数理解析研究所講究録
1498 巻 2006 年 59-73

Stability of traveling waves in curvature flows in the whole plane

東京工業大学 大学院情報理工学研究科 奈良光紀 (Mitsunori Nara)
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology

1. INTRODUCTION

In this note, we study the long time behavior of solutions to a Cauchy problem of the form

$$\frac{u_t}{\sqrt{1+u_x^2}} = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + k, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{2}$$

where $k \in \mathbb{R}$ is a given constant. Especially, we are concerned with asymptotic stability of two types of traveling waves: *the traveling lines* and *the V-shaped fronts*.

Our motivation to study this problem comes from the theory of interfacial phenomena, which is a part of mathematical science and is concerned with formation and development of a "shape" in biological, chemical, and physical fields. One of the mathematical models that describes the motion of interfaces is a *curvature flow*.

Let $D(t)$ be a moving domain in $\mathbb{R}^2$ with a smooth boundary $\Gamma(t) = \partial D(t)$. Let $\boldsymbol{\nu}$ be the unit normal vector on $\Gamma(t)$ pointing from $D(t)$ to $D(t)^{\mathrm{C}}$. We consider an interface $\Gamma(t)$ governed by a curvature flow with constant driving force $k \in \mathbb{R}$. Namely, we have

$$V = -H + k, \tag{3}$$

where V and H are the normal velocity and the curvature of $\Gamma(t)$, respectively. That is, V is the velocity of $\Gamma(t)$ along $\boldsymbol{\nu}$, and $H = \mathrm{div}\ \boldsymbol{\nu}$. This model appears in several fields. One of them is the dynamics of interfaces in an excitable media, for example, Belousov-Zhabotinsky reaction [3, 27]. Equation (3) also appears in the dynamics of interfaces in the Allen-Cahn equations. See [5] for instance. Moreover it appears in the reaction-diffusion systems of a competition type. See [10].

In this note, we deal with the case where an initial curve is given by a function $y = u_0(x), x \in \mathbb{R}$, and a moving curve is expressed by $y = u(x,t), x \in \mathbb{R}, t > 0$. Under these assumptions, (3) is rewritten as the initial value problem (1) - (2).

1.1. **Traveling waves in the curvature flow.** A moving curve $\Gamma(t)$ is called a *traveling wave* of (3) with the velocity $|\boldsymbol{v}|$, if $\Gamma(t)$ satisfies

$$\Gamma(t) = \Gamma(0) + \boldsymbol{v}t, \quad t > 0$$

for some vector $\boldsymbol{v} \in \mathbb{R}^2$. When $k \neq 0$, a stationary circle with radius $1/k$ is a traveling wave with $|\boldsymbol{v}| = 0$. Except for this circle and self-crossing ones, a traveling wave of $V = -H + k\,(k \neq 0)$ is one of the following two cases [3, 23]:

(i) *The traveling line*, that is, a line with normal velocity k.

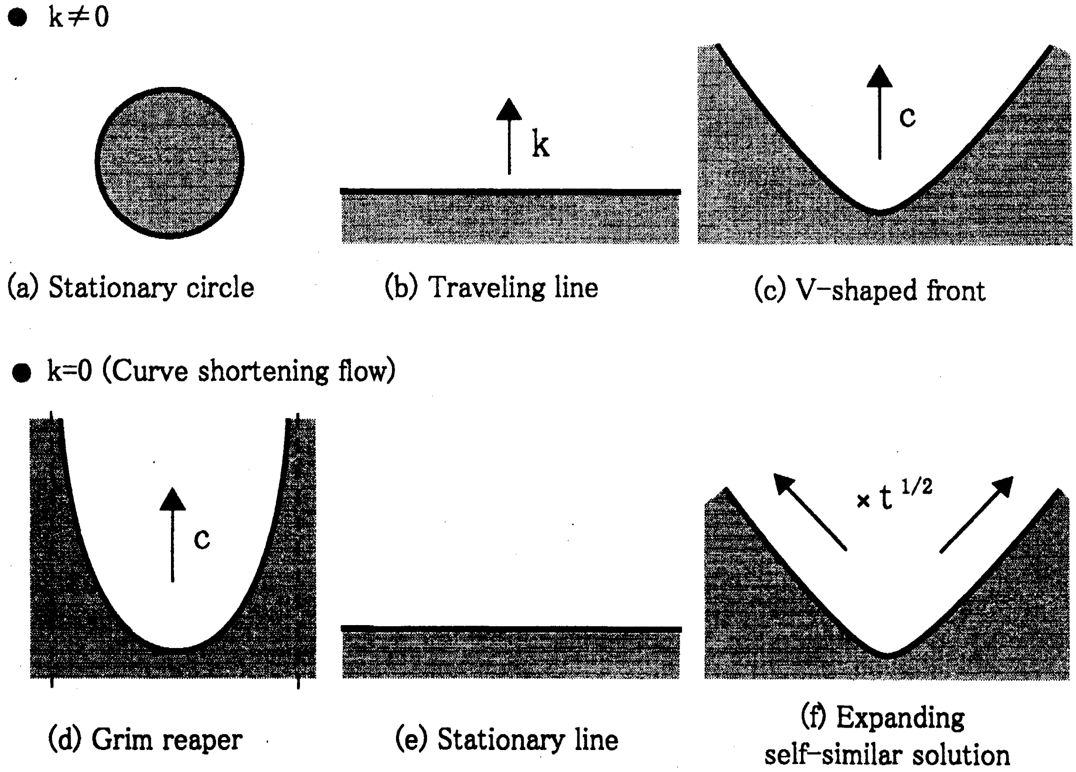

FIGURE 1. Characteristic waves of the curvature flow.

(ii) *The V-shaped front*, which is convex and is asymptotic to the traveling lines at infinity.

As is mentioned later, the traveling lines and the V-shaped fronts can be represented by graphical forms under an appropriate rotation of coordinates.

On the other hand, $k = 0$ means the so-called *curve shortening flow*, which is the mean curvature flow in $\mathbb{R}^2$. In this case, every line is a stationary wave, and there exists a traveling wave that is called the *Grim Reaper*. Furthermore the curve shortening flow has an extracting self-similar solution, which is convex and is asymptotic to the stationary lines as $|x| \to \infty$ as in [9, 18].

Letting $k \in \mathbb{R}$ be an arbitrary constant, we study asymptotic stability of the traveling (or stationary) lines. In addition, we consider asymptotic stability of the V-shaped fronts for $k > 0$. Especially we are interested in the large time behavior of these traveling waves for some initial perturbations that do not decay as $|x| \to \infty$.

1.2. **Profiles of the traveling line and the V-shaped front.** We assume that a traveling wave in $\mathbb{R}^2$ moves along the y-axis without loss of generality. To obtain profiles of the traveling lines and the V-shaped fronts, we substitute $u(x,t) = U(x) + ct$ to (1), and obtain an ordinary differential equation

$$c = \frac{U''}{1 + (U')^2} + k\sqrt{1 + (U')^2}, \quad x \in \mathbb{R}. \tag{4}$$

For any $k \in \mathbb{R}$ and $m \in \mathbb{R}$, traveling lines are obtained as

$$U(x) + ct = mx + ct, \quad c = k\sqrt{1 + m^2}.$$

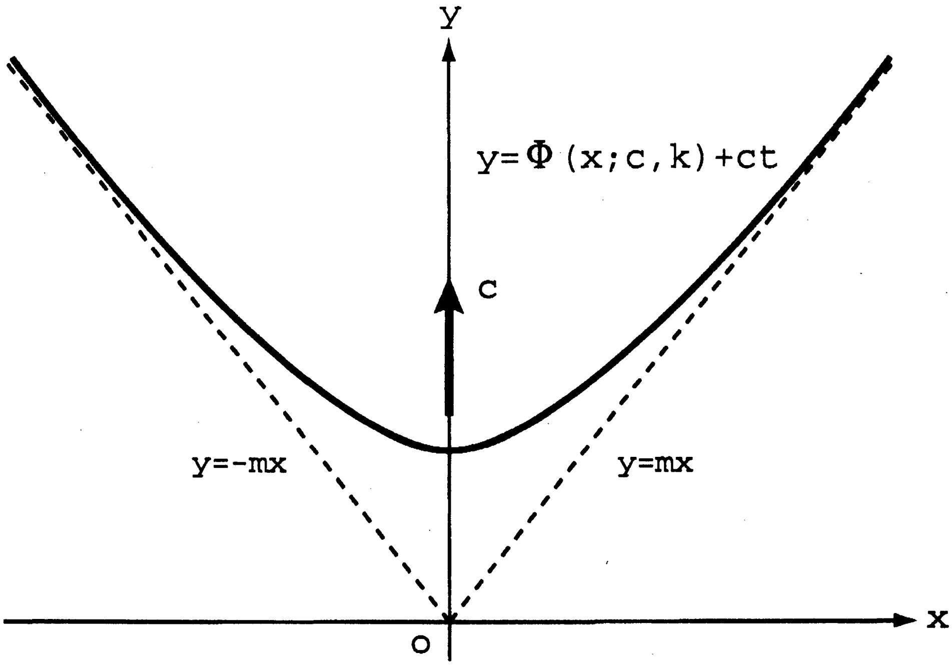

FIGURE 2. The graph of a V-shaped front.

These traveling lines have velocity k in the normal direction and velocity c in the y-direction.

On the other hand, the V-shaped front is another solution of (4). The exact representation of the profile of the V-shaped front $\Phi(x; c, k)$ is written as follows.

Proposition 1.1 (Ninomiya and Taniguchi [23]). *For $c > k > 0$, (4) has a solution $\Phi(x; c, k)$ represented by*

$$\begin{aligned} x(\theta) &= \frac{\theta}{c} + \frac{k}{c\sqrt{c^2-k^2}} \log \left| \frac{1+\sqrt{\dfrac{c+k}{c-k}}\tan\dfrac{\theta}{2}}{1-\sqrt{\dfrac{c+k}{c-k}}\tan\dfrac{\theta}{2}} \right|, \\ y(\theta) &= \frac{1}{c}\log\left(\frac{2(c^2-k^2)}{c\,(c\cos\theta - k)}\right) + \frac{\sqrt{c^2-k^2}}{ck}\arctan\left(\frac{\sqrt{c^2-k^2}}{k}\right), \end{aligned}$$

for $\theta \in (-\arctan m, \arctan m)$. Here $m = \sqrt{c^2-k^2}/k$. Moreover $\Phi(x; c, k)$ is strictly convex with

$$\Phi_{xx}(x; c, k) > 0, \quad x \in \mathbb{R}.$$

From Proposition 1.1, we find that $\Phi(x; c, k)$, or $\Phi(x)$ in short, satisfies

$$\Phi', \Phi'', \text{ and } \Phi''' \text{ are continuous and bounded on } \mathbb{R}, \tag{5}$$

$$\lim_{x\to-\infty} \Phi(x) = -mx, \quad \lim_{x\to+\infty} \Phi(x) = mx, \tag{6}$$

$$\lim_{x\to-\infty} \Phi'(x) = -m, \quad \lim_{x\to+\infty} \Phi'(x) = m, \quad \lim_{|x|\to\infty} \Phi''(x) = 0. \tag{7}$$

We note that, for each $c > k > 0$, the problem (1) - (2) has three traveling waves: the traveling lines $y = \pm mx + ct$, $m = \sqrt{c^2 - k^2}/k$, and the V-shaped front $y = \Phi(x; c, k) + ct$ that is asymptotic to the traveling lines $\pm mx + ct$ as $x \to \pm\infty$.

1.3. **Stability of traveling waves.** To study stability of the traveling line $y = mx + ct$, we consider a function $\overline{u}(x,t) = u(x,t) - (mx + ct)$. Then we have a quasi-linear parabolic equation

$$\begin{aligned}\overline{u}_t &= \frac{\overline{u}_{xx}}{1 + (\overline{u}_x + m)^2} + k\left(\sqrt{1 + (\overline{u}_x + m)^2} - \sqrt{1 + m^2}\right) \\ &= (\arctan(\overline{u}_x + m))_x + k\left(\sqrt{1 + (\overline{u}_x + m)^2} - \sqrt{1 + m^2}\right),\end{aligned}$$

where $c = k\sqrt{1 + m^2}$. Similarly, for the V-shaped front $\Phi(x; c, k)$, we have

$$\overline{u}_t = (\arctan(\overline{u}_x + \Phi'))_x + k\left(\sqrt{1 + (\overline{u}_x + \Phi')^2} - \sqrt{1 + m^2}\right).$$

In these equations, every constant, that is, $\overline{u}(x,t) \equiv$ (const.), implies a traveling wave with translation in y-direction. Thus every constant is a stationary solution of these equations.

For the Cauchy problem of the heat equation

$$h_t = h_{xx}, \qquad x \in \mathbb{R},\ t > 0, \tag{8}$$

$$h(x,0) = \varphi(x), \quad x \in \mathbb{R}, \tag{9}$$

it is well known that every constant is a stationary solution and is asymptotically stable in $L^\infty(\mathbb{R})$ for spatially decaying initial perturbations. To be more precise, a stationary solution $h(x,t) \equiv \mu$ of (8) - (9) is asymptotically stable in $L^\infty(\mathbb{R})$ if and only if the initial value $\varphi(x)$ satisfies

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{1}{2R} \left| \int_{x-R}^{x+R} \varphi(y) - \mu \, dy \right| = 0.$$

For the proof, see [11, 19, 20, 29]. In relation to this criterion, Collet & Eckmann [8] showed the example of an initial perturbation for which the constant solution of (8) - (9) loses the asymptotic stability, where the bounded initial value does not decay but oscillates slower and slower as $|x| \to \infty$.

Proposition 1.2 (Collet and Eckmann [8]). *Let $L_n = n!$ and define an even function $\varphi^*(x) \in C^\infty(\mathbb{R})$ that satisfies $|\varphi^*(x)| \le 1$ for $x \in \mathbb{R}$ and*

$$\varphi^*(x) = (-1)^n, \quad x \in [\, L_n + 2^n,\ L_{n+1} - 2^{n+1}\,]$$

for $n \ge 5$. Then the solution $h(x,t)$ of (8) - (9) with $h(x,0) = \varphi^*(x)$ satisfies

$$\liminf_{t\to\infty} h(0,t) = -1, \quad \limsup_{t\to\infty} h(0,t) = 1.$$

In [21], we also obtained the following example for the curvature flow (1) - (2). In this example, the initial perturbation looks like $\varphi^*(x)$ in Proposition 1.2, that is, it does not decay but oscillates slower and slower at infinity.

Example 1.3 (Nara and Taniguchi [21]). *Define a function $f(x)$ as*

$$\begin{aligned} f(x) = 0 \quad &\text{if} \quad (2n)^2 \le |x| < (2n+1)^2, \\ f(x) = 1 \quad &\text{if} \quad (2n+1)^2 \le |x| < (2n+2)^2, \end{aligned}$$

for $n = 0, 1, 2, \ldots$. Then the solution of (1) - (2) *with $u_0(x) = (\eta * f)(x)$ does not converge uniformly to the traveling line $u(x,t) = kt + \mu$ for any fixed μ. Here $\eta*$ is Friedrichs' mollifier, that is, $\eta(x) \in C_0^\infty(\mathbb{R})$, $\eta(x) \geq 0, \|\eta\|_{L^1(\mathbb{R})} = 1$, and*

$$(\eta * f)(x) = \int_{\mathbb{R}} \eta(x-y) f(y) dy.$$

It is also known that the V-shaped front of the Cauchy problem (1) - (2) is not asymptotically stable for similar perturbations as in [24]. Moreover Poláčik & Yanagida [28] studied related works on a supercritical semi-linear diffusion equation.

Such counter-examples may cause difficulty in considering asymptotic stability of constant solutions in the Cauchy problem of a parabolic PDE with an initial perturbation that does not decay at infinity.

1.4. **Outline of this note.** In Section 2, we consider asymptotic stability of the traveling lines. In Section 3, we give some examples of spatially non-decaying initial perturbations for which traveling lines lose the asymptotic stability. In Section 4, we consider the asymptotic stability of the V-shaped fronts. Finally in Section 5 we show the outline of proofs.

Our results in this note for the traveling lines and the V-shaped fronts are based on the discussions in [21] and [22]. We omit most of proofs. See [21] and [22] for further details.

1.5. **Notation.** In what follows, $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$, $W^{1,\infty}(\mathbb{R})$, denote the Lebesgue or Sobolev spaces. For $\gamma \in (0,1)$, $C^\gamma(\mathbb{R})$ denotes the Hölder space, that is, the space of functions that are bounded and uniformly Hölder continuous with exponent γ on $\mathbb{R}$. $C^{2+\gamma}(\mathbb{R})$ means the space of functions with $u, u', u'' \in C^\gamma(\mathbb{R})$. For a domain $R_T = \mathbb{R} \times [0,T]$, $C^{\gamma,\gamma/2}(R_T)$ denotes the space of functions that are bounded and uniformly Hölder continuous with exponent γ and $\gamma/2$ with respect to x and t, respectively on R_T. $C^{2+\gamma,1+\gamma/2}(R_T)$ means the space of functions with $u, u_x, u_{xx}, u_t \in C^{\gamma,\gamma/2}(R_T)$.

2. Stability of traveling lines

In this section, we show the asymptotic stability of traveling lines $y = mx + ct$ of (1) - (2). In what follows, let $k \in \mathbb{R}$ and $m \in \mathbb{R}$ be given constants, and put $c = k\sqrt{1+m^2}$. First we consider the stability with spatially decaying initial perturbations. Next we focus on the stability of traveling lines with spatially non-decaying initial perturbations. The key to our discussion is the concept of *an almost periodic function.*

2.1. **Stability with spatially decaying perturbations.** First we give a result for the horizontal traveling line $u(x,t) = kt$, that is, the case of $m = 0$.

Theorem 2.1. *Suppose that $\phi \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} \phi(x) = 0$. Then for the initial value $u_0(x) = \phi(x)$, the solution $u(x,t)$ to the Cauchy problem* (1) - (2) *exists up to $t = \infty$. Moreover it satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt| = 0.$$

Especially, if ϕ belongs to $C^{2+\gamma}(\mathbb{R}) \cap L^1(\mathbb{R})$, the solution $u(x,t)$ satisfies the estimate

$$\sup_{x\in\mathbb{R}} |u(x,t) - kt| \leq C(1+t)^{-\frac{1}{2}}, \quad t > 0,$$

where C is a constant depending only on k and ϕ.

This result is similar to that for the Cauchy problem of the heat equation. By virtue of this result, we also obtain a result for the inclined traveling line $y = mx + ct$, $m \in \mathbb{R}$ as follows.

Theorem 2.2. *Suppose that $\phi \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} \phi(x) = 0$. Then for the initial value $u_0(x) = mx + \phi(x)$, the solution $u(x,t)$ to the Cauchy problem* (1) - (2) *exists up to $t = \infty$. Moreover it satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - (mx + ct)| = 0.$$

2.2. **Stability with spatially non-decaying initial perturbations.** Next we show the asymptotic stability with spatially non-decaying initial perturbations. We begin by recalling the definition of an almost periodic function.

Definition 2.3. *A continuous function $f(x) : \mathbb{R} \to \mathbb{R}$ is called an almost periodic function (in the sense of Bohr) if, for every $\varepsilon > 0$, there exists $\ell(\varepsilon) > 0$ such that, for every $p \in \mathbb{R}$, an interval $[p, p + \ell(\varepsilon)]$ contains at least one number q with*

$$|f(x-q) - f(x)| < \varepsilon \quad \text{for all} \quad x \in \mathbb{R}. \tag{10}$$

For any almost periodic function f, there exists a mean $\mathcal{M}\{f\}$ defined by

$$\mathcal{M}\{f\} = \lim_{R\to\infty} \frac{1}{R} \int_s^{s+R} f(x)\,dx,$$

where the convergence is uniform with respect to $s \in \mathbb{R}$, and the limit is independent of s.

By this definition, every periodic function is an almost periodic function. Moreover if f and g are both almost periodic functions, $f(x) + g(x)$ is an almost periodic function, where $\mathcal{M}\{f+g\} = \mathcal{M}\{f\} + \mathcal{M}\{g\}$ holds true. Note that a non-periodic function $f(x) = \sin x + \sin\sqrt{2}x$ is an almost periodic function with $\mathcal{M}\{f\} = 0$. For further details, see [1, 2, 7] for instance.

The following result is the central point of our discussion for asymptotic stability with spatially non-decaying initial perturbations. This implies that the almost periodicity of an initial perturbation is sufficient for asymptotic stability of traveling lines.

Theorem 2.4. *Assume that $\phi \in C^{2+\gamma}(\mathbb{R})$ is an almost periodic function. Then for the initial value $u_0(x) = mx + \phi(x)$, the solution $u(x,t)$ to the Cauchy problem* (1) - (2) *exists up to $t = \infty$. Moreover it satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - (mx + ct + \mu)| = 0$$

for a constant μ with

$$\inf_{x\in\mathbb{R}} \phi(x) \le \mu \le \sup_{x\in\mathbb{R}} \phi(x).$$

Especially for each ϕ, the constant μ is a nondecreasing function of $k \in \mathbb{R}$ when $m = 0$. In addition, $\mu = \mathcal{M}\{\phi\}$ holds true when $k = 0$.

We show the outline of the proof of Theorem 2.4 in Section 5. The constant μ in Theorem 2.4 may not be determined explicitly if $k \neq 0$. Indeed we have the following fact.

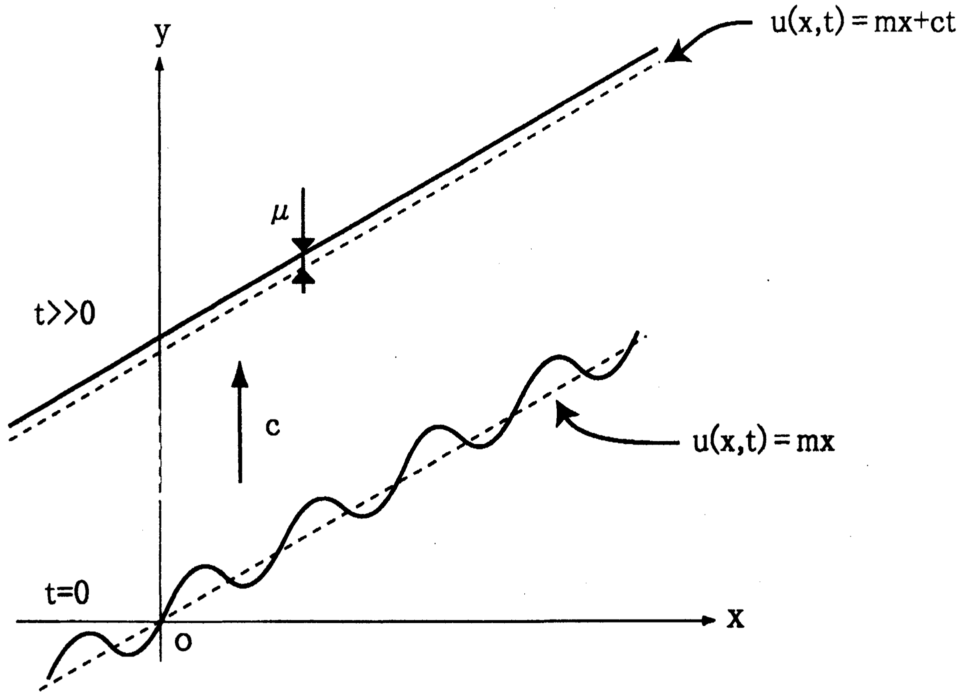

FIGURE 3. Stability with an almost periodic function as an initial perturbation.

Remark 2.5. *Generically $\mu \neq \mathcal{M}\{\phi\}$ holds true if $k \neq 0$. Indeed, for $k > 0$ and $\phi(x) = \sin x$, we have*

$$\mu = \lim_{t\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \phi(x)\,dx + \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} u_t(x,t)\,dx\,dt$$
$$= \frac{1}{2\pi}\int_0^{\infty}\left([\arctan(u_x+m)]_0^{2\pi} + \int_0^{2\pi} k\left(\sqrt{1+(u_x+m)^2} - \sqrt{1+m^2}\right)dx\right)dt$$
$$> \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} k\frac{m\,u_x}{\sqrt{1+m^2}}\,dx\,dt = \frac{k\,m}{2\pi\sqrt{1+m^2}}\int_0^{\infty}[u]_0^{2\pi}\,dt = 0$$

by using the periodic boundary condition at $x = 0, 2\pi$, and the inequality

$$\sqrt{1+(p+m)^2} \geq \left|\frac{mp}{\sqrt{1+m^2}} + \sqrt{1+m^2}\right| \geq \frac{mp}{\sqrt{1+m^2}} + \sqrt{1+m^2}$$

for $p \in \mathbb{R}$. Thus μ differs from $\mathcal{M}\{\phi\} = 0$ in this case.

We can extend Theorem 2.4 to the case where an initial perturbation is asymptotic to an almost periodic function as $|x| \to \infty$. The following theorem gives the bounds for a perturbation depending only on the asymptotic behavior of a given initial perturbation at infinity.

Theorem 2.6. *For some functions $\phi_*(x)$ and $\phi^*(x)$ that belong to $C^{2+\gamma}(\mathbb{R})$, let $u_*(x,t)$ and $u^*(x,t)$ be the solutions of (1) - (2) with the initial values $mx + \phi_*$ and $mx + \phi^*$, respectively. Assume that u_* and u^* satisfy*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u_*(x,t) - (mx+ct+\mu_*)| = 0, \quad \lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u^*(x,t) - (mx+ct+\mu^*)| = 0,$$

for some constants μ_* and μ^*. Then for an initial perturbation $\phi \in C^{2+\gamma}(\mathbb{R})$ with

$$\lim_{x\to-\infty}(\phi(x)-\phi_*(x))=0, \quad \lim_{x\to+\infty}(\phi(x)-\phi^*(x))=0,$$

the solution $u(x,t)$ of (1) - (2) with $u_0(x)=mx+\phi(x)$ exists up to $t=\infty$. Moreover it satisfies

$$\lim_{t\to\infty}\inf_{x\in\mathbb{R}}(u(x,t)-(mx+ct+\min\{\mu_*,\mu^*\})=0,$$
$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}(u(x,t)-(mx+ct+\max\{\mu_*,\mu^*\})=0.$$

This extends a class of initial values for which the stability is determined. The following corollary gives an extended sufficient condition for the asymptotic stability of traveling lines.

Corollary 2.7. *Assume that $f \in C^{2+\gamma}(\mathbb{R})$ is an almost periodic function. And assume that $g \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} g(x)=0$. Then the solution $u(x,t)$ to the Cauchy problem (1) - (2) with the initial value $u_0(x)=mx+f(x)+g(x)$ satisfies*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)-(mx+ct+\mu)|=0$$

for a constant μ that depends only on $k, m,$ and f, and is independent of g. Especially, $\mu=\mathcal{M}\{f\}$ holds true if $k=0$.

3. Examples for asymptotic stability of traveling lines

In this section we show some examples and counter-examples for stability of traveling lines. If a given initial perturbation $\phi(x)$ on the traveling line $mx+ct$ is bounded, we have

$$\inf_{x\in\mathbb{R}}\phi(x) \le u(x,t)-(mx+ct) \le \sup_{x\in\mathbb{R}}\phi(x), \quad x\in\mathbb{R},\, t>0,$$

by using the comparison principle. This implies that a traveling line is always stable for bounded perturbations. The problem is the asymptotic stability for these perturbations.

Example 3.1. *The solution $u(x,t)$ to the Cauchy problem (1) - (2) with the initial value $u_0(x)=mx+\tanh x$ satisfies*

$$\lim_{t\to\infty}\inf_{x\in\mathbb{R}}(u(x,t)-(mx+ct))=-1, \quad \lim_{t\to\infty}\sup_{x\in\mathbb{R}}(u(x,t)-(mx+ct))=1.$$

Though this example is intuitively clear, it is proved rigorously by virtue of Theorem 2.6. Namely, $\tanh x$ is asymptotic to ± 1 at infinity, and the solutions with the initial value $mx+1$ and $mx-1$ are given by $u^*(x,t)=mx+ct+1$ and $u_*(x,t)=mx+ct-1$, respectively. Thus Theorem 2.6 gives Example 3.1. The next example shows difficulty of asymptotic stability for $k\neq 0$ compared with $k=0$.

Example 3.2. *Define the initial perturbation $\phi(x)\in C^{2+\gamma}(\mathbb{R})$ to satisfy*

$$\phi(x)=0 \quad \text{if} \quad x\in(-\infty,-1],$$
$$\phi(x)=\sin x \quad \text{if} \quad x\in[1,+\infty).$$

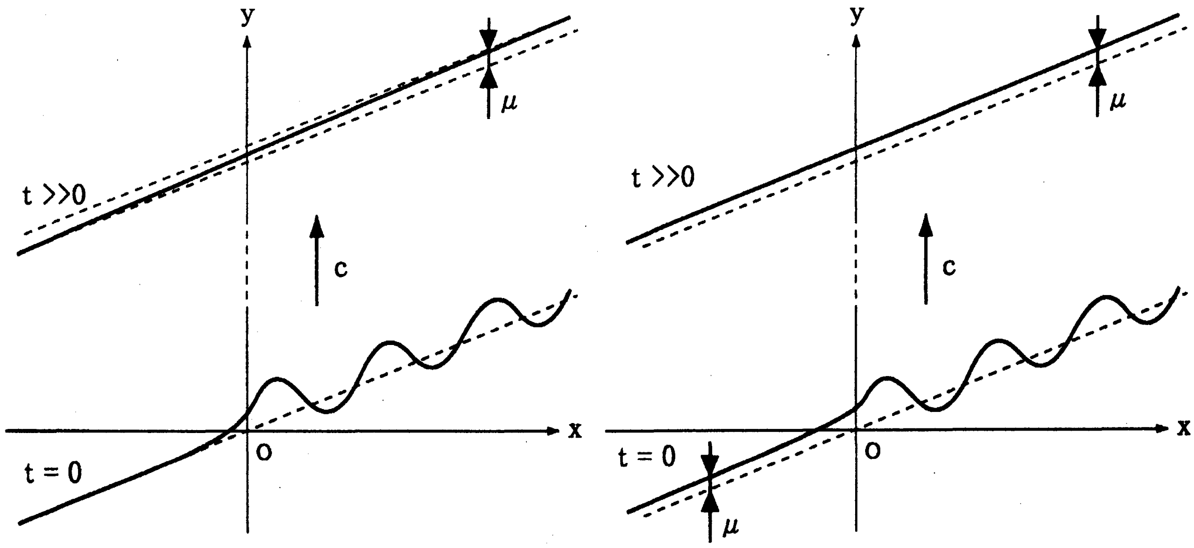

FIGURE 4. Initial values and the solutions in Example 3.2 and 3.3

Then the solution $u(x,t)$ *to the Cauchy problem* (1) - (2) *with* $u_0(x) = mx + \phi(x)$ *satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - mx| = 0 \qquad \text{if } k = 0,$$

$$\lim_{t\to\infty} \inf_{x\in\mathbb{R}} (u(x,t) - (mx+ct)) = 0, \; \lim_{t\to\infty} \sup_{x\in\mathbb{R}} (u(x,t) - (mx+ct)) = \mu \qquad \text{if } k > 0$$

for a positive constant μ.

This example follows from Theorem 2.6 and Remark 2.5. It is due to the fact that a phase shift of the limiting traveling line occurs if $u_0(x) = mx + \sin x$, while it does not occur for $u_0(x) = mx$. Arranging this example, we have the following one.

Example 3.3. *Define the initial perturbation* $\phi(x) \in C^{2+\gamma}(\mathbb{R})$ *to satisfy*

$$\phi(x) = \mu \qquad \text{if} \quad x \in (-\infty, -1],$$
$$\phi(x) = \sin x \quad \text{if} \quad x \in [1, +\infty),$$

where μ *is the constant defined as in Remark 2.5. Then the solution* $u(x,t)$ *to the Cauchy problem* (1) - (2) *with* $u_0(x) = mx + \phi(x)$ *satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - (mx + ct + \mu)| = 0.$$

Example 3.2 and 3.3 show the peculiarity of our problem due to a phase shift of the limiting traveling line. This mechanism is the point of discussion for the asymptotic stability of traveling lines, in addition to Example 1.3 and Proposition 1.2 in the introduction.

4. V-SHAPED FRONTS

In this section, letting $c > k > 0$ be any constants and setting $m = \sqrt{c^2 - k^2}/k > 0$, we study asymptotic stability of the V-shaped front $\Phi(x; c, k)$, which is asymptotic to the traveling lines $\pm mx + ct$ as $x \to \pm\infty$. First we show asymptotic stability of the V-shaped fronts for spatially decaying initial perturbations. Ninomiya and Taniguchi proved the following result.

Theorem 4.1 (Ninomiya and Taniguchi [24]). *Suppose that $\phi \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} \phi(x) = 0$. Then for the initial value $u_0(x) = \Phi(x; c, k) + \phi(x)$, the solution $u(x,t)$ to the Cauchy problem (1) - (2) exists up to $t = \infty$. Moreover it satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - (\Phi(x; c, k) + ct)| = 0.$$

This result is proved by constructing a supersolution and a subsolution. In this problem, the decay estimate is not obtained yet. Next we show the result for spatially non-decaying initial perturbations. In this situation, the key to our problem is the stability of two asymptotic traveling lines $\pm mx + ct$.

Theorem 4.2. *For some functions $\phi_*(x)$ and $\phi^*(x)$ in $C^{2+\gamma}(\mathbb{R})$, let $u_*(x,t)$ and $u^*(x,t)$ be the solutions of (1) - (2) with the initial values $-mx+\phi_*(x)$ and $mx+\phi^*(x)$, respectively. Assume that u_* and u^* satisfy*

$$\lim_{t\to\infty} \sup_{x<0} |u_*(x,t) - (-mx + ct + \mu_*)| = 0, \quad \lim_{t\to\infty} \sup_{x>0} |u^*(x,t) - (mx + ct + \mu^*)| = 0$$

for some constants μ_* and μ^*. Then for an initial perturbation $\phi \in C^{2+\gamma}(\mathbb{R})$ with

$$\lim_{x\to-\infty} (\phi(x) - \phi_*(x)) = 0, \quad \lim_{x\to+\infty} (\phi(x) - \phi^*(x)) = 0,$$

the solution $u(x,t)$ of (1) - (2) with the initial value $u_0(x) = \Phi(x; c, k) + \phi(x)$ exists up to $t = \infty$. Moreover it satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| u(x,t) - \left[\Phi\left(x - \frac{\mu_* - \mu^*}{2m}; c, k \right) + ct + \frac{\mu_* + \mu^*}{2} \right] \right| = 0.$$

Thus the asymptotic stability of the traveling lines $y = \pm mx + ct$ for the initial perturbations ϕ_* and ϕ^* gives the asymptotic stability of the V-shaped front $\Phi(x; c, k)$. The shift of the V-shaped front is generically observed in the case where initial perturbations do not decay at infinity. Combining Theorem 4.2 with Theorem 2.4, we obtain a corollary that gives concrete sufficient condition for the asymptotic stability of the V-shaped front.

Corollary 4.3. *Assume that ϕ_*, $\phi^*(x) \in C^{2+\gamma}(\mathbb{R})$ are both almost periodic functions in the sense of Bohr, and let $u_*(x,t)$ and $u^*(x,t)$ be the solutions of (1) - (2) with the initial values $-mx + \phi_*(x)$ and $mx + \phi^*(x)$, respectively. Then u_* and u^* exist up to $t = +\infty$, and satisfy*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u_*(x,t) - (-mx + ct + \mu_*)| = 0, \quad \lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u^*(x,t) - (mx + ct + \mu^*)| = 0$$

for some constants μ_* and μ^* with

$$\inf_{x\in\mathbb{R}} \phi_*(x) \le \mu_* \le \sup_{x\in\mathbb{R}} \phi_*(x), \quad \inf_{x\in\mathbb{R}} \phi^*(x) \le \mu^* \le \sup_{x\in\mathbb{R}} \phi^*(x).$$

Moreover for an initial perturbation $\phi \in C^{2+\gamma}(\mathbb{R})$ with

$$\lim_{x\to-\infty} (\phi(x) - \phi_*(x)) = 0, \quad \lim_{x\to+\infty} (\phi(x) - \phi^*(x)) = 0,$$

the solution $u(x,t)$ of (1) - (2) with the initial value $u_0(x) = \Phi(x; c, k) + \phi(x)$ satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| u(x,t) - \left[\Phi\left(x - \frac{\mu_* - \mu^*}{2m}; c, k \right) + ct + \frac{\mu_* + \mu^*}{2} \right] \right| = 0.$$

By this result, a V-shaped front with the initial perturbation $\sin x+\sin\sqrt{2}x$ is asymptotically stable since this perturbation is an almost periodic function. Moreover a V-shaped front with an smooth initial perturbation $\phi(x)$ with

$$\phi(x)=\begin{cases}\sin x, & x\in[0,\infty),\\ 0, & x\in(-\infty,-1],\end{cases}$$

is also asymptotically stable. This is clear by letting $\phi_*(x)\equiv 0$ and $\phi^*(x)=\sin x$ in Corollary 4.3. This makes sharp contrast with Example 3.2 for the traveling lines.

5. Proof of Theorem 2.4

In this section, we show the main part of the proof of Theorem 2.4. In what follows, let $k\in\mathbb{R}$ and $m\in\mathbb{R}$ be given constants, and put $c=k\sqrt{1+m^2}$. As is mentioned in the introduction, we consider the function $\overline{u}(x,t)=u(x,t)-(mx+ct)$ instead of $u(x,t)$ in order to analyze the large time behavior of perturbed traveling lines. Here we denote $\overline{u}(x,t)$ by $u(x,t)$ for simplicity. Then we have

$$u_t=(\arctan(u_x+m))_x+k\left(\sqrt{1+(u_x+m)^2}-\sqrt{1+m^2}\right),\quad x\in\mathbb{R},\ t>0,\tag{11}$$

$$u(x,0)=\phi(x),\qquad x\in\mathbb{R}.\tag{12}$$

It suffices to prove the results for this problem instead of the original problem (1) - (2). First we show the global existence and some estimates for solutions of (11) - (12). The following proposition plays important roles in the proof.

Proposition 5.1. *Assume that $\phi\in C^{2+\gamma}(\mathbb{R})$. Then there exists a classical solution $u(x,t)$ to the Cauchy problem* (11) - (12) *that belongs to $C^{2+\gamma,1+\gamma/2}(R_T)$, $R_T=\mathbb{R}\times[0,T]$ for any $T>0$. It satisfies the following estimates*

$$\begin{aligned}\sup_{x\in\mathbb{R},\,t>0}|u(x,t)| &\le \|\phi\|_{L^\infty(\mathbb{R})}, & \sup_{x\in\mathbb{R},\,t>0}|u_x(x,t)+m| &\le \|\phi'+m\|_{L^\infty(\mathbb{R})},\\ \sup_{x\in\mathbb{R},\,t>0}|u_{xx}(x,t)| &\le C, & \sup_{x\in\mathbb{R},\,t>0}|u_t(x,t)| &\le C.\end{aligned}$$

Here C is a constant depending only on k, m, and $\|\phi'\|_{W^{1,\infty}(\mathbb{R})}$.

Remark 5.2. *Existence of global solutions to the problem* (1) - (2) *is already obtained in* [6] *and* [23]. *Chou & Kwong* [6] *proved it for a smooth initial value without the restriction of growth order. Ninomiya & Taniguchi* [23] *also showed that, for an initial value $u_0(x)=\Phi(x;c,k)+\phi(x),\phi\in BC^1$, the solution $u(x,t)$ of* (1) - (2) *exists globally in time and satisfies*

$$u(x,t)-(\Phi(x;c,k)+ct)\in BC^1\quad\text{for each}\ \ t>0,$$

where $BC^1=C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$. In Proposition 5.1, we established the global existence of solutions with more detailed estimates of solutions, which are suitable and essential for our later discussions.

In what follows, we always assume that an initial value or an initial perturbation belongs to $C^{2+\gamma}(\mathbb{R})$ even if it is not mentioned specifically. Here $\gamma\in(0,1)$ is an arbitrary constant. We state some lemmas for the proof.

Lemma 5.3. *Assume $k \leq 0$. Let $M > 0$, $s > 0$, $T > 0$, and $L > 0$ be given constants. Let $u(x,t)$ be the solution to the Cauchy problem* (11) - (12) *with*

$$\|\phi'\|_{L^\infty(\mathbb{R})} \leq M, \quad \sup_{x\in\mathbb{R}} \phi(x) \leq s, \tag{13}$$

$$u(a,t) \leq 0 \quad \text{and} \quad u(b,t) \leq 0 \quad \text{for} \quad 0 \leq t \leq T \tag{14}$$

for $a, b \in \mathbb{R}$ with $a < b$ and $b - a \leq L$. Then there exists a positive constant λ depending only on M, s, T and L with

$$\max_{a\leq x\leq b} u(x,T) \ \leq \ s - \lambda, \tag{15}$$

where λ depends continuously on $s \in (0,+\infty)$ for any fixed M, T and L.

Next we show a simple lemma and provide an important property of the solution of (11) - (12) with an almost periodic function as an initial value. Roughly speaking, for each $t > 0$, such a solution has the same almost periodicity as that of the initial value.

Lemma 5.4. *Suppose that $\phi(x)$ is an almost periodic function that satisfies* (10) *as in Definition 2.3 with $\ell(\varepsilon)$. Let $u(x,t)$ be the solution of* (11) - (12). *Then, for every $p \in \mathbb{R}$, an interval $[p, p+\ell(\varepsilon)]$ contains at least one number q with*

$$|u(x-q,t) - u(x,t)| \ < \ \varepsilon \quad \text{for} \quad x \in \mathbb{R}, \ t > 0.$$

Now we prove the following result, which implies the asymptotic stability of traveling lines for an almost periodic function as an initial perturbation. The proof is done by deriving a contradiction.

Proposition 5.5. *Suppose that $\phi(x)$ is an almost periodic function. Then the solution $u(x,t)$ of* (11) - (12) *satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \mu| = 0 \tag{16}$$

for a constant μ that satisfies

$$\inf_{x\in\mathbb{R}} \phi(x) \ \leq \ \mu \ \leq \ \sup_{x\in\mathbb{R}} \phi(x). \tag{17}$$

Proof. Since all constants are stationary solutions of (11) - (12), the functions $U^+(t)$ and $U^-(t)$ defined by

$$U^+(t) = \sup_{x\in\mathbb{R}} u(x,t), \quad U^-(t) = \inf_{x\in\mathbb{R}} u(x,t)$$

are nonincreasing and nondecreasing, respectively by virtue of the comparison principle. The constants $U^* = \lim_{t\to\infty} U^+(t)$ and $U_* = \lim_{t\to\infty} U^-(t)$ exist. Now we define $\mu = (U^* + U_*)/2$ and $\delta = (U^* - U_*)/2$. Then μ satisfies (17) because we have $U^-(0) \leq U_* \leq \mu \leq U^* \leq U^+(0)$ by the definition.

Since $u(x,t) - \mu$ also satisfies (11) - (12), we may assume $\mu = 0$ without loss of generality. Moreover we may assume $k \leq 0$, since the case of $k \geq 0$ is reduced to the case of $k \leq 0$ by considering $-u(-x,t)$, which also satisfies (11) - (12).

It suffices to show $\delta = 0$. In what follows, we derive a contradiction by assuming $\delta > 0$. Suppose that ϕ satisfies (10) as in Definition 2.3 with $\ell(\varepsilon)$. We define a constant $L = \ell(\delta/4)$ depending only on ϕ and δ.

Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ be arbitrarily fixed. Then we have some points $a \in [x_0 - L - 1, x_0 - 1]$ and $b \in [x_0 + 1, x_0 + L + 1]$ with

$$u(a, t_0) < -\frac{\delta}{4} \quad \text{and} \quad u(b, t_0) < -\frac{\delta}{4}. \tag{18}$$

Indeed, by the definition of δ, we can take some point $x_* \in \mathbb{R}$ with $u(x_*, t_0) < -\delta/2$. By virtue of Lemma 5.4, the interval $[x_* - x_0 + 1, x_* - x_0 + 1 + L]$ contains a number q with

$$|u(x_* - q, t_0) - u(x_*, t_0)| \ < \ \frac{\delta}{4}.$$

Setting $a = x_* - q$, we have $a \in [x_0 - L - 1, x_0 - 1]$ and

$$u(a, t_0) \ < \ u(x_*, t_0) + \frac{\delta}{4} \ < \ -\frac{\delta}{4},$$

which is the first inequality of (18). Similarly we can take a number $b \in [x_0 + 1, x_0 + L + 1]$ for the second inequality of (18).

Now we use the uniform bound for $|u_t|$ obtained by Proposition 5.1. Using this estimate, we obtain

$$u(a, t) \leq 0 \quad \text{and} \quad u(b, t) \leq 0 \quad \text{for} \quad t_0 \leq t \leq t_0 + T$$

for some positive constant T depending only on ϕ and δ. Here we shall find a positive-valued function $\lambda(s)$ for $s > 0$ with

$$u(x_0, t_0 + T) \ \leq \ U^+(t_0) - \lambda\left(U^+(t_0)\right).$$

Using Lemma 5.3, we can choose $\lambda(\cdot)$ so that $\lambda \in C(0, +\infty)$ and that it depends only on $\|\phi'\|_{L^\infty(\mathbb{R})}$, T and $2(L+1)$. If $U^+(t_0) \geq \delta/2$, we get

$$u(x_0, t_0 + T) \ \leq \ U^+(t_0) - \lambda\left(U^+(t_0)\right) \ \leq \ U^+(t_0) - \lambda_0.$$

Here a constant λ_0 is defined by

$$\lambda_0 = \min_{s \in [\delta/2, U^+(t_0)]} \lambda(s).$$

Note that λ_0 is well defined and is positive. Since $x_0 \in \mathbb{R}$ is arbitrary and λ_0 is independent of x_0, we obtain $U^+(t_0 + T) \leq U^+(t_0) - \lambda_0$.

If $U^+(t_0) - \lambda_0 \geq \delta/2$, the same argument can be carried out at $t = t_0 + T$. Namely, for any fixed $x_0 \in \mathbb{R}$, we have

$$u(x_0, t_0 + 2T) \ \leq \ U^+(t_0) - \lambda_0 - \lambda\left(U^+(t_0) - \lambda_0\right) \ \leq \ U^+(t_0) - 2\lambda_0.$$

Consequently, we find that $U^+(t_0 + nT) < \delta/2$ for some large n. It follows that $U^* < \delta/2$ because $U^+(t)$ is a nonincreasing. This contradicts the definition of δ. Thus $\delta = 0$ follows, and the proof of Proposition 5.5 is completed. □

Remark 5.6. *In the case of $m = 0$, that is, in the case of traveling line $u(x, t) = kt$, the constant μ is a nondecreasing function of $k \in \mathbb{R}$ for each ϕ. For any fixed ϕ, let $u_1(x, t)$ and $u_2(x, t)$ be the solutions to the problem* (11) - (12) *with $m = 0$ for $k = k_1$ and $k = k_2$, respectively. Then there exist constants μ_1 and μ_2 with*

$$\limsup_{t \to \infty} \sup_{x \in \mathbb{R}} |u_j(x, t) - \mu_j| = 0, \quad j = 1, 2.$$

It suffice to show $\mu_1 \geq \mu_2$ *if* $k_1 \geq k_2$. *If* $k_1 \geq k_2$, *we have*

$$\begin{aligned} 0 &= (u_1)_t - (\arctan(u_1)_x)_x - k_1(\sqrt{1+(u_1)_x^2} - 1) \\ &\leq (u_1)_t - (\arctan(u_1)_x)_x - k_2(\sqrt{1+(u_1)_x^2} - 1), \end{aligned}$$

which implies that u_1 *is a supersolution of* (11) - (12) *with* $m = 0$ *and* $k = k_2$. *Thus* $u_1(x,t) \geq u_2(x,t)$ *holds true, and hence* $\mu_1 \geq \mu_2$ *follows.*

Remark 5.7. *In the case of* $k = 0$, *that is, in the case of the stationary line* $u(x,t) = mx$ *in the curve shortening flow,* $\mu = \mathcal{M}\{\phi\}$ *holds true. Indeed, for any fixed* $t > 0$, *we have*

$$\begin{aligned} \frac{1}{R}\left|\int_0^R (u(x,t) - \phi(x))dx\right| &= \frac{1}{R}\left|\int_0^R \left(\int_0^t u_t(x,s)\,ds\right) dx\right| \\ = \frac{1}{R}\left|\int_0^R \left(\int_0^t (\arctan(u_x+m))_x\,ds\right) dx\right| &\leq \frac{1}{R}\int_0^t \left|[\arctan(u_x+m)]_0^R\right| ds \leq \frac{\pi t}{R}. \end{aligned}$$

Therefore we obtain $|\mathcal{M}\{u\}(t) - \mathcal{M}\{\phi\}| \leq \lim_{R\to\infty} \pi t/R = 0$. *It follows that*

$$U^-(t) \leq \mathcal{M}\{u\}(t) \equiv \mathcal{M}\{\phi\} \leq U^+(t), \quad t > 0,$$

and hence $U_* = \mathcal{M}\{\phi\} = U^*$ *in the limit* $t \to \infty$.

Proof of Theorem 2.4. Theorem 2.4 follows directly from Proposition 5.1, Proposition 5.5, Remark 5.6, and Remark 5.7. □

REFERENCES

[1] A.S.Besicovitch, *Almost periodic functions*, Dover Publications, Inc., New York, 1955.

[2] H.Bohr, *Almost periodic functions*, Chelsea, New York, NY, 1951.

[3] P.K.Brazhnik, *Exact solutions for the kinematic model of autowaves in two-dimensional excitable media*, Physica D **94** (1996), 205–220.

[4] Y.-G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solu-tions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), 749–786.

[5] X.Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations **96** (1992), 116–141.

[6] K.-S.Chou and Y.-C.Kwong, *On quasilinear parabolic equations which admit global solutions for initial data with unrestricted growth*, Calc. Var. Partial Differential Equations **12** (2001), 281–315.

[7] C.Corduneanu, *Almost periodic functions*, 2nd English edition, Chelsea, New York, NY, 1989.

[8] P.Collet and J.-P.Eckmann, *Space-time behaviour in problems of hydrodynamic type: a case study*, Nonlinearity **5** (1992), 1265–1302.

[9] K.Ecker and G.Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. **130** (1989), 453–471.

[10] S.-I.Ei and E.Yanagida, *Dynamics of interfaces in competition-diffusion systems*, SIAM J. Appl. Math. **54** (1994), 1355–1373.

[11] S.D.Eidel'man, *Parabolic systems*, North-Holland, Amsterdam-London; Wolters-Noordhoff, Groningen, 1969.

[12] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature*, J. Differential Geom. **33** (1991), 635–681.

[13] A.Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[14] M.Gage and R.Hamilton, *The shrinking of convex plane curves by the heat equation*, J. Differential Geom. **23** (1986), 69–96.

[15] M.Grayson, *The heat equation shrinks embedded plane curves to points*, J. Differential Geom. **26** (1987), 285–314.

[16] F.Hamel, R.Monneau and J.-M.Roquejoffre, *Existence and qualitative properties of multi-dimensional conical bistable fronts,* preprint.

[17] G.Huisken, *Flow by mean curvature of convex surface into surface,* J. Differential Geom. **20** (1984), 237–266.

[18] N.Ishimura, *Curvature evolution of plane curves with prescribed opening angle,* Bull. Austral. Math. Soc. **52** (1995), 287–296.

[19] S.Kamenomostskaya, *On stabilization of solution of the Cauchy problem for parabolic equations,* Proceedings of the Royal Society of Edinburgh, **76A**, (1976), 43–53.

[20] V.P.Mihailov, *On stabilization of the solution of the Cauchy problem for the heat conduction equation,* Soviet Math. Dokl., **11** (1970), 34–37.

[21] M.Nara and M.Taniguchi, *Stability of a traveling wave in curvature flows for spatially non-decaying perturbations,* Discrete and Continuous Dynamical Systems, **14** (2006), 203–220.

[22] M.Nara and M.Taniguchi, *Convergence to V-shaped fronts in curvature flows for spatially non-decaying perturbations, to appear in* Discrete and Continuous Dynamical Systems.

[23] H.Ninomiya and M.Taniguchi, *Traveling curved fronts of a mean curvature flow with constant driving force,* Free boundary problems: theory and applications, I, Math. Sci. Appl., GAKUTO Internat. Ser. **13** (2000), 206–221.

[24] H.Ninomiya and M.Taniguchi, *Stability of traveling curved fronts in a curvature flow with driving force,* Methods Appl. Anal. **8** (2001), 429–450.

[25] H.Ninomiya and M.Taniguchi, *Existence and global stability of traveling curved fronts in the Allen-Cahn equations,* J. Differential Equations, **213** (2005), 204–233.

[26] O.A.Ladyženskaja, V.A.Solonnikov and N.N.Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type,* Translations of Mathematical Monograph 23, Amer. Math. Soc., Providence, RI, 1968.

[27] V.Pérez-Muñuzuri, M.Gómez-Gesteira, A.P.Muñuzuri, V.A.Davydov and V.Pérez-Villar, *V-shaped stable nonspiral patterns,* Physical Review E **51-2** (1995), 845–847.

[28] P.Poláčik and E.Yanagida, *Nonstabilizing solutions and grow-up set for a supercritical semilinear diffusion equation,* Differential Integral Equations **17** (2004), 535–548.

[29] V.D.Repnikov and S.D.Eidel'man, *A new proof of the theorem on the stabilization of the solution of the Cauchy problem for the heat equation,* Math. USSR-Sb., **2**, (1967), 135–139.