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Instability of vortex solitons for 2D focusing NLS

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1 Introduction

In the present article, we consider instability of radially symmetric vortex solitons to 2-dimensional nonlinear Schrödinger equations

$$\begin{cases} iu_t + \Delta u + f(u) = 0 & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $n = 2$ and $f(u) = |u|^{p-1}u$. Let $\omega > 0$, $m \in \mathbb{N} \cup \{0\}$, and let $e^{i(\omega t + m\theta)}\phi_{\omega,m}(r)$ be a standing wave solution of (1) belonging to $H^1(\mathbb{R}^2)$. Here r and θ denote polar coordinates in $\mathbb{R}^2$. Then $\phi_{\omega,m}(r)$ is a solution to

$$\begin{cases} \phi'' + \dfrac{1}{r}\phi' - \left(\omega + \dfrac{m^2}{r^2}\right)\phi + f(\phi) = 0 & \text{for } r > 0, \\ \lim\limits_{r\to 0} \dfrac{\phi(r)}{r^m} = \lim\limits_{r\to 0} \dfrac{\phi'(r)}{mr^{m-1}}, \\ \lim\limits_{r\to\infty} \phi(r) = 0. \end{cases} \tag{2}$$

We remark that $e^{im\theta}\phi_{\omega,m}(r)$ is a solution to the scalar field equation

$$\Delta\varphi - \omega\varphi + f(\varphi) = 0 \quad \text{for } x \in \mathbb{R}^2. \tag{3}$$

A standing wave solution of the form $e^{i(\omega t+m\theta)}\phi_{\omega,m}(r)$ appears in the study of nonlinear optics (see references in [13]). If $m = 0$ and $\phi_{\omega,m}(r)$ is positive, then $\phi_{\omega,m}$ is a ground state. Existence and uniqueness of the ground state are well known (see [4], [5], [12] and reference therein).

If $m \neq 0$, Iaia and Warchall proved the existence of smooth solutions to (2) with any prescribed number of zeroes. The uniqueness of positive solutions can been proved by using the classification theorem of positive solutions due to Yanagida and Yotsutani [24].

Theorem 1 ([14]). *Let m be an integer and $1 < p < \infty$. Then there exists a unique positive radially symmetric solution $\phi_{\omega,m}$ to* (2) *that belongs to $H^1(\mathbb{R}^2)$.*

Let $c > 0$ and let Q_c be a positive solution to

$$\begin{cases} Q'' - cQ + f(Q) = 0 & \text{for } x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} Q(x) = 0, \\ Q(0) = \max_{x\in\mathbb{R}} Q(x). \end{cases} \tag{4}$$

Then

$$Q_c(x) = \left(\frac{(p+1)c}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\frac{(p-1)\sqrt{c}}{2}x\right). \tag{5}$$

In [13], Pego and Warchall numerically observe that as spin index m becomes larger, a solution $\phi_{\omega,m}(r)$ to (2) remains small initially and then is approximated by $Q_c(r-\bar{r})$ around $r = \bar{r}$, where $c = \omega + (m^2/\bar{r}^2)$ and $\bar{r}$ is a positive number with $\bar{r} = O(m)$ as $m \to \infty$ (see also [16] and references in [13]). One of our goals in the present paper is to explain this phenomena. Benci and D'Aprile [2] studied (2) in a slightly general setting and locate the asymptotic peak of solutions (see also [7]). Recently, Ambrosetti, Malchiodi and Ni [1] have proved the existence of positive radial solutions concentrating on spheres to a class of singularly perturbed problem

$$\varepsilon^2\Delta u - Vu + |u|^{p-1}u = 0,$$

and obtain their asymptotic profile. Adopting the argument in [1], we obtain the following.

Theorem 2. *Let $p > 1$ and let $\phi_{\omega,m}$ be a positive solution to* (2). *Then there exists an $m_* \in \mathbb{N}$ such that if $m \geq m_*$,*

$$\|\phi_{\omega,m}(\cdot) - Q_c(\cdot - \bar{r})\|_{H^2_r(\mathbb{R}^2)} = O(m^{-1/2}), \tag{6}$$

$$\|\phi_{\omega,m}(\cdot) - Q_c(\cdot - \bar{r})\|_{L^\infty(\mathbb{R}^2)} = O(m^{-1}), \tag{7}$$

where $\bar{r} = 2m/\sqrt{(p-1)\omega}$ and $c = (p+3)\omega/4$.

Remark 1. *Let $r = ms$, $\varepsilon = 1/m$ and $V(r) = \omega + r^{-2}$. Then* (2) *is transformed into*

$$\varepsilon^2\Delta_r\phi - V(r)\phi + f(\phi) = 0.$$

Though [1] assumes the boundedness of $V(r)$ and cannot be applied directly to our problem, a maximum point of $\phi_{\omega,m}(r)$ can be predicted from an auxiliary weighted potential $rV(r)$ introduced by [1].

Let φ_ω be a ground state to (3). As is well known, the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable if $d\|\varphi_\omega\|^2_{L^2(\mathbb{R}^n)}/d\omega > 0$ and unstable if $d\|\varphi_\omega\|^2_{L^2(\mathbb{R}^n)}/d\omega < 0$. See e.g. Berestycki-Cazenave [3], Cazenave-Lions [6], Grillakis-Shatah-Strauss [9], Shatah [18], Shatah-Strauss [19] and Weinstein [23]. Namely, the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable if $1 < p < 1 + 4/n$ and unstable if $p \geq 1 + 4/n$. Grillakis [8] proved that every radially symmetric standing wave solution is linearly unstable if $p \geq 1 + 4/n$. However, to the best our knowledge, it remains unknown whether there exists an unstable standing wave solution with higher energy in the subcritical case $(1 < p < 1 + 4/n)$.

From Theorem 1, we can deduce nondegeneracy of a bound state $e^{im\theta}\phi_{\omega,m}(r)$. Let $N[u] := \int_{\mathbb{R}^2} |u(x)|^2 dx$. Since $\phi_{\omega,m}$ is a least energy solution to (2) in the class

$$X_m = \left\{u \in H^1(\mathbb{R}^2) \mid u = f(r)e^{im\theta}\right\},$$

it follows from Grillakis-Shatah-Strauss [9] that a bound state $e^{i(\omega t+m\theta)}\phi_{\omega,m}(r)$ is stable to the perturbation of the form $e^{im\theta}v(r)$ if $dN[\varphi_{\omega,m}]/d\omega > 0$ and unstable if $dN[\varphi_\omega]/d\omega < 0$. More precisely, we have the following.

Theorem 3. *Let $m \in \mathbb{N} \cup \{0\}$ and $\phi_{\omega,m}$ be a positive radially symmetric solution of* (2) *that belongs to $H^1(\mathbb{R}^2)$.*

(i) *Let $p \geq 3$. Then the standing wave solution $e^{i(\omega t+m\theta)}\phi_{\omega,m}$ of* (1) *is unstable.*

(ii) *Let $1 < p < 3$ and $u_0 \in X_m$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\inf_{\gamma\in\mathbb{R}} \|u_0 - e^{i(m\theta+\gamma)}\phi_{\omega,m}\|_{H^1(\mathbb{R}^2)} < \delta$, the solution of* (1) *satisfies*

$$\sup_{t\geq 0} \inf_{\gamma\in\mathbb{R}} \|u(\cdot,t) - e^{i(m\theta+\gamma)}\phi_{\omega,m}\|_{H^1(\mathbb{R}^2)} < \varepsilon.$$

Theorem 3 implies vortex solitons are stable to symmetric perturbations in the subcritical case $(1 < p < 3)$. It is expected that vortex solitons are unstable even in the subcritical case. Using the limiting profile of vortex solitons as $m \to \infty$, we prove that a standing wave solution $e^{i(\omega t+m\theta)}\phi_{\omega,m}(r)$ is unstable to perturbations in $H^1(\mathbb{R}^2)$ for large m.

Theorem 4. *Let $p > 1$ and $\phi_{\omega,m}$ be as in Theorem 2. Then there exists an $m_* \in \mathbb{N}$ such that if $m \geq m_*$, a standing wave solution $e^{i(\omega t + m\theta)}\phi_{\omega,m}$ is unstable.*

2 Proof of Theorem 4

In this section, we will prove Theorem 4. Let $u(x,t) = e^{i\omega t}(e^{im\theta}\phi_\omega(r) + e^{\lambda t}v)$ and linearize (1) around $v = 0$ and $t = 0$. Then

$$i\lambda v + (\Delta - \omega + \beta_1(r))v + e^{2im\theta}\beta_2(r)\bar{v} = 0, \tag{8}$$

where

$$\beta_1(r) = \frac{p+1}{2}\phi_\omega(r)^{p-1}, \quad \beta_2(r) = \frac{p-1}{2}\phi_\omega(r)^{p-1}.$$

Put $v = e^{i(j+m)\theta}y_+$, $\bar{v} = e^{i(j-m)\theta}y_-$ and complexify (8) into a system

$$\begin{cases} \left(\Delta_r - \omega - \dfrac{(m+j)^2}{r^2} + i\lambda + \beta_1(r)\right) y_+ + \beta_2(r)y_- = 0, \\ \left(\Delta_r - \omega - \dfrac{(m-j)^2}{r^2} - i\lambda + \beta_1(r)\right) y_- + \beta_2(r)y_+ = 0. \end{cases} \tag{9}$$

If λ is an eigenvalue of the linearized operator, there exist a $j \in \mathbb{Z}$ and a solution (y_+, y_-) to (9) that satisfy $(e^{i(j+m)\theta}y_+(r), e^{i(j-m)\theta}y_-(r)) \in H^1(\mathbb{R}^2, \mathbb{C}^2)$. We will show the existence of unstable eigenvalues for j with $1 \ll j \ll m$.

Let $w_1 = y_+ + y_-$, $w_2 = y_+ - y_-$, $\varepsilon = m^{-1}$ and $\delta = j\varepsilon$. Let $s = r - \alpha_0 m$. Then (9) can be rewritten as

$$\mathcal{H}(\varepsilon, \delta)\mathbf{w} = \lambda \mathbf{w}, \tag{10}$$

where $\mathbf{w} = {}^t(w_1, w_2)$,

$$\mathcal{H}(\varepsilon, \delta) = i \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \tag{11}$$

and

$$\begin{aligned} h_{11} = h_{22} &= \frac{-2mj}{r^2}, \\ h_{12} &= \Delta_r - \omega - \frac{m^2 + j^2}{r^2} + \phi_\omega^{p-1} \\ h_{21} &= \Delta_r - \omega - \frac{m^2 + j^2}{r^2} + p\phi_\omega^{p-1}. \end{aligned}$$

We remark that

$$\tau_{-\bar{r}}h_{11} = \tau_{-\bar{r}}h_{22} = \frac{-2\delta}{(\alpha_0+\varepsilon r)^2}$$
$$\tau_{-\bar{r}}h_{12} = \partial_r^2 + \frac{\varepsilon}{\alpha_0+\varepsilon r}\partial_r - \omega - \frac{1+\delta^2}{(\alpha_0+\varepsilon r)^2} + \phi_\omega^{p-1}$$
$$\tau_{-\bar{r}}h_{21} = \partial_r^2 + \frac{\varepsilon}{\alpha_0+\varepsilon r}\partial_r - \omega - \frac{1+\delta^2}{(\alpha_0+\varepsilon r)^2} + p\phi_\omega^{p-1}.$$

Before we investigate the spectrum of $\mathcal{H}(\varepsilon,\delta)$, let us consider the spectrum of a linear operator

$$H(\delta) := i\begin{pmatrix} -2\alpha_0^{-2}\delta & L_- - \alpha_0^{-2}\delta^2 \\ L_+ - \alpha_0^{-2}\delta^2 & -2\alpha_0^{-2}\delta, \end{pmatrix}$$

where $L_+ = \partial_s^2 - c + pQ_c^{p-1}$, $L_- = \partial_s^2 - c + Q_c^{p-1}$, $D(L_+) = D(L_-) = H^2(\mathbb{R})$ and $c = \omega + \alpha_0^{-2}$.

To begin with, we recall some spectral properties of $H(0)$. Let

$$\Phi_1 = \begin{pmatrix} 0 \\ Q_c \end{pmatrix}, \quad \Phi_2 = -i\begin{pmatrix} \partial_c Q_c \\ 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} Q_c' \\ 0 \end{pmatrix}, \quad \Phi_4 = -\frac{i}{2}\begin{pmatrix} 0 \\ sQ_c \end{pmatrix},$$

and

$$\Phi_1^* = \theta_1\sigma_2\Phi_2, \quad \Phi_2^* = \theta_1\sigma_2\Phi_1, \quad \Phi_3^* = \theta_2\sigma_2\Phi_4, \quad \Phi_4^* = \theta_2\sigma_2\Phi_3,$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_1 = 2\left(\frac{d}{dc}\|Q_c\|_{L^2(\mathbb{R})}^2\right)^{-1}, \quad \theta_2 = 4\|Q_c\|_{L^2(\mathbb{R})}^{-2}.$$

Then we have

$$H(0)\Phi_1 = 0, \quad H(0)\Phi_2 = \Phi_1, \quad H(0)\Phi_3 = 0, \quad H(0)\Phi_4 = \Phi_3, \tag{12}$$
$$H(0)^*\Phi_1^* = \Phi_2^*, \quad H(0)^*\Phi_2^* = 0, \quad H(0)^*\Phi_3^* = \Phi_4^*, \quad H(0)^*\Phi_4^* = 0, \tag{13}$$

and $\langle \Phi_i, \Phi_j^*\rangle = \delta_{ij}$ for i, $j = 1,2,3,4$. Here we denote by $\langle\cdot,\cdot\rangle$ the inner product of $L^2(\mathbb{R},\mathbb{C}^2)$.

Proposition 5 (see [22]). *Let $p > 1$ and $p \neq 5$. Then $\lambda = 0$ is a discrete eigenvalue of $H(0)$ with algebraic multiplicity 4.*

Using Proposition 5, we investigate the spectrum of $H(\delta)$.

Lemma 6. *Let $1 < p < 5$. Then there exist a positive number δ_0 and a neighborhood $U \subset \mathbb{C}$ of 0 such that for every $\delta \in (0, \delta_0)$, $\sigma(H(\delta)) \cap U$ consists of algebraically simple eigenvalues $\lambda_i(\delta)$ $(i = 1, 2, 3, 4)$ satisfying*

$$\left|\mathrm{Re}\lambda_1(\delta) - \alpha_0^{-1}\gamma\delta\right| \le \alpha_0^{-1}\gamma\delta/4, \quad \liminf_{\delta\downarrow 0}\left(\delta^{-1}\min_{\substack{1\le i,j\le 4,\\ i\ne j}}|\lambda_i(\delta) - \lambda_j(\delta)|\right) > 0,$$

where

$$\gamma = \left(2\frac{\|Q_c\|^2_{L^2(\mathbb{R})}}{\frac{d}{dc}\|Q_c\|^2_{L^2(\mathbb{R})}}\right)^{1/2}.$$

Proof. Let $P_H(\delta)$ be a projection defined by

$$P_H(\delta) = \frac{1}{2\pi i}\oint_{|\lambda|=\rho_0}(\lambda - H(\delta))^{-1}\,d\lambda,$$

and let $Q_H(\delta) = I - P_H(\delta)$. In view of Proposition 5, there exist positive numbers ρ_0 and δ_0 such that $\mathcal{X}_0 := R(P_H(\delta))$ is 4-dimensional for every $\delta \in (0, \delta_0)$.

Let $\mathcal{X}_0$ be a linear subspace whose basis is $\langle\Phi_1, \Phi_2, \Phi_3, \Phi_4\rangle$. We decompose $H^2(\mathbb{R}; \mathbb{C}^2)$ and $L^2(\mathbb{R}; \mathbb{C}^2)$ as

$$H^2(\mathbb{R};\mathbb{C}^2) = \mathcal{X}_0 \oplus Q_H(0)H^2(\mathbb{R};\mathbb{C}^2), \quad L^2(\mathbb{R};\mathbb{C}^2) = \mathcal{X}_0 \oplus Q_H(0)L^2(\mathbb{R};\mathbb{C}^2).$$

Then

$$H(\delta) = \begin{pmatrix} H_{11}(\delta) & H_{12}(\delta) \\ H_{21}(\delta) & H_{22}(\delta)\end{pmatrix},$$

where

$$H_{11}(\delta) = P_H(0)H(\delta)P_H(0), \quad H_{12}(\delta) = P_H(0)H(\delta)Q_H(0)$$
$$H_{21}(\delta) = Q_H(0)H(\delta)P_H(0), \quad H_{22}(\delta) = Q_H(0)H(\delta)Q_H(0).$$

By a simple computation, we have

$$H_{11}(\delta) = -2i\alpha_0^{-2}\delta I + \begin{pmatrix} 0 & 1+b_2\delta^2 & 0 & 0 \\ b_1\delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+b_4\delta^2 \\ 0 & 0 & b_3\delta^2 & 0 \end{pmatrix},$$

$$H_{12}(\delta) = -i\alpha_0^{-2}\delta^2 P_H(0)\sigma_1 Q_H(0), \quad H_{21}(\delta) = -i\alpha_0^{-2}\delta^2 Q_H(0)\sigma_1 P_H(0),$$

where

$$b_1 = \alpha_0^{-2}\theta_1\|Q_c\|^2_{L^2(\mathbf{R})}, \quad b_2 = -\alpha_0^{-2}\theta_1\|\partial_c Q_c\|^2_{L^2(\mathbf{R})},$$
$$b_3 = -4\alpha_0^{-4}, \quad b_4 = \alpha_0^{-2}\|sQ_c\|^2_{L^2(\mathbf{R})}\|Q_c\|^{-2}_{L^2(\mathbf{R})}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

First, we investigate the spectrum of $H_{11}(\delta)$. Suppose λ is an eigenvalue of the matrix $H_{11}(\delta)$. Then

$$\begin{aligned}&\det(\lambda I - H_{11}(\delta))\\ =&\left\{(\lambda + 2i\alpha_0^{-2}\delta)^2 - b_1\delta^2 - b_1b_2\delta^4\right\}\left\{(\lambda + 2i\alpha_0^{-2}\delta)^2 - b_3\delta^2 - b_3b_4\delta^4\right\} = 0.\end{aligned}$$

Hence there exist eigenvalues $\hat\lambda_i$ $(i = 1,2,3,4)$ of $H_{11}(\delta)$ satisfying

$$\hat\lambda_1 = -\delta\left(2i\alpha_0^{-2} - \alpha_0^{-1}\gamma + O(\delta^2)\right), \quad \hat\lambda_2 = -\delta\left(2i\alpha_0^{-2} + \alpha_0^{-1}\gamma + O(\delta^2)\right),$$
$$\hat\lambda_3 = -4i\alpha_0^{-2}\delta\left(1 + O(\delta^2)\right), \quad \hat\lambda_4 = O(\delta^3).$$

Let $R_{ii}(\lambda,\delta) = (\lambda - H_{ii}(\delta))^{-1}$ for $i = 1,2$ and let

$$R_0(\lambda,\delta) = \begin{pmatrix} R_{11}(\lambda,\delta) & 0 \\ 0 & R_{22}(\lambda,\delta)\end{pmatrix},$$
$$V_0(\lambda,\delta) = \begin{pmatrix} 0 & H_{12}(\lambda,\delta)R_{22}(\lambda,\delta) \\ H_{21}(\lambda,\delta)R_{11}(\lambda,\delta) & 0\end{pmatrix}.$$

We remark that $R_{22}(\lambda,\delta)$ is uniformly bounded for $\lambda \in U$ and $\delta \in (0,\delta_0)$. Suppose that $|\lambda - \hat\lambda_i| = c_1\delta$, where $c_1 \in (0, \alpha_0^{-1}|\gamma|\delta/4)$ is a constant such that $|\hat\lambda_j - \hat\lambda_k| \geq c_1\delta$ for every j, $k = 1,2,3,4$ with $j \neq k$. Then in view of the definitions of $H_{12}(\lambda,\delta)$ and $H_{21}(\lambda,\delta)$, we have

$$\|V_0(\lambda,\delta)\|_{B(L^2(\mathbf{R}))} = O(\delta), \tag{14}$$

and

$$(\lambda - H(\delta))^{-1} = R_0(\lambda,\delta)\sum_{i=0}^{\infty} V_0(\lambda,\delta)^i. \tag{15}$$

Now let

$$P_{H,i}(\delta) = \frac{1}{2\pi i}\oint_{|\lambda-\hat\lambda_i| = c_1\delta}(\lambda - H(\delta))^{-1}d\lambda,$$
$$\widehat{P}_{H,i}(\delta) = \frac{1}{2\pi i}\oint_{|\lambda-\hat\lambda_i| = c_1\delta}R_0(\lambda,\delta)d\lambda.$$

Combining (14) and (15) with the fact that

$$\|R_0(\lambda,\delta)V_0(\lambda,\delta)\|_{B(L^2(\mathbb{R}))} = \left\| \begin{pmatrix} 0 & R_{11}H_{12}R_{22} \\ R_{22}H_{21}R_{11} & 0 \end{pmatrix} \right\|_{B(L^2(\mathbb{R}))} = O(\delta),$$

we have

$$\left\| P_{H,i}(\delta) - \widehat{P}_{H,i}(\delta) \right\| = O(\delta) \quad \text{for every } i = 1,2,3,4.$$

Hence it follows that $R(\widehat{P}_{H,i}(\delta))$ is isomorphic to $R(P_{H,i}(\delta))$ and that $R(P_{H,i}(\delta))$ is 1-dimensional for $i = 1,2,3,4$. Furthermore, we see that eigenvalues of $H(\delta)$ which lie in U satisfy $|\lambda - \hat{\lambda}_i| < c_1\delta$ for an $i \in \mathbb{N}$ with $1 \le i \le 4$.

Since $d\|Q_c\|^2_{L^2(\mathbb{R})}/dc > 0$ for $p \in (1,5)$, we see that γ is a positive number and that there exist eigenvalues λ_1 and λ_2 satisfying

$$\alpha_0^{-1}\gamma\delta/2 < \mathrm{Re}\lambda_1 < 3\alpha_0^{-1}\gamma\delta/2, \quad -3\alpha_0^{-1}\gamma\delta/2 < \mathrm{Re}\lambda_2 < -\alpha_0^{-1}\gamma\delta/2.$$

Thus we complete the proof of Lemma 6. □ □

Proposition 7. *Let j, $m \in \mathbb{N}$, $\varepsilon = m^{-1}$ and $\delta = j\varepsilon$. Let $\beta = \min(p-1,1)/6$. Then there exists an $m_* \in \mathbb{N}$ such that if $m \ge m_*$, the linearized operator $\mathcal{H}(\varepsilon,\delta)$ with $j = [m^\beta]$ has an unstable eigenvalue.*

Proof. In order to prove Proposition 7, we will show the spectrum of $\mathcal{H}(\varepsilon,\delta)$ becomes close to the spectrum of $H(\delta)$ as $\varepsilon \downarrow 0$. Let

$$\mathcal{H}_0 = i \begin{pmatrix} \frac{-2jm}{r^2} & \Delta_r - \omega - \frac{m^2+j^2}{r^2} \\ \Delta_r - \omega - \frac{m^2+j^2}{r^2} & \frac{-2jm}{r^2} \end{pmatrix},$$

and $H_0 = U\mathcal{H}_0U^{-1}$. Let

$$\mathcal{D}(\lambda) = (\tau_{\bar{r}}\tilde{\chi}_0)(\lambda - H_0)^{-1}(\tau_{\bar{r}}\chi_0) + \tau_{\bar{r}}\tilde{\chi}_1(\lambda - H(\delta))^{-1}\chi_1\tau_{-\bar{r}}.$$

Then we have

$$\mathcal{D}(\lambda)U(\lambda - \mathcal{H}(\varepsilon,\delta))U^{-1} = I + R_3 + R_4,$$

where

$$R_3 = i(\tau_{\bar r})\tilde\chi_0(\lambda - H_0)^{-1}\left\{\begin{pmatrix} 0 & [\partial_r^2, \tau_{\bar r}\chi_0] \\ [\partial_r^2, \tau_{\bar r}\chi_0] & 0 \end{pmatrix} - (\tau_{\bar r}\chi_0)\phi_\omega^{p-1}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\right\}$$

$$R_4 = i\tau_{\bar r}\tilde\chi_1(\lambda - H(\delta))^{-1}\left\{\begin{pmatrix} 0 & [\partial_r^2, \chi_1] \\ [\partial_r^2, \chi_1] & 0 \end{pmatrix} - \chi_1(R_{41} + R_{42})\right\}\tau_{-\bar r},$$

$$R_{41} = \begin{pmatrix} \frac{-2\delta}{(\alpha_0+\varepsilon r)^2} + \frac{2\delta}{\alpha_0^2} & -\frac{1+\delta^2-\frac14\varepsilon^2}{(\alpha_0+\varepsilon r)^2} + \frac{1+\delta^2}{\alpha_0^2} \\ -\frac{1+\delta^2-\frac14\varepsilon^2}{(\alpha_0+\varepsilon r)^2} + \frac{1+\delta^2}{\alpha_0^2} & \frac{-2\delta}{(\alpha_0+\varepsilon r)^2} + \frac{2\delta}{\alpha_0^2} \end{pmatrix},$$

$$R_{42} = \begin{pmatrix} 0 & f(\phi_\omega) - f(Q_c) \\ f'(\phi_\omega) - f'(Q_c) & 0 \end{pmatrix}.$$

We remark that

$$\|[\partial_r^2, \chi_i]\|_{B(L^2(\mathbb{R}), H^{-1}(\mathbb{R}))} = O(l^{-1}) \quad \text{for } i = 0, 1,$$
$$\|\chi_1 R_{41}\|_{B(L^2_r(\mathbb{R}^2))} + \|R_{42}\|_{B(L^2_r(\mathbb{R}^2))} = O(\varepsilon^{6\beta} l).$$

We have

$$\sup_{\lambda\in\mathbb{C}, |\lambda|\le\omega/2} \|(\lambda - \mathcal{H}_0)^{-1}\|_{B(H_r^{-2}(\mathbb{R}^2), L^2(\mathbb{R}^2))} < \infty,$$

since

$${}^tO\mathcal{H}_0 O = i\begin{pmatrix} \Delta_r - \omega - \frac{(m+j)^2}{r^2} & 0 \\ 0 & -\Delta_r + \omega + \frac{(m-j)^2}{r^2} \end{pmatrix},$$

where

$$O = \frac{1}{\sqrt2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Lemma 6 yields that for $\delta \in (0, \delta_0)$, there exists a $c > 0$ such that

$$\left\|(\lambda - H(\delta))^{-1}\right\|_{B(L^2_r(\mathbb{R}^2))} \le C\delta^{-1}$$

for every $\lambda \in U$ with $\min_{1\le i\le 4} |\lambda - \lambda_i(\delta)| \ge c\delta$ and that $\mathrm{Re}(\lambda_1(\delta) - c\delta) > 0$. Let $l = \delta^{-3}$. Then it follows from the above that

$$\|R_3\|_{B(L^2_r(\mathbb{R}^2))} = O(\delta^3 + e^{-2\sqrt{c}\delta^{-3}}),$$
$$\|R_4\|_{B(L^2_r(\mathbb{R}^2))} = O(\delta^2 + \varepsilon^{6\beta}\delta^{-4}).$$

Put

$$\begin{aligned}\mathcal{P}_{\mathcal{H},1}(\varepsilon,\delta) =& \frac{1}{2\pi i}\oint_{|\lambda-\lambda_1(\delta)|=c\delta}(\lambda-\mathcal{H}(\varepsilon,\delta))^{-1}d\lambda,\\ \mathcal{P}_{H,1}(\varepsilon,\delta) =& U^{-1}\tau_{\bar r}\tilde\chi_1 P_{H,1}(\delta)\chi_1\tau_{-\bar r}U.\end{aligned}$$

Making use of Cauchy's theorem and noting that $\delta\sim\varepsilon^\beta$, we have

$$\begin{aligned}&\|\mathcal{P}_{\mathcal{H},1}(\varepsilon,\delta)-\mathcal{P}_{H,1}(\varepsilon,\delta)\|_{B(L^2_{\bar r}(\mathbb{R}^2))}\\ =&\frac{1}{2\pi}\left\|\oint_{|\lambda|=c\delta}\left\{(\lambda-\mathcal{H}(\varepsilon,\delta))^{-1}-U^{-1}\mathcal{D}(\lambda)U\right\}d\lambda\right\|_{B(L^2_{\bar r}(\mathbb{R}^2))}\\ \le& C\delta^{-1}\sup_{|\lambda|=c\delta}(\|R_3\|_{B(L^2(-\bar r,\infty))}+\|R_4\|_{B(L^2(-\bar r,\infty))})\\ \le& C(\delta+\varepsilon^{6\beta}\delta^{-5})\\ =&O(\delta).\end{aligned}$$

From the above, we conclude that the range of $\mathcal{P}_{\mathcal{H},1}(\varepsilon,\delta)$ is isomorphic to the range of $P_{H,1}(\delta)$ and that there exists an eigenvalue λ of $\mathcal{H}(\varepsilon,\delta)$ with $\mathrm{Re}\lambda>0$. Thus we complete the proof of Proposition 7. □ □

Now we are in position to prove Theorem 4.

Proof of Theorem 4. Let $\mathfrak{L}$ be the linearized operator of (1) around $e^{i(\omega t+m\theta)}\phi_\omega$. Then

$$\mathfrak{L}=i\begin{pmatrix}\Delta-\omega+\beta_1(r) & e^{2im\theta}\beta_2(r)\\ -e^{-2im\theta}\beta_2(r) & -\Delta+\omega-\beta_1(r)\end{pmatrix}.$$

Proposition 7 tells us that $\mathfrak{L}$ has unstable eigenvalues if $m\in\mathbb{N}$ is large and $p\in(1,5)$. On the other hand, [15] tells us that $\mathfrak{L}$ has an unstable eigenvalue if $p>3$. Hence it follows that $\mathfrak{L}$ has an unstable eigenvalue if $p>1$ and $m\in\mathbb{N}$ is sufficiently large. □ □

Remark 2. *We remark that our method can also be applied to prove that a one-dimensional standing wave solution $e^{ict}Q_c(x_1)$ of (1) is unstable to long-wavelength transversal disturbances.*

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