4.1 in Section 4.

For the reduction from the paraboloc system to the phase field model, see [16]. The relation between the functional I and the reduced functional I_0 may be justified mathematically by the notion of the Gamma convergence. See [13]. The radially symmetric case for the related problems is studied in [7, 8, 11, 14, 15, 17].

The direct method of calculus of variations implies the existence of global minimizers of I_0. This gives the solution of (RP). However it is usually difficult to know the profile of the global minimizers. Here we consider the problem to find a solution of (RP) which does not necessarily correspond to the global minimizers.

In order to state the result, we define the Green's function and its harmonic part.

Definition 1.1 *For each $y \in \Omega$, let $G(x,y)$ be the solution to*

$$\begin{cases} -\Delta_x G(x,y) = \delta(x-y) - \frac{1}{|\Omega|}, & x \in \Omega, \\ \dfrac{\partial G}{\partial n_x}(x,y) = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega G(x,y)\, dx = 0. \end{cases}$$

Set

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + \frac{|x-y|^2}{4|\Omega|} + H(x,y), \quad x,y \in \Omega.$$

Then it is known that $H(x,y)$ is symmetric and harmonic in both x and y. Let $\mathcal{H}(x) = H(x,x)$.

We define the following two conditions.

(A1) $0 \in \Omega$ is a strict local minimum point of $\mathcal{H}$. More precisely, there exists a neighborhood U of 0 in Ω such that $\mathcal{H}(0) < \mathcal{H}(x)$ for all $x \in U\backslash\{0\}$.

(A2) $0 \in \Omega$ is a non-degenerate critical point of $\mathcal{H}$.

Remark. When Ω is a disk, the center of Ω is a unique minimum point of $\mathcal{H}$ and both **(A1)** and **(A2)** are satisfied. The regular part of Green's function subject to the homogeneous Dirichlet boundary condition has a unique non-degenerate minimum point when $\Omega \subset \mathbb{R}^2$ is convex (see [2]). The regular part of Green's function subject to the homogeneous Neumann boundary condition is considered in [9].

We denote by d_{H} Hausdorff metric

$$d_{\mathrm{H}}(K_1, K_2) = \max[\sup\{\mathrm{dist}(x, K_2)\,;\, x \in K_1\}, \sup\{\mathrm{dist}(y, K_1)\,;\, y \in K_2\}],$$

$$S_r(0) = \{x \in \mathbb{R}\,;\, |x| = r\}, \text{ and } B_r(0) = \{x \in \mathbb{R}\,;\, |x| < r\}.$$

Theorem 1.1 *Assume that (A1) or (A2). If*

$$r_0 := \sqrt{\frac{|h^-|\,|\Omega|}{\pi(h^+ - h^-)}} < \mathrm{dist}(0, \partial\Omega),$$

then there exists a constant $D_0 > 0$ such that (RP) has a solution

$$\begin{cases} \Gamma = \Gamma_D, \\ v_1 = v_D, \\ \Omega^\pm = \Omega_D^\pm, \end{cases}$$

for all $D > D_0$ satisfying $d_{\mathrm{H}}(\Gamma_D, S_{r_0}(0)) \to 0$ as $D \to \infty$.

2 Normalization

Let $\Omega, \Omega^+, \Gamma, D, h^\pm, \sigma, v, \kappa$ be as in Section 1 and r_0 be as in the statement of Theorem 1.1. We normalize the problem in what follows. Define the rescaled domains

$$\tilde{\Omega}^+ = \{x \in \mathbb{R}^2\,;\, r_0 x \in \Omega^+\},$$

$$\tilde{\Omega} = \{x \in \mathbb{R}^2 \, ; \, r_0 x \in \Omega\}.$$

Set

$$\tilde{v}(x) = \frac{D}{(h^+ - h^-)r_0^2} v(r_0 x)$$

for $x \in \tilde{\Omega}$. The rescaled sharp interface is

$$\tilde{\Gamma} = \{x \in \mathbb{R}^2 \, ; \, r_0 x \in \Gamma\}.$$

The curvature of $\tilde{\Gamma}$ is

$$\tilde{\kappa} = r_0 \kappa.$$

Define new constants

$$m = \frac{|h^-|}{h^+ - h^-} \in (0, 1)$$

and

$$\beta = \frac{(h^+ - h^-)^2 r_0^3}{D\sigma}.$$

Noting (1.4), the reduced equation (RP) then becomes

$$\begin{cases} -\Delta \tilde{v} = 1_{\tilde{\Omega}^+} - m, & \text{in } \tilde{\Omega}, \\[2mm] \dfrac{\partial \tilde{v}}{\partial n} = 0, & \text{on } \partial\tilde{\Omega}, \\[2mm] \beta \tilde{v} + \tilde{\kappa} = 0, & \text{on } \tilde{\Gamma}. \end{cases} \qquad (2.1)$$

The necessary condition for (2.1) to have a solution is that the average of $1_{\tilde{\Omega}^+} - m$ over $\tilde{\Omega}$ vanishes, i.e., $|\tilde{\Omega}^+| = m|\tilde{\Omega}|(= \frac{m|\Omega|}{r_0^2} = \pi)$.

Define the Green's function $\tilde{G}$ for $\tilde{\Omega}$ as

$$\tilde{G}(x, y) = G(r_0 x, r_0 y), \quad x, y \in \tilde{\Omega},$$

the harmonic part $\tilde{H}$ of $\tilde{G}$ as

$$\tilde{G}(x, y) = -\frac{1}{2\pi} \log |x - y| + \frac{|x - y|^2}{4|\tilde{\Omega}|} + \tilde{H}(x, y), \quad x, y \in \tilde{\Omega},$$

and the diagonal component of $\tilde{H}$ as

$$\tilde{\mathcal{H}}(x) = \tilde{H}(x, x), \quad x \in \tilde{\Omega}.$$

Then $\tilde{G}$ satisfies

$$\begin{cases} -\Delta_x \tilde{G}(x,y) = \delta(x-y) - \frac{1}{|\tilde{\Omega}|}, & x \in \tilde{\Omega}, \\ \dfrac{\partial \tilde{G}}{\partial n_x}(x,y) = 0, & x \in \partial\tilde{\Omega}, \\ \displaystyle\int_{\tilde{\Omega}} \tilde{G}(x,y)\,dx = 0. \end{cases}$$

Since

$$\tilde{H}(x,y) = H(r_0 x, r_0 y) - \frac{1}{2\pi}\log r_0,$$

the properties **(A1)** and **(A2)** are invariant under the above rescaling. Hence in what follows, we assume that $r_0 = 1$. Then Theorem 1.1 follows from Theorem 3.1 in Section 3.

3 Existence of Solution

We consider

$$\begin{cases} -\Delta v = 1_{\Omega^+} - m, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \\ \beta v + \kappa = 0, & \text{on } \Gamma, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth boundary $\partial\Omega$; Ω^+ is an open set in Ω; $\Gamma = \partial\Omega^+ \subset \Omega$ is a C^2-curve embedded in Ω; κ is the curvature of Γ; $\beta > 0$ is a parameter; 1_{Ω^+} denotes the characteristic function of Ω^+; and $|\Omega^+| = \pi = m|\Omega|$. The last condition is equivalent to $r_0 = 1$. In this section we assume that $1 < \text{dist}(0, \partial\Omega)$.

We identify 2π-periodic functions on $\mathbb{R}$ with the functions on $S^1 = \{x \in \mathbb{R}^2 \,;\, |x| = 1\} \cong \mathbb{R}/2\pi\mathbb{Z}$. For $q \in C^2(S^1)$, we use the following notations:

$$\dot{q}(\omega) = \frac{dq}{d\omega}(\omega) = \frac{d}{d\theta}q(\cos\theta, \sin\theta), \quad \omega = (\cos\theta, \sin\theta) \in S^1$$

and

$$\ddot{q}(\omega) = \frac{d^2 q}{d\omega^2}(\omega) = \frac{d^2}{d\theta^2}q(\cos\theta, \sin\theta), \quad \omega = (\cos\theta, \sin\theta) \in S^1.$$

Let $0 < \alpha < 1$. We write

$$[q]_Y = \sup_{\substack{\omega,\hat{\omega} \in S^1 \\ \omega \neq \hat{\omega}}} \frac{|q(\omega) - q(\hat{\omega})|}{|\omega - \hat{\omega}|^\alpha}$$

$X = C^{2,\alpha}(S^1)$ consists of all functions $q \in C^2(S^1)$ for which the norm

$$\|q\|_X = \max_{\omega \in S^1} |q(\omega)| + \max_{\omega \in S^1} |\dot{q}(\omega)| + \max_{\omega \in S^1} |\ddot{q}(\omega)| + [\ddot{q}]_Y$$

is finite. $Y = C^\alpha(S^1)$ consists of all functions $q \in C(S^1)$ for which the norm

$$\|q\|_Y = \max_{\omega \in S^1} |q(\omega)| + [q]_Y$$

is finite.

For $q_1, q_2 \in L^2(S^1)$, denote

$$\langle q_1, q_2 \rangle = \int_{S^1} q_1(\omega) q_2(\omega) \, d\omega = \int_0^{2\pi} q_1(\cos\theta, \sin\theta) q_2(\cos\theta, \sin\theta) \, d\theta.$$

Define $\Phi_0(\omega) = 1/\sqrt{2\pi}$, $\Phi_1(\omega) = \omega_1/\sqrt{\pi}$, and $\Phi_2(\omega) = \omega_2/\sqrt{\pi}$ for $\omega = (\omega_1, \omega_2) \in S^1$. Let $\Pi_0, \Pi_1 : L^2(S^1) \to L^2(S^1)$ denote the projections with respect to $\langle \cdot, \cdot \rangle$ onto $\mathrm{span}\{\Phi_0\}$ and $\mathrm{span}\{\Phi_i\}_{i=0,1,2}$ respectively. Let $\Pi_0^\perp = \mathbf{Id} - \Pi_0$, $\Pi_1^\perp = \mathbf{Id} - \Pi_1$. Then $\Pi_0^\perp, \Pi_1^\perp$ are the projections onto the orthogonal complements of $\mathrm{span}\{\Phi_0\}$ and $\mathrm{span}\{\Phi_i\}_{i=0,1,2}$ respectively.

For $r > 0$, define

$$X_r = \{q \in X \,;\, \|q\|_X \leq r, \langle q, 1 \rangle = 0\}.$$

We can choose a constant $\delta \in (0, 1/2)$ such that $B_{1+\delta}(0) \subset \Omega$ by our assumption. For $q \in X_{\delta/2}$, define

$$\Gamma(q) = \{\sqrt{1 + q(\omega)}\omega \,;\, \omega \in S^1\},$$

$$\Omega^+(q) = \{r\omega \,;\, 0 \leq r \leq \sqrt{1 + q(\omega)}, \omega \in S^1\}.$$

Let $q \in X_{\delta/2}$. Then $\Gamma(q) \subset \Omega$ and $|\Omega^+(q)| = \pi$. Indeed since $\sqrt{1+q} \leq 1 + \frac{1}{2}q \leq 1 + \frac{\delta}{4}$, we have $\Gamma(q) \subset B_{1+\delta/2}(0) \subset \Omega$. In addition, since $\langle q, 1 \rangle = 0$, we have

$$|\Omega^+(q)| = \int_{S^1} \int_0^{\sqrt{1+q(\omega)}} r \, dr d\omega = \int_{S^1} \frac{1 + q(\omega)}{2} \, d\omega = \pi.$$

Let M_β be the map from $X_{\delta/2}$ to Y defined by

$$M_\beta(q)(\omega) := K(q)(\omega) + \beta \int_{\Omega^+(q)} G(\sqrt{1+q(\omega)}\omega, y)\, dy, \qquad \omega \in S^1$$

for $q \in X_{\delta/2}$, where

$$K(q) = \frac{1 + q + \frac{3\dot{q}^2}{4(1+q)} - \frac{1}{2}\ddot{q}}{\left[1 + q + \frac{\dot{q}^2}{4(1+q)}\right]^{3/2}}$$

is the curvature of $\Gamma(q)$. Indeed, set $x_1(\theta) = r(\theta)\cos\theta$, $x_2(\theta) = r(\theta)\sin\theta$ with

$$r(\theta) = \sqrt{1 + q(\cos\theta, \sin\theta)}.$$

Then the curvature of $\Gamma(q)$ can be computed as follows.

$$\frac{\dot{x}_1 \ddot{x}_2 - \ddot{x}_2 \dot{x}_1}{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}} = \frac{r^2 + 2\dot{r}^2 - r\ddot{r}}{(r^2 + \dot{r}^2)^{3/2}} = \frac{1 + q + \frac{3\dot{q}^2}{4(1+q)} - \frac{1}{2}\ddot{q}}{\left[1 + q + \frac{\dot{q}^2}{4(1+q)}\right]^{3/2}}.$$

In order to solve (3.1), we need only find a function $q \in X_{\delta/2}$ such that $\Pi_0^\perp M_\beta(q) = 0$. Indeed, if $q \in X_{\delta/2}$ is a solution of $\Pi_0^\perp M_\beta(q) = 0$, then there exists a constant C such that

$$M_\beta(q) \equiv C.$$

Now set

$$v(x) = \int_{\Omega^+(q)} G(x, y)\, dy - \frac{1}{\beta}C, \quad x \in \Omega.$$

Then v satisfies

$$\begin{cases} -\Delta v = 1_{\Omega^+(q)} - m, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. From $|\Omega^+(q)| = \pi$ and $-\Delta_x G(x, y) = \delta(x - y) - \frac{1}{|\Omega|}$, we have $-\Delta v = 1_{\Omega^+} - \frac{|\Omega^+(q)|}{|\Omega|} = 1_{\Omega^+} - m$. Moreover $\partial v/\partial n = 0$ on $\partial\Omega$ follows from $\partial G/\partial n_x(x, y) = 0$ for $x \in \partial\Omega, y \in \Omega$. $\square$

Hence we see that

$$\Gamma = \Gamma(q), \quad v(x) = \int_{\Omega^+(q)} G(x, y)\, dy - \frac{1}{\beta}C, \quad \Omega^+ = \Omega^+(q)$$

solves our equation (3.1).

Our main result is the following:

Theorem 3.1 *Suppose either (A1) or (A2). Then there exists a constant $\beta_0 > 0$ such that $\Pi_0^\perp M_\beta(q) = 0$ has a solution $q = q_\beta \in X_{\delta/2}$ for all $\beta \in (0, \beta_0)$ satisfying $q_\beta \to 0$ in X as $\beta \to 0$.*

The proof consists of two steps:
(i) $\Pi_1^\perp M_\beta(q) = 0$ and (ii) $(\Pi_1 - \Pi_0)M_\beta(q) = 0$.

4 Linearized non-degeneracy

We linearize M_β. For $t > -1$, $p \in \mathbb{R}$, $s \in \mathbb{R}$, set

$$L(t, p, s) = \frac{1 + t + \frac{3p^2}{4(1+t)} - \frac{1}{2}s}{\left[1 + t + \frac{p^2}{4(1+t)}\right]^{3/2}}.$$

Then K is C^1 on $X_{\delta/2}$ and there holds

$$K'(q)\zeta = L_s(q, \dot{q}, \ddot{q})\ddot{\zeta} + L_p(q, \dot{q}, \ddot{q})\dot{\zeta} + L_t(q, \dot{q}, \ddot{q})\zeta \quad \text{for } \zeta \in \Pi_0^\perp X.$$

Moreover since

$$L_s(q, \dot{q}, \ddot{q}) = -\frac{1}{2}\left[1 + q + \frac{\dot{q}^2}{4(1+q)}\right]^{-3/2},$$

$$L_p(q, \dot{q}, \ddot{q}) = \frac{3\dot{q}}{16}\left\{4 - \frac{\dot{q}^2}{(1+q)^2} + \frac{2\ddot{q}}{1+q}\right\}\left[1 + q + \frac{\dot{q}^2}{4(1+q)}\right]^{-5/2},$$

we have $\frac{d}{d\omega}L_s(q, \dot{q}, \ddot{q}) = L_p(q, \dot{q}, \ddot{q})$. Hence it follows that for $\zeta \in \Pi_0^\perp X$

$$K'(q)\zeta = L_s(q, \dot{q}, \ddot{q})\ddot{\zeta} + L_p(q, \dot{q}, \ddot{q})\dot{\zeta} + L_t(q, \dot{q}, \ddot{q})\zeta$$

$$= \frac{d}{d\omega}[L_s(q, \dot{q}, \ddot{q})\dot{\zeta}] + L_t(q, \dot{q}, \ddot{q})\zeta.$$

Since

$$M_\beta(q)(\omega) = K(q)(\omega) + \beta \int_{S^1} \int_{-1}^{q(\hat{\omega})} G(\sqrt{1 + q(\omega)}\omega, \sqrt{1 + \hat{q}}\hat{\omega}) \frac{d\hat{q}}{2} d\hat{\omega}$$

for $\omega \in S^1$ and $q \in X_{\delta/2}$, we see that M_β is also C^1 and

$$[M'_\beta(q)\zeta](\omega) = [K'(q)\zeta](\omega) + \frac{\beta}{2} \int_{S^1} G(\sqrt{1+q(\omega)}\omega, \sqrt{1+q(\hat{\omega})}\hat{\omega})\zeta(\hat{\omega})\, d\hat{\omega}$$

$$+ \beta\zeta(\omega) \int_{S^1} \int_{-1}^{q(\hat{\omega})} \frac{\nabla_x G(\sqrt{1+q(\omega)}\omega, \sqrt{1+\hat{q}}\hat{\omega}) \cdot \omega}{2\sqrt{1+q(\omega)}} \frac{d\hat{q}}{2}\, d\hat{\omega}$$

$$= \frac{d}{d\omega}[L_s(q,\dot{q},\ddot{q})\zeta] + L_t(q,\dot{q},\ddot{q})\zeta$$

$$+ \frac{\beta}{2} \int_{S^1} G(\sqrt{1+q(\omega)}\omega, \sqrt{1+q(\hat{\omega})}\hat{\omega})\zeta(\hat{\omega})\, d\hat{\omega}$$

$$+ \frac{\beta\zeta(\omega)}{2\sqrt{1+q(\omega)}} \int_{\Omega^+(q)} \omega \cdot \nabla_x G(\sqrt{1+q(\omega)}\omega, y)\, dy$$

for $\omega \in S^1$, $q \in X_{\delta/2}$, and $\zeta \in \Pi_0^\perp X$.

For small ε, the singular perturbation problem (1.3) has a solution $(u_\varepsilon, v_\varepsilon)$ which have an internal transition layer near Γ provided that q is a solution of $\Pi_0^\perp M_\beta(q) = 0$ and $M'_\beta(q)$ is non-degenerate. See [11]. We can show that this non-degeneracy condition holds under the condition (**A2**).

However in the case of FitzHugh–Nagumo type, we can apply the Gamma convergence theory in order to obtain the layered solution. First we define the energy functional.

Definition 4.1 *For $q \in X_{\delta/2}$, define*

$$E_\beta[q] := \frac{1}{\beta}|\Gamma(q)| + \frac{1}{2} \int_\Omega |\nabla v|^2\, dx$$

where

$$v(x) = \int_{\Omega^+(q)} G(x,y)\, dy, \quad x \in \Omega.$$

Note that

$$\int_\Omega |\nabla v|^2\, dx = -\int_\Omega v\Delta v\, dx = \int_\Omega v(1_{\Omega^+(q)} - m)\, dx$$

$$= \int_{\Omega^+(q)} \int_{\Omega^+(q)} G(x,y)\, dxdy.$$

Lemma 4.1 *Let $T : \mathbb{I} \to X_{\delta/2}$ be a C^1-map from an open interval $\mathbb{I} \subset \mathbb{R}$ to $X_{\delta/2}$. Then*

$$\frac{d}{dt}E_\beta[T(t)] = \frac{1}{2\beta}\langle M_\beta(q), T'(t)\rangle.$$

This implies that the solution q_β is a critical point of E_β in $X_{\delta/2}$. When

(A3) $0 \in \Omega$ is a non-degenerate local minimum point of $\mathcal{H}$, i.e., a critical point of $\mathcal{H}$ at which the Hessian matrix of $\mathcal{H}$ is positive definite.

is satisfied, we can see that q_β is an isolated local minimizer of E_β in $X_{\delta/2}$. In this case we can show the existence of layered solution of (1.3) using the idea in [5]. In the case of FitzHugh–Nagumo type, we can also establish the existence of the layered solution using the spectrum estimate for the Allen–Cahn operator for generic interfaces obtained in [1, 3].

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