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MATHEMATICAL ANALYSIS FOR CHEMOTAXIS GROWTH SYSTEM

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1. INTRODUCTION

We are concerned with the following initial value problem for a quasilinear parabolic system

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla\{u\nabla\chi(\rho)\} + f(u) & \text{in}\quad \Omega\times(0,\infty), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - c\rho + du & \text{in}\quad \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on}\quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ \ \rho(x,0) = \rho_0(x) & \text{in}\quad \Omega. \end{cases}$$

This problem was presented by Mimura and Tsujikawa [1] to describe a process of pattern formation by some chemotactic biological individuals (for detail see [2, 3]). Here, $u(x,t)$ denotes the population density of biological individuals at a position $x \in \Omega \subset \mathbb{R}^2$ and time $t \in [0,\infty)$ respectively, and $\rho(x,t)$ the concentration of chemical substance at $x \in \Omega$ and $t \in [0,\infty)$. Biological individuals have, besides random walking, a tendency to move toward higher concentration of the chemical substance. Such an effect is called chemotaxis in biology. The chemical substance is produced by the biological individuals themselves. $a > 0$ and $b > 0$ are diffusion rates of u and ρ respectively. $c > 0$ and $d > 0$ are degradation and production rates of ρ respectively. $\chi(\rho)$ is a function of $0 < \rho < \infty$ representing the chemotactic sensitivity of biological individuals. The functions $\nu\rho$, $\nu\rho^2$, $\frac{\nu\rho}{\rho+1}$, $\nu\log\rho$ etc. are prototypes of the sensitivity function, where $\nu > 0$ is a chemotactic coefficient. $f(u)$ is a function of $0 \le u < \infty$ representing the growth rate of u. The functions $(1-u)u$, $(1-u)(u-\theta)u\,(0<\theta<1)$ etc. are prototypes of the growth function. Initial functions are taken from the space

$$(1.2)\qquad \mathcal{K} = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\ 0 \le L_2(\Omega) \quad \text{and} \quad 0 \le \rho \in H_N^2(\Omega) \right\},$$

where $H_N^2(\Omega)$ is a closed subspace of $H^2(\Omega)$ consisting of functions satisfying the Neumann boundary conditions on the boundary $\partial\Omega$. Ω is a two-dimensional bounded domain.

The authors have already published various mathematical results for this model. In this report we intend to review these results which have been obtained not only by the present authors but also in collaboration with M. Aida, M. Efendiev and K. Osaki.

Throughout this report, we assume that Ω is a bounded, $\mathcal{C}^2$ or convex domain of $\mathbb{R}^2$. $\chi(\rho)$ is a smooth function for $0 \le \rho < \infty$ satisfying

$$(1.3)\qquad \sup_{0\le\rho<\infty} |\chi^{(i)}(\rho)| < \infty, \qquad i = 1,\, 2.$$

$f(u)$ is a smooth function for $0 \le u < \infty$ satisfying the conditions $f(0) = 0$ and

$$f(u) = (-\mu' u + \nu')u \qquad \text{for sufficiently large } u \tag{1.4}$$

with some constants $\mu', \nu' > 0$.

In Section 5, we shall consider the case when the sensitivity function $\chi(\rho)$ has a singularity at $\rho = 0$.

2. Global solutions

2.1. Local solutions. Local solutions to (1.1) can be constructed by directly applying the theory of abstract parabolic evolution equations.

Setting an underlying space

$$X = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^1(\Omega)^* \quad \text{and} \quad \rho \in H^1(\Omega) \right\},$$

we formulate the problem (1.1) as the Cauchy problem of an abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{2.1}$$

in the space X. Here, $A = \begin{pmatrix} A_1 & 0 \\ -d & A_2 \end{pmatrix}$ is a linear operator of X, A_1 (res. A_2) being a realization of the differential operator $-a\Delta + 1$ (resp. $-b\Delta + c$) in $H^1(\Omega)^*$ (resp. $H^1(\Omega)$) under the Neumann boundary conditions. And $F(U)$ is a nonlinear operator of X of the form $F(U) = \begin{pmatrix} -\nabla \cdot \{u \nabla \chi(\rho)\} + u + f(u) \\ 0 \end{pmatrix}$. U_0 is an initial value from $\mathcal{K}$.

If $F(U)$ satisfies the Lipschitz condition

$$\begin{aligned} \|F(U) - F(V)\| \le{}& \varphi(\|A^\beta U\| + \|A^\beta V\|) \\ & \times \{\|A^\eta (U - V)\| + (\|A^\eta U\| + \|A^\eta V\|)\|A^\beta (U - V)\|\}, \quad U, V \in \mathcal{D}(A^\eta) \end{aligned} \tag{2.2}$$

with some exponents $0 < \beta < \eta < 1$ and some increasing function $\varphi(\cdot)$, then the problem (2.1) has, for any $U_0 \in \mathcal{D}(A^\beta)$, a unique local solution $U(t)$ in the function space

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A^\beta)) \cap \mathcal{C}^1((0, T_{U_0}]; X),$$

here the positive time $T_{U_0} > 0$ is determined by the norm $\|A^\beta U_0\|$ alone, see [4]. In the present case, the condition (2.2) is fulfilled with exponents $\beta = \frac{1}{2} < \eta < 1$ and the domain $\mathcal{D}(A^\beta) = \mathcal{D}(A^{\frac{1}{2}})$ is characterized by the product space of $u \in L_2(\Omega)$ and $\rho \in H^2_N(\Omega)$. Therefore, for any initial value $U_0 \in \mathcal{K}$, the problem (2.1) has a unique local solution on an interval $[0, T_{U_0}]$. Furthermore, by the truncation method, it is possible to show that the nongegativity of initial functions implies that of the local solution. In this way, we conclude the following results. For each initial value $U_0 \in \mathcal{K}$, (2.1) and hence (1.1) has a unique local solution $U(t)$ in the function space

$$\begin{cases} 0 \le u \in \mathcal{C}((0, T_{U_0}]; H^1(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; L^2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; H^1(\Omega)^*), \\ 0 \le \rho \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A_2)) \cap \mathcal{C}([0, T_{U_0}]; H^2_N(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; H^1(\Omega)), \end{cases} \tag{2.3}$$

where $T_{U_0} > 0$ is determined by the norm $\|u_0\|_{L_2} + \|\rho_0\|_{H^2}$ alone.

2.2. *A priori* **estimates.** In order to construct a global solution we have to prove *a priori* estimates for local solutions. Let $U_0 \in \mathcal{K}$ and let $U(t)$ be any nonnegative local solution of (2.1) with the initial value U_0 on interval $[0, T_U]$ in the function space (2.3) where T_{U_0} is replaced by T_U. Then for the local solution it holds that

$$\|A^{\frac{1}{2}}U(t)\| \leq p(\|A^{\frac{1}{2}}U_0\|), \quad 0 \leq t \leq T_U, \tag{2.4}$$

where $p(\cdot)$ is some increasing continuous function independent of $U(t)$. This energy estimate is established by utilizing the second mean value theorem of integration and the embedding estimate

$$\|u\|_{L_3}^3 \leq \varepsilon \|u\|_{H^1}^2 \|u \log(u+1)\|_{L_1} + C_\varepsilon \|u\|_{L_1}, \qquad 0 \leq u \in H^1(\Omega)$$

with an arbitrary number $\varepsilon > 0$. For the detailed proof, see [5, 6].

2.3. **Global solutions.** Now that we have the a priori estimate (2.4) for local solutions, a global solution to (2.1) can be constructed. In fact, let $U_0 \in \mathcal{K}$ and let $U(t)$ be any nonnegative local solution of (2.1) on interval $[0, T_U]$ in the function space (2.3). Since $U(T_U) \in \mathcal{D}(A) \subset \mathcal{D}(A^{\frac{1}{2}})$, we can extend the local solution to a new nonnegative local solution over an interval $[0, T_U + \tau]$, here the extension length $\tau > 0$ is determined by $\|A^{\frac{1}{2}}U(T_U)\|$ alone and therefore determined by $p(\|A^{\frac{1}{2}}U_0\|)$ alone. Repeating this procedure, we obtain for each $U_0 \in \mathcal{K}$ a global solution $U(t)$ in the function space

$$\begin{cases} 0 \leq u \in \mathcal{C}((0,\infty); H^1(\Omega)) \cap \mathcal{C}((0,\infty); L^2(\Omega)) \cap \mathcal{C}^1((0,\infty); H^1(\Omega)^*), \\ 0 \leq \rho \in \mathcal{C}((0,\infty); \mathcal{D}(A_2)) \cap \mathcal{C}((0,\infty); H^2_N(\Omega)) \cap \mathcal{C}^1((0,\infty); H^1(\Omega)). \end{cases} \tag{2.5}$$

Furthermore, we can establish an absorbing global estimate

$$\|u(t)\|_{H^1} + \|A_2\rho(t)\|_{H^1} \leq (1 + t^{-\frac{1}{2}})e^{-\delta t} p(\|u_0\|_{L_2} + \|\rho_0\|_{H^2}) + C, \quad 0 < t < \infty, \tag{2.6}$$

with some exponent $\delta > 0$ and a constant $C > 0$ independent of initial values U_0's.

As well, we can establish a Lipschitz continuity of solution with respect to the initial values in the sense that, for any fixed $0 < T < \infty$ and $0 < R < \infty$,

$$\begin{aligned} t^{\frac{1}{2}}[\|u(t) - \widetilde{u}(t)\|_{H^1} + \|A_2\{\rho(t) - \widetilde{\rho}(t)\}\|_{H^1}] + \|u(t) - \widetilde{u}(t)\|_{L_2} + \|\rho(t) - \widetilde{\rho}(t)\|_{H^2} \\ \leq C_{T,R}[\|u_0 - \widetilde{u}_0\|_{L_2} + \|\rho_0 - \widetilde{\rho}_0\|_{H^2}], \qquad 0 < t \leq T;\ U_0, \widetilde{U}_0 \in \mathcal{K}_R, \end{aligned} \tag{2.7}$$

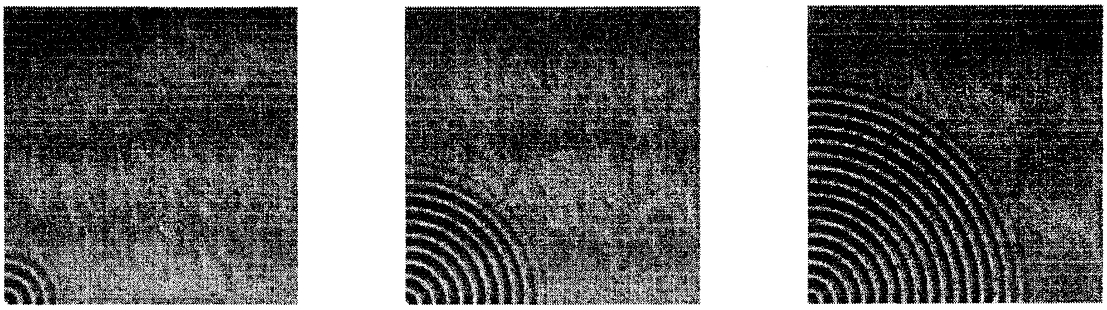
where $\mathcal{K}_R = \{U_0 \in \mathcal{K};\ \|A^{\frac{1}{2}}U_0\| \leq R\}$.

3. Numerical results

In this section, we present various numerical results for the problem (1.1), see [7, 8, 9].

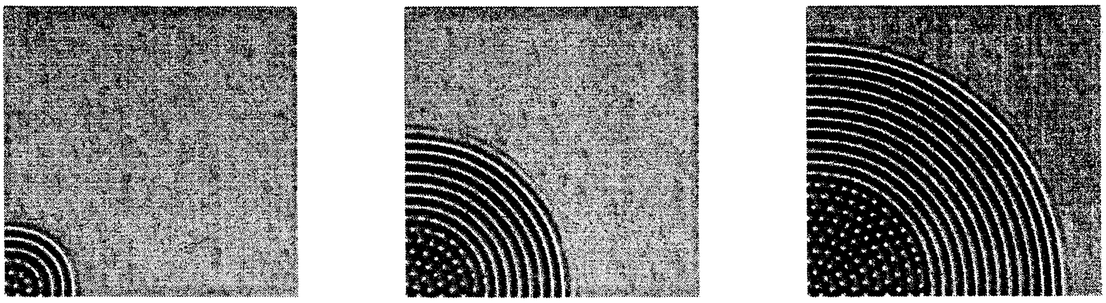
First, two kinds of target patterns are presented, cf. [8]. Let $\Omega = [0, 32] \times [0, 32]$ and take the coefficients as $a = 0.0625$, $b = 1.0$, $c = 64.0$, $d = 1.0$ and the functions as $\chi(\rho) = \nu\rho$ and $f(u) = u(u - 0.0625)(1 - u)$. When $\nu = 11.0$, we have target pattern with continuous rings, Fig. 1. When $\nu = 32.0$, we have target pattern with perforated rings, Fig. 2. These are transit patterns, when the most outside ring reaches at the boundary of Ω, the target patterns disappear and change types. These numerical solutions show good agreement with the experimental results due to [3].

Secondly, we present some stationary patterns, cf. [9]. Let $\Omega = [0, 16] \times [0, 16]$ and $a = 0.0625$, $b = 1.0$, $c = 64.0$, $d = 1.0$, and take the functions as $\chi(\rho) = \nu\rho$ and $f(u) =$



(a) $t = 16$ (b) $t = 32$ (c) $t = 48$

FIGURE 1. Target pattern with continuous rings for $\nu = 11.0$.

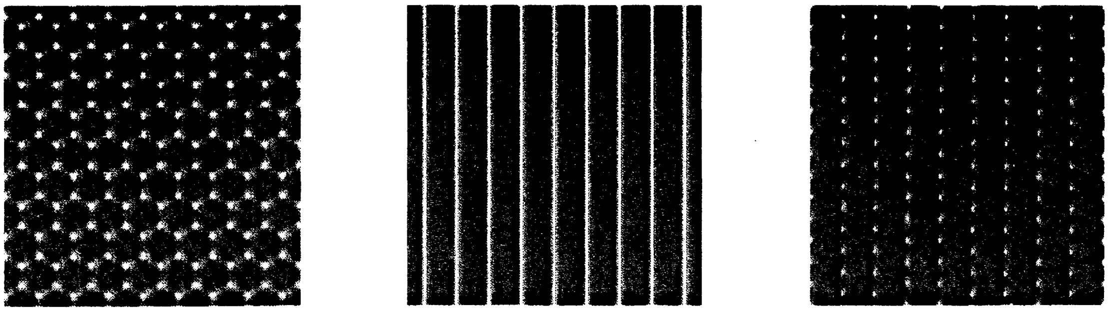


(a) $t = 8$ (b) $t = 16$ (c) $t = 24$

FIGURE 2. Target pattern with perforated rings for $\nu = 32.0$.

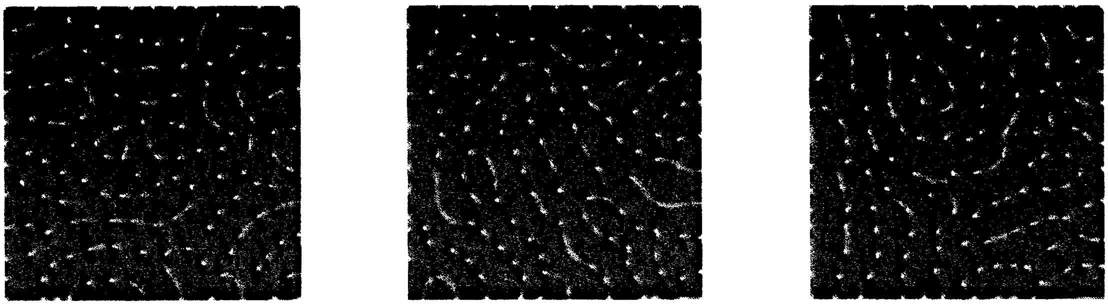
$u^2(1-u)$. When ν is sufficiently small, the homogeneous stationary solution $(\overline{u}, \overline{\rho}) \equiv (1, 1/32)$ is stable and no pattern solution is observed. When $\nu = 6.2$, the homogeneous stationary solution is no longer observed numerically and an inhomogeneous stationary solution with honeycomb structure comes out as in Fig. 3 (a). As ν increases from 6.2 to 8.5, the inhomogeneous stationary solution changes its types from honeycomb to perforated stripe as Fig. 3 shows.

When ν still increases, such an inhomogeneous and ordered stationary solution is no longer found. That may lose its stability or completely vanish. Instead some moving pattern comes out. When $\nu = 9.0$, moving perforated labyrinthine pattern is observed as in Fig. 4. When $\nu = 11.0$, a chaotic spot pattern is observed as in Fig. 5. Each spot continues to move in a chaotic manner, a few spots are combined here and there in Ω, on the other hand some new spots are generated to conserve the total number of spots at every moment.



(a) Honeycomb ($\nu = 6.2$) (b) Stripe ($\nu = 7.2$) (c) Perforated Stripe ($\nu = 8.5$)

FIGURE 3. Stationary patterns for $\nu = 6.2$, $\nu = 7.2$ and $\nu = 8.5$.



(a) $t = 1024$ (b) $t = 2048$ (c) $t = 4096$

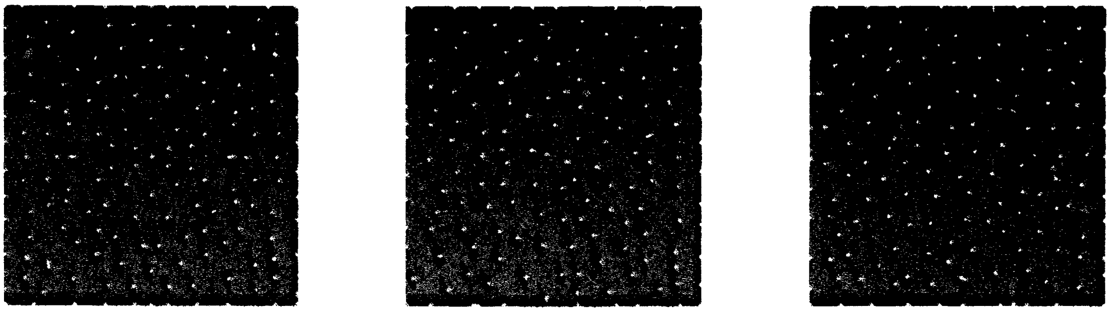
FIGURE 4. Moving perforated labyrinthine pattern for $\nu = 9.0$.

4. DYNAMICAL SYSTEM

From the global solution to the Cauchy problem (2.1) we define a nonlinear semigroup $S(t)U_0 = U(t; U_0)$ acting on $\mathcal{K}$, where $U(t; U_0)$ denotes the global solution of (2.1) with initial value $U_0 \in \mathcal{K}$. We equip $\mathcal{K}$ with the induced distance $d_{\mathcal{K}}$ from the space $\mathcal{D}(A^{\frac{1}{2}})$ which is characterized as a product space of $u \in L_2(\Omega)$ and $\rho \in H^2_N(\Omega)$. In this sense, $(\mathcal{K}, d_{\mathcal{K}})$ is a metric space. As shown by (2.7), $S(t)$ is continuous from $\mathcal{K}$ into itself. So, $(S(t), \mathcal{K}, \mathcal{D}(A^{\frac{1}{2}}))$ is a dynamical system with phase space $\mathcal{K}$ in the universal space $\mathcal{D}(A^{\frac{1}{2}})$.

4.1. **Exponential attractors.** The absorbing estimate (2.6) shows that, if we set

$$\mathcal{B} = \{U \in \mathcal{D}(A^{\frac{1}{2}});\ \|u\|_{H^1} + \|A_2\rho\|_{H^1} \le C + 1\} \cap \mathcal{K},$$



(a) $t = 64$ (b) $t = 128$ (c) $t = 256$

FIGURE 5. Chaotic spot pattern for $\nu = 11.0$.

then this subset of $\mathcal{K}$ is an absorbing set of the semigroup $S(t)$. Obviously, $\mathcal{B}$ is a compact set. These facts then yield that the dynamical system $(S(t), \mathcal{K}, \mathcal{D}(A^{\frac{1}{2}})$ possesses a global attractor $\mathcal{A}$ which is characterized by the following properties:

(1) $\mathcal{A} \subset \mathcal{K}$ and $\mathcal{A}$ is a compact set of $\mathcal{D}(A^{\frac{1}{2}})$;
(2) $\mathcal{A}$ is a strictly invariant set of $S(t)$, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for every $0 \le t \le< \infty$;
(3) For any bounded subset $B \subset \mathcal{K}$, it holds that $\lim_{t\to\infty} h(S(t)B, \mathcal{A}) = 0$, where $h(B_1, B_2) = \sup_{U\in B_1} \inf_{V\in B_2} d_{\mathcal{K}}(U, V)$ denotes a pseudo distance.

Furthermore we can construct a family of exponential attractors for $(S(t), \mathcal{K}, \mathcal{D}(A^{\frac{1}{2}}))$. A subset $\mathcal{M}$ of $\mathcal{K}$ is called an exponential attractor if it has the following properties:

(1) $\mathcal{A} \subset \mathcal{M} \subset \mathcal{K}$ and $\mathcal{M}$ is a compact set of $\mathcal{D}(A^{\frac{1}{2}})$ with finite fractal dimension;
(2) $\mathcal{M}$ is an invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $0 \le t \le< \infty$;
(3) There exists an exponent $\delta > 0$ such that, for any bounded subset $B \subset \mathcal{K}$, it holds that $h(S(t)B, \mathcal{M}) \le C_B e^{-\delta t}$ with some constant $C_B > 0$ dependent on B.

From the definition we see that $\mathcal{M}$ attracts all the trajectories starting from a bounded set at an exponential rate. On the other hand, we see that the exponential attractor is not uniquely determined in general. Indeed, if $\mathcal{M}$ is an exponential attractor, then $\mathcal{M}_T = S(T)\mathcal{M}$ can also be an exponential attractor for any $0 < T < \infty$. It is known that the exponential attractor enjoys stronger robustness as a limit set of dynamical system rather than the global attractor, see [10, 11].

In constructing the exponential attractor we follow the general method due to [12]. They showed that, if $S(t)$ fulfills a compact Lipschitz condition of the form

$$\|S(t^*)U_0 - S(t^*)V_0\|_Z \le L\|A^{\frac{1}{2}}(U_0 - V_0)\|, \qquad U_0, V_0 \in \mathcal{B}, \tag{4.1}$$

for some time $t^* > 0$, where Z is another Banach space such that Z is compactly embedded in $\mathcal{D}(A^{\frac{1}{2}})$, then one can construct a family of exponential attractors. In the present case it is immediate in view of (2.7) to verify this compact Lipschitz condition, see [6].

As a matter of fact, when Ω is of $\mathcal{C}^3$ class, we verify the squeezing property of $S(t)$ introduced in [13] which is a stronger condition than (4.1). In this case, one can construct a lower dimensional exponential attractor than before, see [5].

4.2. **Instability of homogeneous stationary solution.** In this subsection we assume for simplicity that $\chi(\rho) = \nu\rho$ and that $f(1) = 0$ with $f'(1) < 0$. In addition,

(4.2) $\qquad f(u)$ is real analytic in a neighborhood of $u = 1$.

We immediately verify that $(u, \rho) \equiv (1, d/c)$ is a stationary solution to (2.1). Therefore $\overline{U} = \begin{pmatrix} 1 \\ d/c \end{pmatrix}$ is an equilibrium of the dynamical system $(S(t), \mathcal{X}, \mathcal{D}(A^{\frac{1}{2}}))$. We are concerned with stability and instability of $\overline{U}$.

The stable manifold $\mathcal{M}_{(}\overline{U})$ of $\overline{U}$ is defined by

$$\mathcal{M}_-(\overline{U}) = \{U_0 \in \mathcal{X};\ \lim_{t\to\infty} S(t)U_0 = \overline{U}\}.$$

Meanwhile, the unstable manifold $\mathcal{M}_+(\overline{U})$ is defined by

$$\begin{aligned}\mathcal{M}_+(\overline{U}) = \{U_0 \in \mathcal{X};\ \exists U{:}(-\infty, 0] \to \mathcal{X},\ S(t)U(-\tau) = U(t-\tau) \ \text{ for } \ 0 \le t \le \tau,\\ U(0) = U_0 \ \text{ and } \ \lim_{t\to\infty} U(-t) = \overline{U}\}.\end{aligned}$$

By definition it is easy to see that

$$S(t)(\mathcal{M}_-(\overline{U})) \subset \mathcal{M}_-(\overline{U}) \qquad \text{and} \qquad S(t)(\mathcal{M}_+(\overline{U})) = \mathcal{M}_+(\overline{U}).$$

In addition, it is true that $\mathcal{M}_+(\overline{U}) \subset \mathcal{A}$.

We can then show that if $\sqrt{d\nu} < \sqrt{ac} + \sqrt{-bf'(1)}$ (that is, the chemotaxis is smaller than the diffusions of u and ρ), then $\mathcal{M}_+(\overline{U}) = \{\overline{U}\}$ and the equilibrium $\overline{U}$ is stable. In the meantime, if $d\nu$ is sufficiently large, then $\mathcal{M}_+(\overline{U}) \neq \{\overline{U}\}$, namely, $\overline{U}$ becomes unstable.

These results are proved by some representation of $\mathcal{M}_\pm(\overline{U})$. In fact, under (4.2), if the linearized operator $\overline{A} = \begin{pmatrix} -a\Delta - f'(1) & -\nu\Delta \\ -d & -b\Delta + c \end{pmatrix}$ of $A - F(\cdot)$ at $\overline{U}$ considered in X satisfies the spectral separation condition

$$\sigma(\overline{A}) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda = 0\} = \emptyset,$$

then one can prove that $\mathcal{M}_\pm(\overline{U})$ are smooth manifolds in a neighborhood of $\overline{U}$. Furthermore, corresponding the separation $\sigma_\pm = \sigma(\overline{A}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \lesseqgtr 0\}$, not only the universal space X but also the operator $\overline{A}$ are decomposed into a direct sum in such a way that $X = X_+ + X_-$ and $\overline{A} = \overline{A}_+ + \overline{A}_-$, $\overline{A}_\pm$ being closed linear operators of $X_\pm$ with $\sigma(\overline{A}_\pm) = \sigma_\pm$, respectively. In addition, in a neighborhood of $\overline{U}$, there exists one to one correspondence between the points of $\mathcal{M}_\pm(\overline{U})$ and the points in $\overline{U} + X_\pm$, and $\mathcal{M}_\pm(\overline{U})$ are tangential to $X_\pm$, respectively, at $\overline{U}$. By these representation results, we verify a lower estimate of the unstable manifold $\mathcal{M}_+(\overline{U})$, namely, $\dim \mathcal{M}_+(\overline{U}) \ge \dim X_+$. Furthermore, we conclude in the present case that, as $d\nu \to \infty$, $\dim \mathcal{M}_+(\overline{U}) \to \infty$. Consequently, as $d\nu \to \infty$, $\dim \mathcal{M} \ge \dim \mathcal{A} \ge \dim \mathcal{M}_+(\overline{U}) \to \infty$. For the detailed proof, see [9].

5. Case of singular sensitivity function

In this section we assume instead of (1.3) that the sensitivity function satisfies the condition

(5.1) $$\sup_{\delta \le \rho < \infty} |\chi^{(i)}(\rho)| < \infty, \qquad 0 < \delta < 1;\ i = 1, 2.$$

A typical example is the function $\chi(\rho) = \nu \log \rho$ or $-\nu/\rho$.

5.1. **Dynamical system.** In the present case, the space of initial functions is chosen as

$$\mathcal{K}_+ = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} ;\ 0 \le L_2(\Omega) \quad \text{and} \quad 0 < \rho \in H^2_N(\Omega),\ \inf_{x\in\Omega} \rho_0(x) > 0 \right\},$$

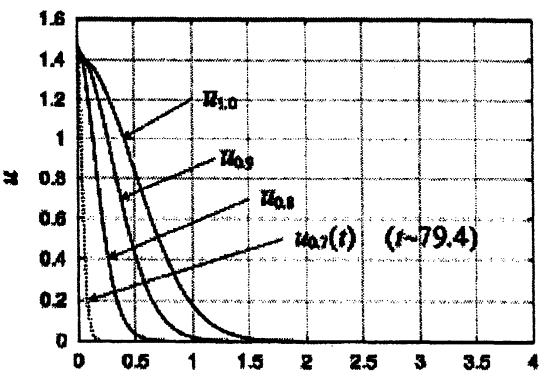
By the similar arguments, we can construct for each initial value $U_0 \in \mathcal{K}_+$ a unique global solution in the same function space as in (2.5). So, a dynamical system $(S(t), \mathcal{K}_+, \mathcal{D}(A^{\frac{1}{2}}))$ is defined as before. But we observe in a particular case ($f'(0) < 0$) that there is a trajectory which tends to $O = (0, 0)$ as $t\infty$. Since O is a boundary point of the phase space $\mathcal{K}_+$ which does not lie in, this means that $(S(t), \mathcal{K}_+, \mathcal{D}(A^{\frac{1}{2}}))$ admits no longer any compact limit set.

Concerning the asymptotic behavior of trajectory $S(t)U_0$, we can say the following. When $\inf_{0\le t<\infty} \|u(t)\|_{L_1} > 0$, the ω-limit set $\omega(U_0)$ of $S(t)U_0$ is nonempty and is contained in $\mathcal{K}_+$. Meanwhile, when $\inf_{0\le t<\infty} \|u(t)\|_{L_1} = 0$, the there exists a time sequence $t_n \nearrow \infty$ such that $S(t_n)U_0$ tends to $O = (0, 0)$ in a suitable norm, see [14].

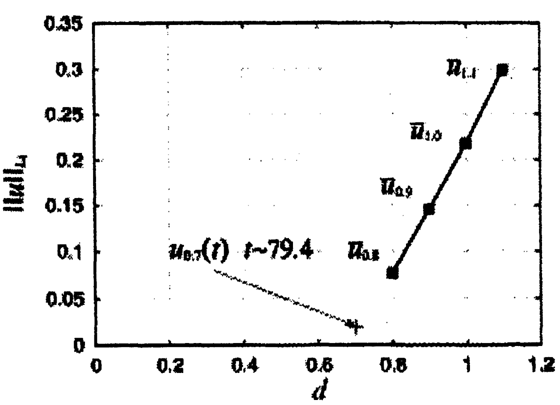
5.2. **Numerical results.** We are interested in the question of whether there exists a solution to (1.1) which tends to O as $t \to \infty$ or not, if $f'(0) > 0$. We shall present here some numerical results.

Let $\Omega = (0, 4)$ and $a = 0.25$, $b = 1$, $c = 6.25$, and take the functions as $\chi(\rho) = -\frac{0.125}{\rho}$ and $f(u) = u(1 - u)$.

For $d \ge 0.8$, we found a numerically stable stationary solution $\overline{U}_d$, Fig. 6(a) shows the graphs of $\overline{u}_d$'s. When $d = 0.7$, we computed the solution $U_{0.7}$ which starts from $\overline{U}_{0.8}$. $U_{0.7}$ is seen to approach to O for a while with L^1-norm of $u_{0.7}$ decaying as t, cf. Fig. 6(b). But when t is about 79.4, our computation of $U_{0.7}$ had lost its stability.



(a) Graphs



(b) L_1-norms

FIGURE 6. Stationary solutions $\overline{u}_d$ and $u_{0.7}(t)$ ($t \sim 79.4$).

This may not be satisfactory evidence to draw the conclusion that no stable stationary solution $\overline{U}_d$ exists for $d = 0.7$ and the solution $U_{0.7}$ tends to O as $t \to \infty$. But, we could say at least that $U_{0.7}$ does get close to O and that $U_{0.7} \to O$ as $t \to \infty$ if and only if the stable stationary solution $\overline{U}_d$ does not exist for $d = 0.7$. For the detailed arguments, see [14].

REFERENCES

[1] M. Mimura and T. Tsujikawa, *Aggregating pattern dynamics in a chemotaxis model including growth*, Physica A **230** (1996), 499-543.

[2] E. O. Budrene and H. C. Berg, *Complex patterns formed by motile cells of Escherichia coli*, Nature **349** (1991), 630–633.

[3] D. E. Woodward, R. Tyson, M. R. Myerscough, J. D. Murray, E. O. Budrene and H. C. Berg, *Spatio-temporal patterns generated by Salmonella typhimurium*, Biophys. J. **68** (1995), 2181-2189.

[4] K. Osaki and A. Yagi, *Global existence for a chemotaxis-growth system in* $\mathbb{R}^2$, Adv. Math. Sci. Appli. **12**(2002), 587-606.

[5] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, *Exponential attractor for a chemotaxis-growth system of equations*, Nonlinear Analysis **51** (2002), 119-144.

[6] M. Aida, M. Efendiev and A. Yagi, *Quasilinear abstract parabolic evolution equations and exponential attractors*, Osaka J. Math. **42**(2005), 101-132.

[7] M. Aida, *Global behaviour of solutions and pattern formation for chemotaxis-growth equations*, Doctoral thesis, Osaka University, December 2003 (in Japanese).

[8] M. Aida and A. Yagi, *Target pattern solutions for chemotaxis-growth system*, Math. Sci. Japonicae **59**(2004), 577-590.

[9] M. Aida, T. Tsujikawa, M. Efendiev, A. Yagi and M. Mimura, *Lower estimate of attractor dimension for chemotaxis growth system*, J. London Math. Soc., to appear.

[10] M. Aida and A. Yagi, *Global stability of approximation for exponential attractors*, Funkcialaj Ekvacioj **47** (2004), 251-276.

[11] M. Efendiev and A. Yagi, *Continuous dependence of exponential attractors for a parameter*, J. Math. Soc. Japan **57**(2005), 167–181.

[12] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion systems in* $\mathbb{R}^3$, C. R. Acad. Sci. Paris, Série I **330**(2000), 713-718.

[13] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, 1994, John Wiley & Sons, Chichester, New York.

[14] M. Aida, K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, *Chemotaxis and growth system with singular sensitivity function*, Nonlinear Analysis: Real World Applications **6**(2005), 323–336.