

Permanence of the periodic logistic system with periodic impulsive perturbations

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Abstract

Sufficient conditions for permanence of the periodic logistic system with periodic impulsive perturbations are obtained via comparison theory of impulsive differential equations.

Keywords: Impulses; Logistic system; Permanence; Periodic perturbation

1 Introduction

Usually, it is difficult to analyze the impulsive differential equations arisen from applications due to numerous theoretical and technical difficulties except that in some cases the models can be rewritten as simple discrete-time mapping or difference equations when the corresponding continuous models can be solved explicitly, eg. [1, 2]. This is the reason that numeric simulations are frequently used in applications. Recently, many investigations focus on the global dynamics of impulsive systems, see, for example [3–7].

Liu and Chen [4] studied the following logistic system with impulsive perturbations.

$$x'(t) = x(t)(r(t) - a(t)x(t)), t \neq \tau_k, k \in N, \quad (1.1)$$

$$\Delta x(\tau_k) = b_k x(\tau_k), k \in N, \quad (1.2)$$

where N is the set of positive integers, $\tau_0 \triangleq 0 < \tau_1 < \dots < \tau_k < \tau_{k+1} < \dots$, $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k)$, $r(\cdot), a(\cdot) \in PC[R, R]$ and $PC[R, R] = \{\phi : R \mapsto R, \phi \text{ is continuous for } t \neq \tau_k, \phi(\tau_k^+) \text{ and } \phi(\tau_k^-) \text{ exist and } \phi(\tau_k) = \phi(\tau_k^-), k \in N\}$. Suppose that (1.1) is ω -periodic and (1.2) is T -periodic, i. e.,

$$r(t + \omega) = r(t), a(t + \omega) = a(t), t \in R \quad (1.3)$$

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and T is the least positive constant such that there are l τ_k s in the interval $(0, T)$ and

$$\tau_{k+l} = \tau_k + T, b_{k+l} = b_k, k \in N. \quad (1.4)$$

The following additional restrictions on system (1.1), (1.2) are natural for biological meanings.

$$r(t) > 0, a(t) > 0, t \in R_+, \quad (1.5)$$

$$1 + b_k > 0, b_k \neq 0, k \in N. \quad (1.6)$$

When $b_k > 0$, the perturbation stands for planting of the species, while $b_k < 0$ stands for harvesting. We suppose that conditions (1.3)-(1.6) always hold in this paper. By the basic theories of impulsive differential equations in [8, 9], system (1.1), (1.2) has a unique solution $x(t) = x(t, x_0) \in PC[R, R]$ for each initial value $x(0) = x_0 \in R_+$ and further $x(t) > 0, t \in R_+$ if $x(0) = x_0 > 0$.

Let $\gamma \triangleq \omega/T$. When γ is rational, [4] showed that system (1.1), (1.2) has a unique positive periodic solution, which is a global attractor of all positive solutions if the following condition holds.

$$\mu = \prod_{0 < \tau_k < T} \left(\frac{1}{1 + b_k} \right)^\gamma e^{-\int_0^\omega r(\tau) d\tau} < 1. \quad (1.7)$$

And if (1.7) is reversed, then the zero solution is a global attractor. When γ is irrational, system (1.1), (1.2) has no periodic solutions. [4] established sufficient conditions for the positive solutions of system (1.1), (1.2) attracting each other and suggested that system (1.1), (1.2) has a positive global attractor which is not periodic. This is quite different from the corresponding continuous system. However, to guarantee the existence of a positive global attractor permanence should be established. The purpose of this paper is to show that system (1.1), (1.2) is permanent if (1.7) holds. Therefore ensure the existence result of Conjecture 3 in [4].

2 Permanence

We first give the definition of permanence.

Definition 2.1. *System (1.1), (1.2) is called permanent iff there exist positive constants $M > \delta$, such that any positive solution $x(t)$ of system (1.1), (1.2) satisfies*

$$\delta \leq \liminf_{t \rightarrow \infty} x(t) \leq \limsup_{t \rightarrow \infty} x(t) \leq M.$$

From Definition 2.1, we can see that permanence means that each positive solutions is ultimately bounded both above and below by some positive constants independent of the initial values of solution. [4, Theorem 3.1] has already established the following result of ultimate upper bound.

Lemma 2.1. *System (1.1), (1.2) is uniformly ultimately bounded above, i. e., there exists a constant $M > 0$ such that $x(t) \leq M$ for t sufficiently large, where $x(t)$ is any solution of system (1.1), (1.2) with $x(0) = x_0 > 0$.*

We now establish the ultimate lower bound of system (1.1), (1.2). If we express $nT, n \in N$ by ω , the following lemma is obviously valid.

Lemma 2.2. *For any $n \in N$, nT can be expressed by ω as*

$$nT = q_n \omega + s_n,$$

where $q_n \in N \cup \{0\}, s_n \in R_+, 0 \leq s_n < \omega$. Moreover, $\lim_{n \rightarrow \infty} n/q_n = \gamma, \lim_{n \rightarrow \infty} q_n = \infty$.

Remark 2.1. *If there exists $n_0 \in N$ such that $s_{n_0} = 0$, then γ is rational. And if $s_n > 0$ for any $n \in N$, then γ is irrational.*

Theorem 2.1. *Suppose that (1.7) holds. Then there exists a $\delta > 0$ such that*

$$\liminf_{t \rightarrow \infty} x(t) \geq \delta,$$

where $x(t)$ is any solution of system (1.1), (1.2) with initial value $x(0) = x_0 > 0$.

Proof. By (1.7), we can choose $\delta_1 > 0$ be sufficiently small such that

$$\prod_{0 < \tau_k < T} (1 + b_k)^\gamma e^{\int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} > 1.$$

As a consequence, by Lemma 2.2, there exist $\theta > 0, n_0 \in N$ such that

$$\prod_{0 < \tau_k < T} (1 + b_k)^{\frac{n}{q_n}} e^{\int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} > 1 + \theta, \quad (2.1)$$

for $n \geq n_0$. Denote

$$h = \min\{0, r(\tau) - a(\tau)\delta_1, \tau \in [0, \omega]\}, H = \max\{0, r(\tau) - a(\tau)\delta_1, \tau \in [0, \omega]\}.$$

Thus $h \leq 0 \leq H$. We will prove the result as the following two steps. We may suppose that $x_0 \leq \delta_1$ since step 1 can be skipped if $x_0 > \delta_1$.

Step 1. There exists a $t_0 > 0$ such that $x(t_0) > \delta_1$.

Suppose for the contrary that

$$x(t) \leq \delta_1,$$

for all $t \geq 0$. Then by (1.1), we have

$$x'(t) \geq x(t)(r(t) - a(t)\delta_1), t \geq 0, t \neq \tau_k, k \in N.$$

By the comparison result of scalar impulsive differential equations [8, 9], Lemma 2.2 and (2.1), we find that

$$\begin{aligned} x(nT) &\geq x_0 \prod_{0 < \tau_k < nT} (1 + b_k) e^{\int_0^{nT} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &= x_0 \prod_{0 < \tau_k < nT} (1 + b_k) e^{\int_0^{q_n \omega} (r(\tau) - a(\tau)\delta_1) d\tau} e^{\int_{q_n \omega}^{q_n \omega + s_n} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &= x_0 \prod_{0 < \tau_k < T} (1 + b_k)^n e^{q_n \int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} e^{\int_0^{s_n} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &= x_0 \left(\prod_{0 < \tau_k < T} (1 + b_k)^{\frac{n}{q_n}} e^{\int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} \right)^{q_n} e^{\int_0^{s_n} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &\geq x_0 e^{h\omega} (1 + \theta)^{q_n}, \end{aligned}$$

for $n \geq n_0$. Hence $x(nT) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Thus there exists a $t_0 > 0$ such that $x(t_0) > \delta_1$.

Step 2. Establish a positive ultimate lower bound $\delta \leq \delta_1$.

Let $t_0 > 0$ such that $x(t_0) > \delta_1$. If $x(t) \geq \delta_1$ for all $t \geq t_0$, then our aim is obtained for any positive constant $\delta \leq \delta_1$. We shall consider those solutions which leave region $\{x \mid x \leq \delta_1\}$ and reenter it. Let $t_1 = \inf\{t > t_0 \mid x(t) \leq \delta_1\}$. Then $x(t) > \delta_1, t \in [t_0, t_1]$ and $x(t_1) \geq \delta_1$. Suppose that $t_1 \in (n_1T, (n_1 + 1)T]$ for some $n_1 \in N \cup \{0\}$. Let $b = \min\{\prod_{t \leq \tau_k \leq T} (1 + b_k) \mid t \in [0, T]\}$. By (1.7), as (2.1), we can choose an $m \in N, m > \gamma$ such that

$$\prod_{0 < \tau_k < T} (1 + b_k)^{\frac{\gamma}{1+\gamma/m}} e^{\int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} > 1 + \theta$$

and

$$be^{h(\omega+T) - H\omega} (1 + \theta)^{\frac{m}{\gamma} - 1} > 1.$$

Denote $n_2 = (n_1 + 1 + m)$. By Lemma 2.2, we have

$$\frac{n_2 - n_1 - 1}{q_{n_2} - q_{n_1+1}} = \frac{m\omega}{mT + s_{n_1+1} - s_{n_2}} \geq \frac{m\omega}{mT + \omega} = \frac{\gamma}{1 + \gamma/m}$$

and

$$q_{n_2} - q_{n_1+1} = \frac{1}{\omega} ((n_2 - n_1 - 1)T + s_{n_1+1} - s_{n_2}) \geq \frac{1}{\omega} (mT - \omega) = \frac{m}{\gamma} - 1.$$

We claim that there must exist $t_2 \in (t_1, n_2T]$ such that $x(t_2) > \delta_1$. Otherwise, $x(t) \leq \delta_1$ for $t \in (t_1, n_2T]$. Thus by

$$x'(t) \geq x(t)(r(t) - a(t)\delta_1), t \in (t_1, n_2T], t \neq \tau_k, k \in N,$$

we have

$$\begin{aligned} x(n_2T) &\geq x(t_1) \prod_{t_1 \leq \tau_k < n_2T} (1 + b_k) e^{\int_{t_1}^{n_2T} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &= x(t_1) \prod_{t_1 \leq \tau_k \leq (n_1+1)T} (1 + b_k) \prod_{(n_1+1)T < \tau_k < n_2T} (1 + b_k) \\ &\quad e^{\int_{t_1}^{(n_1+1)T} (r(\tau) - a(\tau)\delta_1) d\tau} e^{\int_{q_{n_1+1}\omega}^{q_{n_2}\omega} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &\quad e^{-\int_{q_{n_1+1}\omega}^{q_{n_1+1}\omega + s_{n_1+1}} (r(\tau) - a(\tau)\delta_1) d\tau} e^{\int_{q_{n_2}\omega}^{q_{n_2}\omega + s_{n_2}} (r(\tau) - a(\tau)\delta_1) d\tau} \\ &\geq \delta_1 be^{h(\omega+T) - H\omega} \left(\prod_{0 < \tau_k < T} (1 + b_k)^{\frac{n_2 - n_1 - 1}{q_{n_2} - q_{n_1+1}}} e^{\int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} \right)^{q_{n_2} - q_{n_1+1}} \\ &\geq \delta_1 be^{h(\omega+T) - H\omega} \left(\prod_{0 < \tau_k < T} (1 + b_k)^{\frac{\gamma}{1+\gamma/m}} e^{\int_0^\omega (r(\tau) - a(\tau)\delta_1) d\tau} \right)^{q_{n_2} - q_{n_1+1}} \\ &\geq \delta_1 be^{h(\omega+T) - H\omega} (1 + \theta)^{q_{n_2} - q_{n_1+1}} \\ &\geq \delta_1 be^{h(\omega+T) - H\omega} (1 + \theta)^{\frac{m}{\gamma} - 1} \\ &> \delta_1, \end{aligned}$$

which is a contradiction. Thus there exists $t_2 \in (t_1, n_2T]$ such that $x(t_2) > \delta_1$. Let $t_3 = \inf\{t > t_1 \mid x(t) > \delta_1\}$. Then $x(t) \leq \delta_1$ for $t \in (t_1, t_3]$.

Let $b_1 = \min\{\prod_{t_1 \leq \tau_k < t_2} (1+b_k) \mid 0 < t_1 < t_2 \leq (m+1)T\}$ and $\delta = \min\{\delta_1, \delta_1 b_1 e^{h(m+1)T}\}$. Obviously, δ is independent of any positive solution. Note that $x(t_1) \geq \delta_1$, we have for any $t \in (t_1, t_3]$,

$$x'(t) \geq x(t)(r(t) - a(t)\delta_1), t \neq \tau_k, k \in N$$

and

$$\begin{aligned} x(t) &\geq x(t_1) \prod_{t_1 \leq \tau_k < t} (1+b_k) e^{\int_{t_1}^t (r(\tau) - a(\tau)\delta_1) d\tau} \\ &\geq \delta_1 b_1 e^{h(t-t_1)} \\ &\geq \delta_1 b_1 e^{h(m+1)T} \\ &\geq \delta. \end{aligned}$$

Since $x(t_3^+) > \delta_1$, the same argument can be continued. We can conclude that $x(t) \geq \delta$ for all $t \geq t_0$. The proof is complete. \square

Lemma 2.1 and Theorem 2.1 indicate that system (1.1), (1.2) is permanent with conditions (1.3)-(1.7). Lemma 2.1 is proved in [4] by the method of Liapunov function, which relies on condition (1.5). This condition means the birth rate is always larger than death rate and the density dependence always exists. It maybe unreasonable for some species living in a periodic changing environment, for example, birth may take place seasonally. From the proof of Theorem 2.1, we can see clearly that only $a(t) \geq 0$ is necessary. Suppose that

$$a(t) \geq 0, \int_0^\omega a(\tau) d\tau > 0. \quad (2.2)$$

Then we can choose $M_1 > 0$ be sufficient large such that

$$\mu_1 = \prod_{0 < \tau_k < T} (1+b_k)^\gamma e^{\int_0^\omega (r(\tau) - a(\tau)M_1) d\tau} < 1. \quad (2.3)$$

Using (2.3), the ultimate upper bound can be established by the method similar to the proof of Theorem 2.1. Hence we have the following theorem. Its proof will be omitted.

Theorem 2.2. *Suppose that (1.3), (1.4), (1.6) and (2.2) hold. Then there exists a constant $M > 0$ such that $x(t) \leq M$ for t sufficiently large, where $x(t)$ is any solution of system (1.1), (1.2) with $x(0) = x_0 > 0$.*

Theorems 2.1 and 2.2 establish the permanence of system (1.1), (1.2).

Theorem 2.3. *Suppose that (1.3), (1.4), (1.6), (1.7) and (2.2) hold. Then system (1.1), (1.2) is permanent.*

Remark 2.2. *Condition (1.5) is replaced by (2.2), which only contains restrictions for $a(t)$. In fact, condition (1.7) already has restrictions for $r(t)$. Note that it is unnecessary that the intrinsic rate $r(t)$ be always nonnegative here, which is reasonable for species with seasonal birth.*

3 Conclusion remark

When γ is rational, [4] proved that if (1.7) holds, system (1.1), (1.2) has a unique positive global attractor which is a positive periodic solution. The system is then obviously permanent. In this paper, we proved that with this condition, system (1.1), (1.2) is permanent whether γ is rational or not. Our results ensure the existence of positive global attractor in [4, Conjecture 3]. With condition (1.7), [4] proved that the positive solutions of system (1.1), (1.2) attracts each other in the sense of lower limit. Thus the permanence result in this paper also strongly suggests that the global attractivity results in [4, Conjectures 1 and 2] are valid. Since the positive global attractor of system (1.1), (1.2) is not periodic, which is different from the corresponding continuous system, it is interesting to study further its structure.

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