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量子力学における点変換, S^1 上の量子力学, およびフーリエ型の変換

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1 Introduction

Let $-\infty \leq a < b \leq \infty$. Let f be a diffeomorphism of (a, b) onto $\mathbb{R}$:

$$\xi = f(x), \qquad \xi \in \mathbb{R}, \qquad x \in (a, b).$$

Let

$$f(c) = 0, \qquad a < c < b,$$

and set

$$\widetilde{x} = f^{-1}(-\xi).$$

We deal with the following operator in $L^2(a, b)$:

$$\mathcal{D} = \frac{1}{f'}\frac{\partial}{\partial x} - \frac{f''}{2\,(f')^2} - q\frac{\sqrt{|f'|}}{f}R\frac{1}{\sqrt{|f'|}}. \tag{1.1}$$

Here, $q > -1/2$ and R denotes the reflection operator given by

$$Rv(x) = Rv\left(f^{-1}(\xi)\right) = v\left(f^{-1}(-\xi)\right) = v(\widetilde{x}).$$

The expression for our operator therefore becomes

$$\mathcal{D}u(x) = \frac{1}{f'(x)}\frac{\partial u}{\partial x}(x) - \frac{f''(x)}{2f'(x)^2}u(x) - q\frac{\sqrt{|f'(x)|}}{f(x)}\frac{u(\widetilde{x})}{\sqrt{|f'(\widetilde{x})|}}.$$

Remark 1.1. Our operator is a linear differential operator in a bounded or unbounded open interval (a, b). Moreover, its coefficients are variable coefficients, and the last one is singular since $f(x) = 0$ at $x = c$.

Remark 1.2. We denote by f the multiplication by f and regard it as a linear operator in $L^2(a, b)$. Then our operators $\mathcal{D}$ and f satisfy Wigner's commutation relations [21] in quantum mechanics:

$$\{\mathcal{D}, [f, \mathcal{D}]\} = -2\mathcal{D}, \qquad \{f, [f, \mathcal{D}]\} = -2f,$$

where $\{A, B\} = AB + BA$.

We now give some examples of our operator (1.1). In fact, our operator appears in many quantum-mechanical systems.

Example 1. Let $a = -\infty$, $b = \infty$, $f(x) = x$, and let $q = 0$. Then, by (1.1),

$$\mathcal{D} = \frac{\partial}{\partial x},$$

and hence the operator $-i\mathcal{D}$ corresponds to the momentum operator for a quantum-mechanical particle. Here we use the unit $\hbar = 1$.

Example 2. Let $a = -\infty$, $b = \infty$, and let $q = 0$. In this case, each function f gives rise to a point transformation in quauntum mechanics. We [12] first defined and discussed a point transformation as a canonical transformation in quauntum mechanics from the viewpoint of mathematics. Our operator $-i\mathcal{D}$ then corresponds to the new momentum operator given by the point transformation. For more details, see section 2.

Example 3. Let $a = 0$, $b = \infty$, $f(x) = \ln x$, and let $q = 0$. Our operator $-i\mathcal{D}$ then corresponds to the dilatation operator in quantum mechanics and also corresponds to the generator of the dilation operator appearing in wavelets analysis (see e.g. [4, 6]). We [13] studied the essential selfadjointmess of $-i\mathcal{D}$ and showed that the Mellin transform transforms $-i\mathcal{D}$ into the multiplication by y $(y \in \mathbb{R})$.

Example 4. Let $a = 0$, $b = \pi$, $f(x) = -\ln\tan(x/2)$, and let $q = 0$. Our operator $-i\mathcal{D}$ in this case corresponds to the momentum operator appearing in quantum mechanics on S^1 based on Dirac Formalism [3, 11]. Watanabe [17, 18, 19] discussed the selfadjointmess of $-i\mathcal{D}$ and constructed an integral transform that transforms $-i\mathcal{D}$ into the multiplication by y $(y \in \mathbb{R})$. For more details, see section 3. See also Soltani [15] for related material.

Example 5. Let $a = -\infty$, $b = \infty$, and let $f(x) = x$. Our operator $-i\mathcal{D}$ then corresponds to the momentum operator of a bose-like oscillator governed by Wigner's commutation relations mentioned above. See also Yang [22], and Ohnuki and Kamefuchi [9, 10] for related material.

Example 6. Let a, b and f be as in Example 5. Then the operators $-\mathcal{D}^2$ and $-\mathcal{D}^2 + x^2$ correspond to the Hamitonians appearing in the two-body problems of the Calogero model [1], the Calogero-Moser model [1, 7] and the Sutherland model [16]. Each model describes a quantum-mechanical system of many identical particles in one dimension with long-range interactions, and has attracted considerable interest because it is exactly solvable.

We denote by $(\cdot\,,\,\cdot)_{L^2(a,b)}$ the inner product of $L^2(a,\,b)$, and by $\|\cdot\|_{L^2(a,b)}$ its norm. We also denote by $(\cdot\,,\,\cdot)_{L^2(\mathbb{R})}$ the inner product of $L^2(\mathbb{R})$, and by $\|\cdot\|_{L^2(\mathbb{R})}$ its norm.

2 Point transformations in quauntum mechanics

In classical mechanics the coordinate transformation

$$\text{(2.1)} \qquad \begin{cases} f : x = (x_1, x_2, \dots, x_d) \mapsto X = (X_1, X_2, \dots, X_d), \\ X_\alpha = f_\alpha(x) \qquad (\alpha = 1, 2, \dots, d) \end{cases}$$

is called a point transformation, where x belongs to some domain D in $\mathbb{R}^d$:

$$\text{(2.2)} \qquad x \in D$$

and the existence of f^{-1} is assumed. In classical mechanics the domain D does not always coincide with $\mathbb{R}^d$; it is sufficient for D to involve the trajectory of a physical system under consideration.

It is known that the point transformation can be extended to a canonical transformation (see e.g. Whittaker [20, p.293])

$$(x_1, \dots, x_d, p_1, \dots, p_d) \mapsto (X_1, \dots, X_d, P_1, \dots, P_d),$$

which is called an extended point transformation and is given by

$$\tag{2.3} \begin{cases} X_\alpha = f_\alpha(x), \\ P_\alpha = \displaystyle\sum_{\beta=1}^{d} \frac{\partial x_\beta}{\partial X_\alpha} p_\beta. \end{cases}$$

Here the canonical momenta p_α and P_α are conjugate to x_α and X_α, respectively. Let $[A,\, B]_{\mathrm{cl}}$ stand for the classical Poisson bracket for $A(x,\, p)$ and $B(x,\, p)$:

$$[A,\, B]_{\mathrm{cl}} = \sum_{\alpha=1}^{d} \left(\frac{\partial A}{\partial x_\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial B}{\partial x_\alpha}\frac{\partial A}{\partial p_\alpha} \right).$$

The canonical variables x_α and p_α obey the relations

$$[x_\alpha,\, p_\beta]_{\mathrm{cl}} = \delta_{\alpha\beta}, \quad [x_\alpha,\, x_\beta]_{\mathrm{cl}} = [p_\alpha,\, p_\beta]_{\mathrm{cl}} = 0.$$

Then it is known that the new canonical variables X_α and P_α given by (2.3) also obey the same relations

$$\tag{2.4} [X_\alpha,\, P_\beta]_{\mathrm{cl}} = \delta_{\alpha\beta}, \quad [X_\alpha,\, X_\beta]_{\mathrm{cl}} = [P_\alpha,\, P_\beta]_{\mathrm{cl}} = 0.$$

An example of a point transformation in classical mechanics is the coordinate transformation $f : (x_1,\, x_2) \mapsto (r,\, \theta)$ from cartesian to plane polar coordinates. Here, $(x_1,\, x_2) \in \mathbb{R}^2$. The existence of f together with f^{-1} implies $(x_1,\, x_2) \in D = \mathbb{R}^2 \setminus \{(0,\, 0)\}$. We are thus led to the extended point transformation $(x_1,\, x_2,\, p_1,\, p_2) \mapsto (r,\, \theta,\, p_r,\, p_\theta)$. Here the canonical momenta p_r and p_θ are conjugate to r and θ, respectively. In quantum mechanics, however, the situation is quite different. It is known that the continuous spectrum of each canonical variable in quantum mechanics coincides with $\mathbb{R}$. Therefore, the point transformation $f : (x_1,\, x_2) \mapsto (r,\, \theta)$ from cartesian to plane polar coordinates is no longer allowed within the frame work of quantum mechanics. Hence the extended point transformation $(x_1,\, x_2,\, p_1,\, p_2) \mapsto (r,\, \theta,\, p_r,\, p_\theta)$ is not allowed any longer. In fact, if it were allowed, then r, θ, p_r and p_θ would satisfy the canonical commutation relations. But this is not the case, because this clearly contradicts positivity of r and boundedness of θ.

So it is highly desirable to define point transformations both in classical mechanics and in quantum mechanics from the viewpoint of mathematics.

Definition 2.1 (Ohnuki and Watanabe [12]).
We say that the map f is a point transformation in classical mechanics if f is a C^2-diffeomorphism and satisfies (2.1) and (2.2).

Remark 2.2. Let $(x_1, x_2) \in \mathbb{R}^2$. The coordinate transformation $f : (x_1, x_2) \mapsto (r, \theta)$ from cartesian to plane polar coordinates is a point transformation in classical mechanics. The domain D of the map f does not contain the origin, i.e., $D = \mathbb{R}^2 \setminus \{(0, 0)\}$. Hence, r, θ, p_r and p_θ are canonical variables in classical mechanics:

$$[r, p_r]_{\rm cl} = [\theta, p_\theta]_{\rm cl} = 1, \quad [r, \theta]_{\rm cl} = [r, p_\theta]_{\rm cl} = [\theta, p_r]_{\rm cl} = [p_r, p_\theta]_{\rm cl} = 0.$$

In quantum mechanics the operators x_α and p_α are assumed to obey the canonical commutation relations

$$[x_\alpha, p_\beta] = i\,\delta_{\alpha\beta}, \quad [x_\alpha, x_\beta] = [p_\alpha, p_\beta] = 0,$$

where $[A, B] = AB - BA$. Let x_α be the multiplication by x_α. Then it follows from the canonical commutation relations that

$$p_\alpha = -i\,\frac{\partial}{\partial x_\alpha}. \tag{2.5}$$

Definition 2.3 (Ohnuki and Watanabe [12]). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective map satisfying

$$\begin{cases} f : x = (x_1, x_2, \ldots, x_n) \mapsto X = (X_1, X_2, \ldots, X_n), \\ X_\alpha = f_\alpha(x) \qquad (\alpha = 1, 2, \ldots, n). \end{cases}$$

We say that the map f is a point transformation in quantum mechanics if f is a C^3-diffeomorphism.

Remark 2.4. Let $(x_1, x_2) \in \mathbb{R}^2$. The coordinate transformation $f : (x_1, x_2) \mapsto (r, \theta)$ from cartesian to plane polar coordinates is *not* a point transformation in quantum mechanics. This is because the domain of the map f does not coincide with $\mathbb{R}^2$. Therefore, r, θ, p_r and p_θ are not canonical variables in quantum mechanics, and hence one can not impose the following relations

$$[r, p_r] = [\theta, p_\theta] = i, \quad [r, \theta] = [r, p_\theta] = [\theta, p_r] = [p_r, p_\theta] = 0.$$

The coordinate transformation from cartesian to plane polar coordinates is therefore a point transformation in classical mechanics, but not in quantum mechanics.

Definition 2.5 (DeWitt [2]). The new canonical variables X_α and P_α in quantum mechanics are given by

$$\begin{cases} X_\alpha = f_\alpha(x), \\ P_\alpha = \dfrac{1}{2} \displaystyle\sum_{\beta=1}^{d} \left(\frac{\partial x_\beta}{\partial X_\alpha}\, p_\beta + p_\beta\, \frac{\partial x_\beta}{\partial X_\alpha} \right). \end{cases}$$

Combining Definition 2.5 with (2.5) yields

$$P_\alpha = -i \sum_{\beta=1}^{d} \frac{\partial x_\beta}{\partial X_\alpha} \frac{\partial}{\partial x_\beta} - \frac{i}{2} \sum_{\beta=1}^{d} \frac{\partial}{\partial x_\beta} \left(\frac{\partial x_\beta}{\partial X_\alpha} \right).$$

The operators X_α and P_α act on functions of x_α's. When $d = 1$, the operator P_α is nothing but our operator $-i\mathcal{D}$ for $a = -\infty$, $b = \infty$ and $q = 0$. Our operator therefore corresponds to the new momentum operator given by the point transformation in this case.

Theorem 2.6 (Ohnuki and Watanabe [12]).
(a) *The operator* $\dot{P}_\alpha = P_\alpha \upharpoonright C_0^3(\mathbb{R}^d)$ *is essentially selfadjoint.*
(b) *The set* $\mathbb{R}$ *coincides with the continuous spectrum of the selfadjoint operator* $P_\alpha = \overline{\dot{P}_\alpha}$*:*

$$\sigma(P_\alpha) = \sigma_c(P_\alpha) = \mathbb{R}.$$

(c) *The operators* X_α *and* P_α *satisfy the canonical commutation relations:*

$$[X_\alpha,\, P_\beta]u = i\,\delta_{\alpha\beta}\, u, \quad [X_\alpha,\, X_\beta]u = [P_\alpha,\, P_\beta]u = 0$$

for $u \in C_0^2(\mathbb{R}^d)$.

Remark 2.7. The operator X_α is selfadjoint, and $\sigma(X_\alpha) = \sigma_c(X_\alpha) = \mathbb{R}$. Combining this fact with Theorem 2.6 we arrive at the conclusion that Definitions 2.3 and 2.5 are suitable.

3 Quauntum mechanics on S^d

In Dirac formalism [3, 11] for a classical mechanical particle constrained to move on the d-sphere S^d (embedded to $\mathbb{R}^{d+1}$), one imposes the relations: $(\alpha,\, \beta = 1, \ldots, d+1)$

$$\begin{aligned} \{x_\alpha\,, x_\beta\}^* &= 0, \quad \{x_\alpha\,, p_\beta\}^* = \delta_{\alpha\beta} - \frac{1}{r^2} x_\alpha x_\beta, \\ \{p_\alpha\,, p_\beta\}^* &= \frac{1}{r^2}\,(p_\alpha x_\beta - p_\beta x_\alpha) \end{aligned}$$

with

$$\text{the primary constraint} \quad \sum_{\alpha=1}^{d+1} {x_\alpha}^2 - r^2 = 0, \quad \text{the secondary constraint} \quad \sum_{\alpha=1}^{d+1} x_\alpha p_\alpha = 0.$$

Here, r is a constant, $\{\,\cdot\,,\,\cdot\,\}^*$ denotes the Dirac bracket, and x_α's and p_α's stand for the coordinates and the momenta of the particle, respectively.

To proceed to quantum theory, one replaces the Dirac bracket $\{\,\cdot\,,\,\cdot\,\}^*$ by the commutator $i^{-1}\,[\,\cdot\,,\,\cdot\,]$. But one has no knowledge of the order for x_α and p_α in the products $x_\alpha p_\beta$. Ohnuki and Kitakado [11] then replaces the relations above by the following commutation relations:

$$\begin{aligned} &[\,x_\alpha\,, x_\beta\,] = 0, \quad [\,x_\alpha\,, p_\beta\,] = i\left(\delta_{\alpha\beta} - \frac{1}{r^2} x_\alpha x_\beta\right), \\ &[\,p_\alpha\,, p_\beta\,] = \frac{i}{r^2}\,(p_\alpha x_\beta - p_\beta x_\alpha) \end{aligned} \tag{3.1}$$

with

$$\sum_{\alpha=1}^{d+1} {x_\alpha}^2 - r^2 = 0, \qquad \sum_{\alpha=1}^{d+1} (x_\alpha p_\alpha + p_\alpha x_\alpha) = 0. \tag{3.2}$$

Let us deal with the case where the configuration space is S^1 (embedded to $\mathbb{R}^2$). Setting $x_1 = r\cos x$ and $x_2 = r\sin x$ $(-\pi \le x \le \pi)$, Ohnuki and Kitakado [11] derived the expressions for the operators p_1 and p_2 satisfying (3.1) and (3.2):

$$\begin{cases} p_1 = \dfrac{1}{r}\left(i \sin x \dfrac{\partial}{\partial x} + \dfrac{i}{2} \cos x - \alpha \sin x\right), \\ p_2 = -\dfrac{1}{r}\left(i \cos x \dfrac{\partial}{\partial x} - \dfrac{i}{2} \sin x - \alpha \cos x\right), \end{cases}$$

where $0 \le \alpha < 1$. For simplicity, we set $r = 1$ and $\alpha = 0$. Then

$$\begin{cases} p_1 = i\left(\sin x \dfrac{\partial}{\partial x} + \dfrac{1}{2} \cos x\right), \\ p_2 = -i\left(\cos x \dfrac{\partial}{\partial x} - \dfrac{1}{2} \sin x\right), \end{cases} \tag{3.3}$$

where $-\pi \le x \le \pi$. The operator p_1 in $L^2(0, \pi)$ is nothing but our operator $-i\mathcal{D}$ for $a = 0$, $b = \pi$, $f(x) = -\ln\tan(x/2)$ and $q = 0$. The operator p_2 in $L^2(-\pi/2, \pi/2)$ is nothing but our operator $-i\mathcal{D}$ for $a = -\pi/2$, $b = \pi/2$, $f(x) = \dfrac{1}{2}\ln\dfrac{1+\sin x}{1-\sin x}$ and $q = 0$.

Corollary 3.1 (Watanabe [17, 18, 19]). *Let p_1 be as in* (3.3).
(a) *The operator p_1 is selfadjoint in $L^2(0, \pi)$, and is also selfadjoint in $L^2(-\pi, 0)$. Consequently, it is selfadjoint in $L^2(-\pi, \pi)$, and the spectrum of the selfadjoint operator p_1 in $L^2(-\pi, \pi)$ satisfies*

$$\sigma(p_1) = \sigma_c(p_1) = \mathbb{R}.$$

(b) *The selfadjoint operator p_1 in $L^2(-\pi, \pi)$ is unitarily equivalent to the selfadjoint operator $\{-i(\partial/\partial y)\} \oplus \{-i(\partial/\partial y)\}$ in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where $y \in \mathbb{R}$.*

Remark 3.2. Similar results hold for the operator p_2.

4 An integral transform associated with our operator

In this section we construct an integral transform associated with our operator $\mathcal{D}$ based on the Hankel transform.

For $n \in \{0\} \cap \mathbb{N}$, let (cf. [9, (4.31)] and [10, (23.80)])

$$\begin{cases} u_{2n}(x) = K_n^{q+\frac{1}{2}} \sqrt{|f'(x)|}\, |f(x)|^q\, L_n^{q-\frac{1}{2}}(f(x)^2) \exp\left(-\dfrac{f(x)^2}{2}\right), \\ u_{2n+1}(x) = K_n^{q+\frac{3}{2}} \sqrt{|f'(x)|}\, f(x)\, |f(x)|^q\, L_n^{q+\frac{1}{2}}(f(x)^2) \exp\left(-\dfrac{f(x)^2}{2}\right). \end{cases}$$

Here $K_n^\nu = (-1)^n\sqrt{n!/\Gamma(n+\nu)}$ with Γ the gamma function, and L_n^ν is a generalized Laguerre polynomial. Note that $u_n \in L^2(a, b)$.

Remark 4.1. Ohnuki and Kamefuchi [9, 10] obtained the functions u_n when $a = -\infty$, $b = \infty$ and $f(x) = x$.

Let V be the set of finite linear combinations of u_n's. A straightforward calculation gives the following.

Lemma 4.2 (Ohnuki and Watanabe [14]). *The set $\{u_n\}_{n=0}^{\infty}$ is a complete orthonormal set of $L^2(a, b)$. Consequently, V is dense in $L^2(a, b)$.*

Using Nelson's analytic vector theorem [8] we can show the following.

Proposition 4.3 (Ohnuki and Watanabe [14]). *The operator $(-i\mathcal{D}) \restriction V$ is essentially selfadjoint, and so is the multiplication operator $f \restriction V$.*

Set

$$\varphi(y, x) = \frac{\sqrt{|yf(x)f'(x)|}}{2} \left\{ J_{q-1/2}(\,|yf(x)|\,) + i \operatorname{sgn}(yf(x)) J_{q+1/2}(\,|yf(x)|\,) \right\},$$

where $x \in (a, b)$, $y \in \mathbb{R}$ and J_ν denotes the Bessel function of the first kind.

Remark 4.4. Ohnuki and Kamefuchi [9, 10] obtained the function $\varphi(y, x)$ when $a = -\infty$, $b = \infty$ and $f(x) = x$.

We consider the following integral transform:

$$Uu(y) = \int_a^b \overline{\varphi(y, x)} u(x)\, dx, \qquad u \in V,$$

where $y \in \mathbb{R}$. Note that $Uu \in L^2(\mathbb{R})$. The operator U satisfies

$$(Uu_1, Uu_2)_{L^2(\mathbb{R})} = (u_1, u_2)_{L^2(a,b)}, \qquad u_1, u_2 \in V.$$

Combining this fact with Lemma 4.2 gives the following.

Theorem 4.5 (Ohnuki and Watanabe [14]). *The transform U becomes a unitary operator from $L^2(a, b)$ to $L^2(\mathbb{R})$.*

A straightforward calculation gives that our transform U transforms our operator $-i\mathcal{D}$ into the multiplication by y:

Proposition 4.6 (Ohnuki and Watanabe [14]).

$$U\,(-i\mathcal{D})\,U^* = y.$$

This proposition immediately implies the following.

Corollary 4.7 (Ohnuki and Watanabe [14]).
Let $-i\mathcal{D}$ be the selfadjoint operator in $L^2(a, b)$ given above. Then the operator $\mathcal{D}^2$ generates an analytic semigroup $\{\exp(t\mathcal{D}^2) : t > 0\}$ on $L^2(a, b)$.

Remark 4.8. If $a = -\infty$, $b = \infty$, $f(x) = x$ and $q = 0$, then

$$\mathcal{D} = \frac{\partial}{\partial x}, \quad \varphi(y, x) = \frac{1}{\sqrt{2\pi}} \exp(iyx).$$

Here, $(x, y) \in \mathbb{R} \times \mathbb{R}$. Our transform U reduces to the Fourier transform in this case, and hence can be regarded as a generalized Fourier transform.

Remark 4.9. We constructed our transform on the basis of the study of the Hankel transform. Kilbas and Borovco [5] considered a more general integral transform including the Hankel transform.

5 An embedding theorem of Sobolev type

We define spaces of Sobolev type using our transform, and show an embedding theorem for each space. While the Sobolev space contains information about differentiability of each element, our space contains information both about differentiability of each element and about continuity of each element divided by some functions. Therefore, our embedding theorem provides information both about smoothness of each element and about continuity of each element divided by some functions. So our embedding theorem is a generalization of the Sobolev embedding theorem.

We now define spaces of Sobolev type.

Definition 5.1 (Ohnuki and Watanabe [14]). For $\nu \geq 0$,

$$\mathcal{H}^\nu(a,\, b) = \left\{ u \in L^2(a,\, b) : \int_{\mathbb{R}} \left(1 + |y|^2\right)^\nu |Uu(y)|^2 \, dy < \infty \right\}.$$

A straightforward calculation gives that each $\mathcal{H}^\nu(a,\, b)$ is a Hilbert space with inner product

$$(u_1,\, u_2)_{\mathcal{H}^\nu(a,\,b)} = \int_{\mathbb{R}} \left(1 + |y|^2\right)^\nu Uu_1(y)\, \overline{Uu_2(y)}\, dy, \qquad u_1,\, u_2 \in \mathcal{H}^\nu(a,\, b)$$

and norm $|\, u\,|_{\mathcal{H}^\nu(a,\,b)} = \sqrt{(u,\, u)_{\mathcal{H}^\nu(a,\,b)}}$.

Remark 5.2. . If $a = -\infty$, $b = \infty$, $f(x) = x$ and $q = 0$, then our transform U reduces to the Fourier transform as mentioned above, and hence $\mathcal{H}^\nu(a,\, b)$ to the usual Sobolev space $H^\nu(\mathbb{R})$ in this case.

Definition 5.1 together with Proposition 4.6 immediately implies the following.

Corollary 5.3 (Ohnuki and Watanabe [14]).
(a) $\mathcal{H}^0(a,\, b) = L^2(a,\, b)$.
(b) $\mathcal{H}^{\nu'}(a,\, b) \subset \mathcal{H}^\nu(a,\, b), \qquad \nu' \geq \nu$.
(c) $|\, u\,|_{\mathcal{H}^\nu(a,\,b)} \leq |\, u\,|_{\mathcal{H}^{\nu'}(a,\,b)}, \qquad u \in \mathcal{H}^{\nu'}(a,\, b), \qquad \nu' \geq \nu$.
(d) *Let* $|y|^\nu$ *be the selfadjoint multiplication operator and* $D(|y|^\nu)$ *its domain. Then* $U\mathcal{H}^\nu(a,\, b) = D(|y|^\nu)$.

Definition 5.4 (Ohnuki and Watanabe [14]). Let f be as above. For $\beta \in \{0\} \cap \mathbb{N}$, we define

$$S_f^\beta(a,\, b) = \left\{ u(x) : u,\, \frac{u}{f^\beta} \in C(a,\, b) \right\}.$$

Remark 5.5. If $u \in S_f^\beta(a,\, b)$, then u/f^β is continuous on $(a,\, b)$.

The following is our embedding theorem.

Theorem 5.6 (Ohnuki and Watanabe [14]).
Let $q \geq 0$. *Suppose* $\nu > \dfrac{1}{2}$ *and* $\nu \neq m + \dfrac{1}{2}$ $(m \in \mathbb{N})$. *Then*

$$\mathcal{H}^\nu(a,\, b) \subset C^\alpha(a,\, b) \cap S_f^\beta(a,\, b),$$

where α and β are nonnegative integers satisfying ($k \in \{0\} \cap \mathbb{N}$)

$$\alpha = \begin{cases} [\nu - \frac{1}{2}] & (q = 2k), \\ \min\left([\nu - \frac{1}{2}], q - 1\right) & (q = 2k+1), \\ \min\left([\nu - \frac{1}{2}], [q]\right) & (otherwise) \end{cases}$$

and

$$\beta = \begin{cases} \min\left([\nu - \frac{1}{2}], q\right) & (q = 2k), \\ \min\left([\nu - \frac{1}{2}], q - 1\right) & (q = 2k+1), \\ \min\left([\nu - \frac{1}{2}], [q]\right) & (otherwise). \end{cases}$$

Remark 5.7. If $a = -\infty$, $b = \infty$, $f(x) = x$ and $q = 0$, then our transform U and our space $\mathcal{H}^\nu(a, b)$ reduce to the Fourier transform and to the Sobolev space $H^\nu(\mathbb{R})$, respectively. Moreover, $\alpha = [\nu - \frac{1}{2}]$ and $\beta = 0$ in this case. Our embedding theorem thus reduces to the usual Sobolev embedding theorem:

$$H^\nu(\mathbb{R}) \subset C^{[\nu - 1/2]}(\mathbb{R}).$$

So our embedding theorem is a generalization of the Sobolev embedding theorem.

6 An application

We apply our results to the following problem in $L^2(0, \pi)$ with a singular variable coefficient. We look for $u(t, \cdot) \in \mathcal{H}^2(0, \pi)$ of the the problem.

$$(6.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \sin^2 x \dfrac{\partial^2 u}{\partial x^2}(t,x) + 2 \sin x \cos x \dfrac{\partial u}{\partial x}(t,x) \\ \qquad + \dfrac{1 - 3\sin^2 x}{4} u(t,x) - \dfrac{90\, u(t,x)}{\left(\ln \tan \frac{x}{2}\right)^2}, \quad t > 0, \quad x \in (0, \pi), \\ u(0,x) = u_0(x), \quad x \in (0, \pi). \end{cases}$$

Here, $u_0 \in L^2(0, \pi)$ satisfies

$$u_0(\pi - x) = u_0(x).$$

Note that the coefficient $90/(\ln \tan \frac{x}{2})^2$ is singular at $x = \pi/2$. Since there is such a singular coefficient, at first sight, it cannot be expected that the solution $u(t, \cdot)$ is infinitely differentiable on $(0, \pi)$, nor that the functions: $x \mapsto u(t, x)/\{\ln \tan(x/2)\}^\beta$ are continuous on $(0, \pi)$. Here, β are nonnegative integers. But, as is shown just below, this is not the case.

If $a = 0$, $b = \pi$, $f(x) = -\ln \tan(x/2)$ and $q = 10$, then

$$\mathcal{D}^2 = \sin^2 x \frac{\partial^2}{\partial x^2} + 2 \sin x \cos x \frac{\partial}{\partial x} + \frac{1 - 3\sin^2 x}{4} - \frac{90}{\left(\ln \tan \frac{x}{2}\right)^2}.$$

Hence the problem becomes

$$\begin{cases} \dfrac{du}{dt} = \mathcal{D}^2 u, & t > 0, \\ u(0) = u_0 . \end{cases}$$

By Corollary 4.7, $\mathcal{D}^2$ generates an analytic semigroup $\{\exp(t\mathcal{D}^2) : t > 0\}$ on $L^2(0, \pi)$. Theorem 5.6 thus implies the following.

Corollary 6.1. *Let u_0 be as above, and let $m \in \mathbb{N}$. Then there is a unique solution*

$$u \in C\left([0, \infty); L^2(0, \pi)\right) \cap C^1\left((0, \infty); \mathcal{H}^{2m}(0, \pi)\right)$$

of the problem (6.1) *satisfying*

$$u(t, \cdot) = \exp(t\mathcal{D}^2)u_0 \in C^\infty(0, \pi) \cap S^{10}_{-\ln\tan(x/2)}(0, \pi).$$

Remark 6.2. From Corollary 6.1 we see that the solution $u(t, \cdot)$ is infinitely differentiable on $(0, \pi)$ and that the function: $x \mapsto u(t, x)/\{\ln\tan(x/2)\}^{10}$ is continuous on $(0, \pi)$.

Remark 6.3. We can write the solution above in an explicit form.

See Ohnuki and Watanabe [14] for more applications.

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