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THE FREENESS OF A_2 AND B_2-TYPE ARRANGEMENTS AND LATTICE COHOMOLOGIES

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ABSTRACT. In general, it is difficult to determine whether a given arrangement is free or not. As examples, we show some arrangements, i.e., A_2 and B_2-type arrangements, whose freeness is determined by combinatorics. To consider this problem, Yuzvinsky introduced the sheaf of the module of logarithmic vector fields on intersection lattices of arrangements and found a freeness criterion for arrangements by using the cohomology of that sheaf ([Yu1] and [Yu2]). In this article we generalize Yuzvinsky's criterion is to that for multiarrangements. Moreover, we consider an endomorphism sheaf on lattices and try to find a new freeness criterion.

0. INTRODUCTION

A *hyperplane arrangement* (or simply an *arrangement*) is a finite collection of affine hyperplanes in a fixed vector space V over a field $\mathbb{K}$. This is a very simple geometric object, but there are a lot of interesting problems on arrangements. One of the intensively studied objects in the arrangement theory is a module of logarithmic vector fields $D(\mathcal{A})$ associated to $\mathcal{A}$. Roughly speaking, $D(\mathcal{A})$ consists of vector fields tangent to each hyperplanes in $\mathcal{A}$ (for details see Definition 1.1). We say an arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free $\mathrm{Sym}(V^*)$-module. Around this algebraic object, one of the most interesting problems is that called Terao conjecture, which asserts the freeness of $\mathcal{A}$ depends only on the combinatorics of the arrangement. Generally, to determine whether a given arrangement is free or not is a difficult problem and one way to find a free arrangement is to use the addition-deletion theorem (see Theorem 1.2). For example, the author classified the freeness and the stability of arrangements associated to some root systems in [A1] and [A2], which will be shown in this article. To consider the freeness of arrangements, Yuzvinsky employed the cohomology theory on posets and obtained a criterion for the freeness of arrangements in terms of the vanishing of cohomologies. By using that criterion, he showed in an algebraic variety which parameterizes all arrangements with a fixed combinatorics, the set of free arrangements is a Zariski open set in it ([Yu3]). In this article, we extend Yuzvinsky's freeness criterion to that for multiarrangements, whose freeness is more difficult than simple arrangements. Moreover, we introduce the concept of endomorphism sheaves of arrangements and try to construct a new freeness criterion for arrangements.

The organization of this article is as follows. In Section 1 we introduce some basic definitions and results of the hyperplane arrangement theory. In Section 2 we consider, as examples of (non-)free arrangements, the A_2 and B_2-type arrangements. In Section 3 the freeness criterion by Yuzvinsky is reviewed and the criterion is extended to that for multiarrangements. In Section 4 the endomorphism sheaf of an arrangement is introduced and considered.

Notation. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{K}$ denotes a field. For a vector space V over $\mathbb{K}$, V^* denotes the dual vector space of V and S denotes the symmetric algebra of V^*, i.e., $S := \mathrm{Sym}(V^*)$. $\mathrm{Der}_{\mathbb{K}}(S)$ is the S-module of $\mathbb{K}$-linear derivations of S. For any integer d and a graded S-module M which is finitely generated over S, M_d is a homogeneous part of M with degree d. $\widetilde{M}$ denotes the sheafification of M, so $\widetilde{M}$ is a sheaf on $\mathbf{Proj}(S)$. For a vector bundle E, i.e., a locally free sheaf, on the projective space $\mathbf{P}^n_{\mathbb{K}}$, $c_i(E)$ denotes the i-th Chern class of E and we put the Chern polynomial $c_t(E)$ of E as

$$c_t(E) := \sum_{i=0}^{n} c_i(E)t^i.$$

For a finite set A, its cardinality is denoted by $|A|$.

1. Preliminaries.

We introduce and review some results and definitions which will be used in the rest of this article. First we recall those of hyperplane arrangements, for which we refer the reader to [OT]. Let us fix an l-dimensional $\mathbb{K}$-vector space $V \simeq \mathbb{K}^l$. A *hyperplane arrangement* (or a *simple arrangement*) $\mathcal{A}$ is a finite collection of affine hyperplanes in V. We often say an "arrangement" instead of a " hyperplane arrangement", and call an arrangement in an l-dimensional vector space an "l-arrangement". We say an arrangement $\mathcal{A}$ is *central* if each hyperplane in $\mathcal{A}$ is a vector subspace of V. In this article, we assume all arrangements are non-empty and "central" if not otherwise specified. Note we can regard a central l-arrangement as the arrangement in $\mathbf{P}^{l-1} \simeq \mathbf{P}(V)$. Let $\{X_1, \dots, X_l\}$ be a basis for V^* and put $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X_1, \dots, X_l]$. For each hyperplane $H \in \mathcal{A}$, let us fix a nonzero linear form $\alpha_H \in V^*$ such that its kernel is H, and put

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H.$$

Definition 1.1. *For an arrangement $\mathcal{A}$, the S-module $D(\mathcal{A})$ is defined by*

$$\begin{aligned} D(\mathcal{A}) : &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H \ (\forall H \in \mathcal{A})\} \\ &= \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(Q(\mathcal{A})) \in S \cdot Q(\mathcal{A})\}. \end{aligned}$$

We call $D(\mathcal{A})$ the *module of logarithmic vector fields* (with respect to $\mathcal{A}$). We say a nonzero element $\theta = \sum_{i=1}^{l} f_i \dfrac{\partial}{\partial X_i} \in D(\mathcal{A})$ is *homogeneous of degree p* if $f_i \in S_p$ for $1 \leq i \leq l$. An arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module. When $\mathcal{A}$ is free, there exists a homogeneous basis $\{\theta_1, \dots, \theta_l\}$ for $D(\mathcal{A})$. Then the *exponents* of the free arrangement $\mathcal{A}$ are defined by

$$\exp(\mathcal{A}) := (\deg(\theta_1), \dots, \deg(\theta_l)).$$

It is known that $\exp(\mathcal{A})$ do not depend on the choice of a basis.

Next, we define a multiarrangement, which was introduced and studied by Ziegler in [Z].

Definition 1.2 ([Z]). *We say a pair $(\mathcal{A}, m)$ is a multiarrangement if $\mathcal{A}$ is a simple arrangement and*

$$m : \mathcal{A} \to \mathbb{Z}_{>0}$$

is a map from $\mathcal{A}$ to positive integers. The map m is called a multiplicity function.

A simple arrangement $\mathcal{A}$ can be thought of as a multiarrangement with $m \equiv 1$. By the same way as for simple arrangements, we define the module of logarithmic vector fields $D(\mathcal{A}, m)$ for a multiarrangement $(\mathcal{A}, m)$.

Definition 1.3. *For a multiarrangement $(\mathcal{A}, m)$, the S-module $D(\mathcal{A}, m)$ is defined by*

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ (\forall H \in \mathcal{A})\}.$$

Let $H_0 \in \mathcal{A}$ be a hyperplane in an arrangement $\mathcal{A}$. The restriction of $\mathcal{A}$ to H_0 is a simple arrangement $\mathcal{A} \cap H_0 := \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. This restriction has a natural structure of the multiarrangement $(\mathcal{A} \cap H_0, m)$, i.e., the multiplicity function $m : \mathcal{A} \cap H_0 \to \mathbb{Z}_{>0}$ is defined by

$$m : \mathcal{A} \cap H_0 \ni H' \mapsto |\{H \in \mathcal{A} \mid H \cap H_0 = H'\}| \in \mathbb{Z}.$$

For details, see [Z] or [Yo]. It is known that $D(\mathcal{A}, m)$ is a reflexive module (e.g., see Theorem 4.75 in [OT] and Theorem 5 in [Z]). We can define the freeness and exponents of the multiarrangements by the same way as for simple arrangements. The exponents of a free multiarrangement are sometimes called *multi-exponents.* In this article we often consider the sheafification of $D(\mathcal{A})$. The Chern polynomial of $\widetilde{D(\mathcal{A})}$ can be calculated from the combinatorics of $\mathcal{A}$. To see this, let us introduce some notations. The *characteristic polynomial* of an arrangement $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

where $L(\mathcal{A})$ is a lattice which consists of the intersections of elements of $\mathcal{A}$, ordered by reverse inclusion, $\hat{0} := V$ is the unique minimal element of $L(\mathcal{A})$ and $\mu : L(\mathcal{A}) \longrightarrow \mathbb{Z}$ is the Möbius function defined as follows:

$$\begin{aligned} \mu(\hat{0}) &= 1, \\ \mu(X) &= -\sum_{Y<X} \mu(Y), \text{ if } \hat{0} < X. \end{aligned}$$

It is known that for a central arrangement $\mathcal{A}$, its characteristic polynomial $\chi(\mathcal{A}, t)$ can be divided by $(t-1)$. Moreover, the *reduced characteristic polynomial* $\chi_0(\mathcal{A}, t)$ is defined by

$$\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}, t)/(t-1)$$

and the *Poincaré polynomial* $\pi(\mathcal{A}, t)$ by

$$\pi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{codim} X}.$$

The polynomials $\chi(\mathcal{A}, t)$ and $\pi(\mathcal{A}, t)$ are related as follows:

$$\chi(\mathcal{A}, t) = t^l \pi(\mathcal{A}, -1/t),$$

and these polynomials are important concepts in the theory of hyperplane arrangements. Actually there are a lot of combinatorial or geometric interpretations of the characteristic polynomial. For details, see [OT]. We can use $\pi(\mathcal{A}, t)$ to calculate the Chern polynomial.

Theorem 1.1 ([MS], Theorem 4.1). *For a polynomial $F(t) \in \mathbb{Z}[t]$, let $\overline{F(t)}$ denote the class of $F(t)$ in $\mathbb{Z}[t]/(t^l)$. Let $\mathcal{A}$ be a central l-arrangement and assume $\widetilde{D(\mathcal{A})}$ is a vector bundle on $\mathbf{P}(V)$. Then it holds that*

$$c_t(\widetilde{D(\mathcal{A})}) = \overline{\pi(\mathcal{A}, -t)}.$$

In particular, if $l = 3$ and

$$\chi_0(\mathcal{A}, t) = t^2 - c_1 t + c_2,$$

then for any central 3-arrangement $\mathcal{A}$ it holds that

$$c_t(\widetilde{D(\mathcal{A})}) = (1 - c_1 t + c_2 t^2)(1-t).$$

To show the freeness of arrangements, we often use the addition-deletion theorem. Let $\mathcal{A} \neq \emptyset$ be an arrangement, $H \in \mathcal{A}$ be a hyperplane, $\mathcal{A}' := \mathcal{A} \setminus H$ and let $\mathcal{A}'' := \mathcal{A}' \cap H$.

Theorem 1.2 ([OT], Theorem 4.51). *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple defined above. Any two of the following statements imply the third:*

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp(\mathcal{A}) &= (b_1, \dots, b_{l-1}, b_l), \\ \mathcal{A}' \text{ is free with } \exp(\mathcal{A}') &= (b_1, \dots, b_{l-1}, b_l - 1), \\ \mathcal{A}'' \text{ is free with } \exp(\mathcal{A}'') &= (b_1, \dots, b_{l-1}). \end{aligned}$$

Next, let us consider the theory of 3-arrangements. Let $\mathcal{A}$ be an arrangement in a three-dimensional vector space V. Then the sheaf $\widetilde{D(\mathcal{A})}$ is a rank three vector bundle on $\mathbf{P}^2$ since $\widetilde{D(\mathcal{A})}$ is reflexive (e.g., see [H]). Fix a basis $\{X, Y, Z\}$ for V^* in such a way that the hyperplane $\{Z = 0\}$ is an element of $\mathcal{A}$. Regarding $\{Z = 0\}$ as the infinite line in $\mathbf{P}^2$, we define the *deconing* $d\mathcal{A}$ of a 3-arrangement $\mathcal{A}$ with respect to $\{Z = 0\}$ as

$$d\mathcal{A} := \{dH := H|_{Z=1} \mid H \in \mathcal{A} \setminus \{Z = 0\}\}.$$

Let us define $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[X, Y, Z]$. We define the *module of reduced logarithmic vector fields* $D_0(\mathcal{A})$ as follows:

Definition 1.4. *The S-module $D_0(\mathcal{A})$ is defined by*

$$D_0(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(Z) = 0\}.$$

Note that for any (central) arrangement $\mathcal{A}$, there exists an derivation

$$\theta_E := X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z} \in D(\mathcal{A}).$$

We call this derivation θ_E the *Euler derivation.* It is obvious that

$$D_0(\mathcal{A}) \simeq D(\mathcal{A})/(S \cdot \theta_E).$$

Hence the structure of $D_0(\mathcal{A})$ does not depend on the choice of the coordinates of V. Moreover, in the notation of Theorem 1.1, it holds that

$$c_t(\widetilde{D_0(\mathcal{A})}) = 1 - c_1 t + c_2 t^2.$$

As we saw above, we can restrict a given arrangement $\mathcal{A}$ on the plane $H_0 := \{Z = 0\} \in \mathcal{A}$. Moreover, we can obtain a multiarrangement $(\mathcal{A} \cap H_0,\ m)$ and the restriction homomorphism

$$\varphi : D_0(\mathcal{A}) \to D(\mathcal{A} \cap H_0,\ m),$$

defined as follows:

$$D_0(\mathcal{A}) \ni \theta \mapsto \theta|_{Z=0} \in D(\mathcal{A} \cap H_0,\ m).$$

For the details of this homomorphism, see [Z]. We can compute the codimension (as $\mathbb{K}$-vector spaces) of the image of φ from the characteristic polynomial of $\mathcal{A}$ and the exponents of $D(\mathcal{A} \cap H_0,\ m)$ by the following theorem, which is a variant of Theorem 3.2 in [Yo].

Theorem 1.3 (Yoshinaga). *With the above notation, let $\{\theta_1, \theta_2\}$ be a basis for a free $S/(S \cdot Z)$-module $D(\mathcal{A} \cap H_0, m)$ such that* $\deg(\theta_i) = d_i$ $(i = 1, 2)$. *Then the dimension of* $\operatorname{coker}(\varphi)$ *(as a $\mathbb{K}$-vector space) is finite and is given by*

$$\chi_0(\mathcal{A}, 0) - d_1 d_2.$$

In particular, $\mathcal{A}$ is free if and only if

$$\chi_0(\mathcal{A}, 0) = d_1 d_2.$$

2. Freeness of A_2 and B_2-type arrangements.

Generally it is difficult to consider whether a given arrangement is free or not. In this section we introduce some families of arrangements and determine when they are free and not free. In this section we assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero, and consider only 3-arrangements. Let V be a three-dimensional vector space over $\mathbb{K}$. Fix a basis $\{X, Y, Z\}$ for V^* and put $S := \mathrm{Sym}(V^*)$.

Definition 2.1. *A family of arrangements $\{\mathcal{A}(k)\}_{k \in \mathbf{Z}_{>0}}$ in V is called a family of A_2-type arrangements if there exist integers a, b, c, f such that $\mathcal{A}(k)$ is defined as follows:*

$$\begin{aligned} X &= (-k+1)Z, \ldots, (k+c-1)Z \ (c \geq 0), \\ Y &= (-k+1)Z, \ldots, (k+f-1)Z \ (f = 0 \text{ or } 1), \\ Y + X &= (-k+a)Z, \ldots, (k+a+b-1)Z \ (b \geq -1), \\ Z &= 0, \end{aligned}$$

Moreover, we call each arrangement $\mathcal{A}(k)$ in a family of A_2-type arrangements $\{\mathcal{A}(k)\}$ an A_2-type arrangement.

Definition 2.2. *A family of arrangements $\{\mathcal{B}(k)\}_{k \in \mathbf{Z}_{>0}}$ in V is called a family of B_2-type arrangements if there exist integers a, b, c, d, e, f such that $\mathcal{B}(k)$ is defined as follows:*

$$\begin{aligned} X &= (-k+1)Z, \ldots, (k+c-1)Z \ (c \geq 0), \\ Y &= (-k+1)Z, \ldots, (k+f-1)Z \ (f \geq 0), \\ Y + X &= (-k+a)Z, \ldots, (k+a+b-1)Z \ (b \geq -1), \\ Y - X &= (-k+d)Z, \ldots, (k+d+e-1)Z \ (e \geq -1), \\ Z &= 0, \end{aligned}$$

Moreover, we call each arrangement $\mathcal{B}(k)$ in a family of B_2-type arrangements $\{\mathcal{B}(k)\}$ a B_2-type arrangement.

It is obvious that these arrangements are generalizations of the classical Coxeter arrangements of type A_2 and B_2. The author classified the freeness and non-freeness of these arrangements in [A1] and [A2] as follows:

Theorem 2.1 ([A1], Theorem 0.4 and 0.5). *Let $\{\mathcal{A}(k)\}$ be the family of A_2-type arrangements such that $\mathcal{A}(1)$ is defined by*

$$\begin{aligned} X &= 0, Z, \dots, cZ \ (c \geq 0), \\ Y &= 0, Z, \dots, fZ \ (f = 0 \text{ or } 1), \\ Y+X &= (a-1)Z, \dots, (a+b)Z \ (b \geq -1), \\ Z &= 0. \end{aligned}$$

Let us put

$$N := 2a + b - c - f.$$

(a) *For sufficiently large k, $\mathcal{A}(k)$ is free if and only if*

$$N = 0, 1, 2.$$

Moreover, let $(1, d_1^k, d_2^k)$ be the exponents of the arrangement $\mathcal{A}(k)$. Then $|d_1^k - d_2^k| = 0$ or 1.

(b) *For sufficiently large k, $D_0(\widetilde{\mathcal{A}(k)})$ is stable if and only if*

$$N \leq -1 \text{ or } N \geq 3.$$

Theorem 2.2 ([A2]). *Let $\{\mathcal{B}(k)\}$ be the family of B_2-type arrangements such that $\mathcal{B}(1)$ is defined by*

$$\begin{aligned} X &= 0, Z, \dots, cZ \ (c \geq 0), \\ Y &= 0, Z, \dots, fZ \ (f \geq 0), \\ Y+X &= (a-1)Z, \dots, (a+b)Z \ (b \geq -1), \\ Y-X &= (d-1)Z, \dots, (d+e)Z \ (e \geq -1), \\ Z &= 0. \end{aligned}$$

Let us put

$$\begin{aligned} B_1 &:= 2(a + \frac{1}{2}b - \frac{1}{2}c - \frac{1}{2}f - \frac{1}{2}), \\ B_2 &:= 2(d + \frac{1}{2}e + \frac{1}{2}c - \frac{1}{2}f - \frac{1}{2}), \\ B_3 &:= 2(\frac{1}{2}c - \frac{1}{2}f), \end{aligned}$$

and put

$$M := B_1^2 + B_2^2 + B_3^2.$$

Then for sufficiently large k, $\{\mathcal{B}(k)\}$ is free if and only if

$$M = 0, 1, 2$$

or

$$a + d \text{ is even and } M = 3,$$

or

$$a + d \text{ is even and } M = 4.$$

Moreover, let $(1, d_1^k, d_2^k)$ be the exponents of the arrangement $\mathcal{B}(k)$ and $d^k := |d_1^k - d_2^k|$. Then d^k is two if and only if $M = 0$, $a + d$ is even, b

and e are both odd numbers and $\sum_{i=1}^{3} B_i$ is even. Otherwise d^k is zero or one.

In particular, we can see the freeness of these arrangements are determined by the combinatorics. We have completely classified the stability and semistability of B_2-type arrangements, see [A2] for details. In the rest of this section we show the freeness of an A_2-type arrangement and review that of a B_2-type. For the complete proofs and details of Theorem 2.1 and 2.2, see [A1] and [A2].

First let us prove (a) in Theorem 2.1. To show that, we need to calculate the Chern polynomial of $\widetilde{D_0(\mathcal{A}(k))}$, which can be obtained from the characteristic polynomial by Theorem 1.1.

Lemma 2.3. *With the notation in Definition 2.1, for sufficiently large k, it holds that*

$$\begin{aligned} c_t(\widetilde{D_0(\mathcal{A}(k))}) = 1 &- (6k+b+c-2)t \\ &+ ((3k+\frac{1}{2}b+\frac{1}{2}c-1)^2+(a+\frac{1}{2}b-\frac{1}{2}c-\frac{1}{2})^2-\frac{1}{4})t^2. \end{aligned}$$

Then the freeness can be seen by Wakamiko's result in [W], which asserts multi-exponents of 2-multiarrangements consisting of three lines are of the form (u,u) or $(u,u+1)$ ($u \in \mathbb{Z}$), combined with Theorem 1.3. Moreover, by using the same argument as above, we can show that $\mathcal{A}(k)$ is free for all $k \in \mathbb{Z}$ if the condition (a) in Theorem 2.1 is satisfied.

Next let us review the proof of Theorem 2.2 in brief. First let us consider the "if" part. Since there are no results on the multi-exponents of 2-multiarrangements which consist of four lines, we can not use the same argument as in the A_2-case. So we use the addition-deletion theorem (Theorem 1.2).

In this article we only consider the case when $M=0$. This is equivalent to $c=f,\ 2a+b=2c+1,\ 2d+e=1$. First we assume $a+d$ is odd. We show this implies $\mathcal{B}(k)$ is free by the induction on c. Assume $c=f=0$. It is known that the arrangement defined by

$$\begin{aligned} X &= (-k+1)Z,(-k+2)Z,\dots,(k-1)Z, \\ Y &= (-k+1)Z,(-k+2)Z,\dots,(k-1)Z, \\ Y+X &= (-k+1)Z,(-k+2)Z,\dots,(k-1)Z, \\ Y-X &= (-k+1)Z,(-k+2)Z,\dots,(k-1)Z, \\ Z &= 0, \end{aligned}$$

is free with exponents $(1,4k-3,4k-1)$. We call this arrangement $\mathcal{B}_0(k)$. We prove by induction on $a+d$ and using the addition-deletion theorem. The condition implies $a \le 1$ and $d \le 1$. Hence $a+d$ is maximal when $(a,b,d,e) = (1,-1,0,1)$ or $(0,1,1,-1)$. In this case $a+d=1$. First we consider the former case. At first we add the plane

$$H_1(k) := \{Y - X = kZ\}$$

to $\mathcal{B}_0(k)$ and secondly

$$H_2(k) := \{Y - X = -kZ\}$$

to $\mathcal{B}_0(k) \cup H_1(k)$. Note the family $\{\mathcal{B}_0(k) \cup H_1(k) \cup H_2(k)\}$ is the family of B_2-type arrangements which is defined by the condition $(a,b,d,e) = (1,-1,0,1)$. It is easy to see that

$$|\mathcal{B}_0(k) \cap H_1(k)| = 1 + (k) + (2k - 1 - (-k+1) + 1) = 4k,$$

so the addition-deletion theorem shows the family is free with

$$\exp(\mathcal{B}_0(k) \cup H_1(k)) = (1, 4k-2, 4k-1)$$

when $(a,b,c,d,e,f) = (1,-1,0,0,1,0)$. Similarly, we can see that

$$|(\mathcal{B}_0(k) \cup H_1(k)) \cap H_2(k)| = 4k,$$

so the addition-deletion theorem shows

$$\exp(\mathcal{B}_0(k) \cup H_1(k) \cup H_2(k)) = (1, 4k-1, 4k-1).$$

For the rest of this article we express the above process as follows:

add/delete a line	$\mathcal{B} \cap \{\text{the line}\}$	exponents
add $H_1(k)$	$4k$	$(1, 4k-2, 4k-1)$
add $H_2(k)$	$4k$	$(1, 4k-1, 4k-1)$

The second case $(a,b,d,e) = (0,1,1,-1)$ can be proved by the same way, so the first step of induction is completed. Let us assume that

$$\exp(\mathcal{B}(k)) = (1, 4k-a-d, 4k-a-d)$$

for $\mathcal{B}(k)$ defined as above. Since $a+d$ is odd, it suffices to show the freeness is invariant under the following three transforms of (a,d):

(P1) $(a,d) \mapsto (a-2,d)$.
(P2) $(a,d) \mapsto (a,d-2)$.
(P3) $(a,d) \mapsto (a-1,d-1)$.

Noting that the plane $Y + X = (k+a+b-1)Z$ is equal to $Y + X = (k-a)Z$ in this case, (P1) is shown as follows:

add/delete a line	$\mathcal{B} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y + X = (k-a+1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+2)$
add $Y + X = (-k+a-2)Z$	$4k-a-d+3$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y + X = (k-a+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$

(P2) is as follows:

add/delete a line	$\mathcal{B} \cap \{\text{the line}\}$	exponents
add $Y - X = (-k+d-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y - X = (k-d+1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+2)$
add $Y - X = (-k+d-2)Z$	$4k-a-d+3$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (k-d+2)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$

(P3) is as follows:

add/delete a line	$\mathcal{B} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$4k-a-d+1$	$(1, 4k-a-d, 4k-a-d+1)$
add $Y - X = (-k+d-1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+1)$
add $Y + X = (k-a+1)Z$	$4k-a-d+2$	$(1, 4k-a-d+1, 4k-a-d+2)$
add $Y - X = (k-d+1)Z$	$4k-a-d+3$	$(1, 4k-a-d+2, 4k-a-d+2)$

These tables show the condition $c = f = 0,\ 2a + b = 1,\ 2d + e = 1$ implies the freeness. Let us assume the statement is true for $c - 1 \geq 0$. We show that when c the arrangement is free with the exponents $(1, 4k-a-d+2c, 4k-a-d+2c)$. Let us put

$$A := 4k - a - d + 2c.$$

We add/delete four lines to/from the case when c to reduce to the case $c - 1$ as follows:

add/delete a line	$\mathcal{B} \cap \{\text{the line}\}$	exponents
add $Y + X = (-k+a-1)Z$	$A+1$	$(1, A, A+1)$
delete $Y = (k+c-1)Z$	$A+1$	$(1, A, A)$
delete $X = (k+c-1)Z$	$A+1$	$(1, A-1, A)$
add $Y + X = (-k+a-2)Z$	$A+1$	$(1, A, A)$

i.e., we made the change of variables $(a, b, c) \mapsto (a-2, b+2, c-1)$. By the condition on a and b, this kind of addition/deletion can be completed without restrictions. So the case $c = f,\ 2a + b = 2c + 1,\ 2d + e = 1$ and $a + d$ is odd is proved. When $a + d$ is even, the same argument can be used to show the freeness and so is for the other cases.

For the "only if" part, we need some arguments on the normalization of vector bundles. For details see [A2].

3. Yuzvinsky's freeness criterion and its extension.

As we have seen in previous sections, it is difficult to characterize when an arrangement is free or not free. In this section we introduce Yuzvinsky's freeness criterion for simple arrangements by using lattice cohomologies and generalize it to that for multiarrangements. First we consider in a general situation. Let Q be a finite poset, i.e., a finite set which is partially ordered. We view it as a topological space with the topology consisting of all increasing subsets of Q, i.e., the sets $\{R \subset Q \mid X \in R,\ Y \in Q,\ X \leq Y \Rightarrow Y \in R\}$. Regarding the order

$X \leq Y$ $(X,\ Y \in Q)$ as the arrow $X \to Y$, we can consider Q as a category. Then we can define sheaves on posets as follows:

Definition 3.1. *$\mathcal{F}$ is a sheaf of abelian groups on Q if $\mathcal{F}$ is a covariant functor from Q to the category of abelian groups.*

For each $X \in Q$ the stalk of $\mathcal{F}$ at X is denoted by $\mathcal{F}(X)$ and the restriction homomorphism $\rho_{Y,X} : \mathcal{F}(X) \to \mathcal{F}(Y)$ $(X \leq Y \in Q)$ is canonically induced from the covariant functor $\mathcal{F}$. For each open set $U \subset Q$, the global section $H^0(U, \mathcal{F})$ of $\mathcal{F}$ on U is defined by

$$H^0(U, \mathcal{F}) := \varprojlim_{X \in U^{op}} \mathcal{F}(X),$$

where U^{op} is an opposite category of U. By this definition it is easy to see that the sheaf $\mathcal{F}$ satisfies usual sheaf properties. From now on we consider the sheaf of $S := \mathbb{K}[X_1, \dots, X_l]$-modules on Q. By the usual way, the flasque resolution $\mathcal{F}^\bullet$ of $\mathcal{F}$ is defined, and we define the i-th cohomology group $H^i(Q, \mathcal{F})$ of the sheaf $\mathcal{F}$ as $H^i(Q, \mathcal{F}) := H^i(H^0(Q, \mathcal{F}^\bullet))$ (see [Yu1] for details).

In this article, the most important example of the poset Q is an intersection lattice $L(\mathcal{A})$ of an arrangement $\mathcal{A}$. The order is defined as

$$X \leq Y \iff X \supset Y \ (X,\ Y \in L(\mathcal{A})).$$

Let us put $U := \bigcap_{H \in \mathcal{A}} H$, which is non-empty since $\mathcal{A}$ is central. Let us put

$$L_0 := L(\mathcal{A})^{op} \setminus U$$

and put

$$L_X := \{Y \in L_0 \mid X \subsetneq Y\}.$$

An example of the sheaf $\mathcal{F}$ on L_0 is $\mathcal{D}$, which is defined by the following manner:

$$L_0 \ni X \mapsto \mathcal{D}(X) := D(\mathcal{A}_X),$$

where $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$. Since $D(\mathcal{A}_X) \subset D(\mathcal{A}_Y)$ for $X \subset Y$, $\mathcal{D}$ is a covariant functor from L_0 to the category of abelian groups. We introduce the locality of a sheaf, which plays an important role in [Yu1] and this article.

Definition 3.2. *Let $\mathcal{F}$ be a sheaf of S-modules on L_0 and $P \in$ Spec(S). Then the sheaf $\mathcal{F}_P$ is defined as*

$$\mathcal{F}_P(X) := \mathcal{F}(X)_P \ (X \in L_0).$$

Moreover, we say a sheaf $\mathcal{F}$ is local if for all $P \in$ Spec(S) and for all $X \in L_0$ the restriction morphism $\mathcal{F}(X)_P \to \mathcal{F}(X(P))_P$ is an isomorphism, where

$$X(P) := \bigcap_{X \subset H,\ \alpha_H \in P} H.$$

Now we can state the freeness criterion by Yuzvinsky.

Theorem 3.1 ([Yu2], Theorem 1.1). *An arrangement $\mathcal{A}$ is free if and only if $H^i(L_X, \mathcal{D}) = 0$ for every $X \in L(\mathcal{A})$ and every i, $0 < i <$* $\operatorname{codim} X - 1$.

The aim of this section is to generalize this criterion to that for multiarrangements. Let $(\mathcal{A}, m)$ be a multiarrangement. Let us define the sheaf $\mathcal{D}(m)$ on $L_0 := L(\mathcal{A})$ by

$$L_0 \ni X \mapsto \mathcal{D}(m)(X) := D(\mathcal{A}_X, m_X),$$

where m_X is a multiplicity on $\mathcal{A}_X$ such that $m(H) = m_X(H)$ for all $H \in \mathcal{A}_X$. Then we can show the following main theorem in this section.

Theorem 3.2. *A multiarrangement $(\mathcal{A}, m)$ is free if and only if it holds that $H^i(L_X, \mathcal{D}(m)) = 0$ for every $X \in L(\mathcal{A})$ and every i, $0 < i <$* $\operatorname{codim} X - 1$.

To show Theorem 3.2 we use the similar argument to that in [Yu1]. For that we need some results for multiarrangements as follows:

Lemma 3.3. *The sheaf $\mathcal{D}(m)$ is local.*

Proof. It is easy to show from definitions. □.

Lemma 3.4. *Let V_i $(i = 1, 2)$ be a vector space and $(\mathcal{A}_i, m_i)$ be a multiarrangement in V_i. Let us define a multiarrangement $(\mathcal{A}_1 \oplus \mathcal{A}_2, m_1 \oplus m_2) =: (\mathcal{A}, m)$ in a vector space $V_1 \oplus V_2 =: V$ by the following manner:*

$$\begin{aligned} \mathcal{A}_1 \oplus \mathcal{A}_2 : &= \{H_1 \oplus V_2 \text{ or } V_1 \oplus H_2 \mid H_i \in \mathcal{A}_i\}, \\ m(H_1 \oplus V_2) : &= m_1(H_1), \\ m(V_1 \oplus H_2) : &= m_2(H_2). \end{aligned}$$

Then it holds that

$$D(\mathcal{A}, m) \simeq S \cdot D(\mathcal{A}_1, m_1) \oplus S \cdot D(\mathcal{A}_2, m_2),$$

where $S := Sym(V^*)$.

Proof. Note that $\mathrm{Der}_{\mathbb{K}}(S) \simeq S \cdot \mathrm{Der}_{\mathbb{K}}(S_1) \oplus S \cdot \mathrm{Der}_{\mathbb{K}}(S_2)$ $(S_i := \mathrm{Sym}(V_i^*)$ $(i = 1, 2))$. Put $Q_i := \prod_{H \in \mathcal{A}_i} \alpha_H^{m_i(H)}$ and $D_i := D(\mathcal{A}_i, m_i)$ $(i = 1, 2)$. Then for all $\theta_1 \in D_1$, $\theta_1(Q_2) = 0$. By using the same argument, it holds that $S \cdot D_1 \oplus S \cdot D_2 \subset D(\mathcal{A}, m)$. Let us show the reverse inclusion. Take $\theta \in D := D(\mathcal{A}, m)$ and decompose $\theta = \theta_1 + \theta_2$ with $\theta_i \in S \cdot \mathrm{Der}_{\mathbb{K}}(S_i)$ $(S_i := \mathrm{Sym}(V_i^*))$. Since the arugments are the same, it suffices to show $\theta_1 \in S \cdot D_1$. Take a linear form α_1 such that $\alpha_1 | Q_1$ and put $m_1(\ker(\alpha_1)) = n_1$. Note $\theta(\alpha_1) = \theta_1(\alpha_1) \in \alpha_1^{n_1} S$. Let $\{g_i\}_{i \in I}$ be a basis for S_2 over $\mathbb{K}$. Then it is easy to see that $\{g_i\}$ is also independent over S_1. Let us put

$$\theta_1 = \sum g_i \eta_i \ (\eta_i \in \mathrm{Der}_{\mathbb{K}}(S_1)).$$

Since it holds that

$$\theta_1(\alpha_1) = \alpha_1^{n_1} \sum g_i h_i \ (h_i \in S_1) = \sum g_i \eta_i(\alpha_1),$$

we can see that $\eta_i(\alpha_1) \in \alpha_1^{n_1} \cdot S_1$ $(\forall i)$. Repeating the same argument as above, we can show $\theta_1 \in S \cdot D_1$. $\square$

Lemma 3.5. *It holds that*

$$\mathrm{pd}_S(D(\mathcal{A}_X, m_X)) \le \mathrm{pd}_S(D(\mathcal{A}, m)) \ (\forall X \in L(\mathcal{A})).$$

Proof. By using Lemma 3.4, we can prove by the similar manner to that of Lemma 2.1 in [Yu1]. $\square$

With the results above, we can apply the argument in [Yu1] to the multiarrangements, and obtain the following (non-)vanishing theorems.

Theorem 3.6. *If $\mathcal{A}$ is essential (that is, $\dim \bigcap_{H \in \mathcal{A}} H = 0$), then it holds that*

$$H^i(L_0, \mathcal{D}(m)) = 0 \ (0 < \forall i < \mathrm{depth}_S(D(\mathcal{A}, m)) - 1).$$

Corollary 3.7. *If $(\mathcal{A}, m)$ is not free but $(\mathcal{A}_X, m_X)$ is free for all $X \in L_0$, then $H^{d-1}(L_0, \mathcal{D}(m)) \neq 0$ $(d := \mathrm{depth}_S(D(\mathcal{A}, m))$.*

Before the proof of Theorem 3.2 we introduce the following lemma.

Lemma 3.8. *If an essential multiarrangement $(\mathcal{A}, m)$ is free, then for all $X \in L(\mathcal{A})$ a multiarrangement $(\mathcal{A}_X, m_X)$ is free.*

Proof. We can show by the similar way to that of Theorem 4.37 in [OT]. $\square$

Now let us prove Theorem 3.2. By the argument in the proof of Theorem 1.1 in [Yu2] and Lemma 3.4, it holds that

$$(1) \qquad H^i(L_X, \mathcal{D}(m)) \simeq H^i(L_X, \mathcal{D}^X(m)) \otimes S_X \ (i > 0),$$

where $\mathcal{D}^X(m)$ is a restriction of the sheaf $\mathcal{D}(m)$ onto L_X. If $(\mathcal{A}, m)$ is free, then by Lemma 3.8 $(\mathcal{A}_X, m_X)$ is also free for all $X \in L(\mathcal{A})$. Hence (1) and Theorem 3.6 shows the vanishing. Conversely assume $(\mathcal{A}, m)$ is not free. Then there exists $X \in L(\mathcal{A})$ such that $(\mathcal{A}_X, m_X)$ is not free but $(\mathcal{A}_Y, m_Y)$ is free for all $Y \supsetneq X$. Then Corollary 3.7 shows $H^{d-1}(L_X, \mathcal{D}^X(m)) \neq 0$ $(d := \mathrm{depth}_S(D(\mathcal{A}, m))$. $\square$

Remark 3.1. *We can show the multiarrangement version of the results in [Yu3], that is, for an essential multiarrangement $(\mathcal{A}, m)$ there exists an integer d depending only on the lattice $L(\mathcal{A})$ and the multiplicity m such that $H^i(L_0, \mathcal{D}(m))_e = 0$ for all $e > d$. However, we have not yet shown the openness of the set of free multiarrangements in a parameterizing space of those with a fixed lattice and multiplicity. For in the argument in [Yu3] Yuzvinsky used the Terao's factorization theorem for a simple arrangement which asserts the exponents of a simple free arrangement are determined only by the lattice. However, it is known that this theorem is not true for multiarrangements.*

4. Endomorphism sheaves of arrangements.

In the sheaf theory on an intersection lattices of arrangements, only few sheaves are considered, i.e., the sheaf $\mathcal{D}$ by Yuzvinsky and $\mathcal{D}(m)$ in this article. We want to consider a new sheaf on L_0 to characterize the freeness. Then what kind of sheaves are suitable for that purpose? To find such sheaves, let us review some splitting criterions for vector bundles on projective spaces.

Theorem 4.1 (Horrocks, [OSS], Theorem 2.3.1, Chapter I). *A vector bundle E on a projective space $\mathbf{P}^n_{\mathbb{K}}$ splits into a direct sum of line bundles if and only if*

$$\bigoplus_{d \in \mathbf{Z}} \bigoplus_{1 \le i \le n-1} H^i(\mathbf{P}^n_{\mathbb{K}}, E(d)) = 0.$$

Theorem 4.2 (Luk-Yau, [LY], Theorem B). *A vector bundle E on a projective space $\mathbf{P}^n_{\mathbb{C}}$ splits into a direct sum of line bundles if and only if*

$$\bigoplus_{d \in \mathbf{Z}} H^1(\mathbf{P}^n_{\mathbb{C}}, \mathcal{E}nd_{\mathcal{O}_{\mathbf{P}_{\mathbb{C}}}}(E)(d)) = 0.$$

There are some other splitting criterions related to an endomorphism sheaf $\mathcal{E}nd(E)_{\mathcal{O}_{\mathbf{P}_{\mathbb{K}}}}$, e.g., by Kempf or Sumihiro. So let us introduce an endomorphism sheaf $\mathcal{E}nd$ on an intersection lattice. A natural definition of it is the following correspondence:

$$X \in L(\mathcal{A}) \mapsto \mathrm{End}_S(D(X)) \ (D(X) := D(\mathcal{A}_X)).$$

However, in general, $\mathrm{End}_S(D(X)) \not\subset \mathrm{End}_S(D(Y))$ for $X \subset Y \in L(\mathcal{A})$. Hence we need some modifications for restriction morphisms.

Definition 4.1. *An endomorphism sheaf $\mathcal{E}nd$ on $L_0 = L(\mathcal{A})^{op} \setminus U$ for an central arrangement $\mathcal{A}$ with $S := Sym(V^*)$-module valued is defined as*

$$L_0 \ni X \mapsto End_S(D(X)),$$

with restrictions

$$\rho_{Y,X} : \mathcal{E}nd(X) \ni f \mapsto \frac{Q_X}{Q_Y} f \in \mathcal{E}nd(Y) \ (X \subset Y \in L_0),$$

where $Q_X := Q(\mathcal{A}_X)$.

It is easy to see that $\mathcal{E}nd$ is a covariant functor from L_0 to the category of abelian groups. We want to obtain a freeness criterion by using $\mathcal{E}nd$. For that, we have to see some properties of $\mathcal{E}nd$.

Lemma 4.3. $\mathrm{depth}_S(End(D)) \ge 2$ $(D := D(\mathcal{A}))$.

Proof. Since D is reflexive, there is a D-regular sequence (a, b) of length two. Then it is easy to see that (a, b) is also $\mathrm{End}_S(D)$-regular. $\square$

Lemma 4.4. *The sheaf $\mathcal{E}nd$ is local.*

Proof. It is easy to see from the definition of $\mathcal{E}nd$ and the locality of the sheaf $\mathcal{D}$. □

Lemma 4.5.

$$\mathrm{pd}_S(\mathcal{E}nd(X)) \leq \mathrm{pd}_S(End_S(D)) \ (\forall X \in L_0).$$

Proof. We can show by the same manner as that of Lemma 2.1 in [Yu1]. □

By using the same argument as in [Yu1] combined with above results, we can show the following (non-)vanishing theorems of the endomorphism sheaves.

Theorem 4.6. *Let $\mathcal{A}$ be an essential l-arrangement.*

(a) *It holds that*

$$H^i(L_0, \mathcal{E}nd) = 0 \text{ for } 0 < \forall i < l - \mathrm{pd}_S(End_S(D)) - 1.$$

(b) *If $\mathrm{pd}_S(\mathcal{E}nd(X)) < \mathrm{pd}_S(End_S(D))$ for all $X \in L_0$, then it holds that $H^{d-1}(L_0, \mathcal{E}nd) \neq 0$, where $d := \mathrm{depth}_S(End_S(D))$ (≥ 2).*

Now we can show the following freeness criterion in terms of $\mathcal{E}nd$.

Theorem 4.7. *A central arrangement $\mathcal{A}$ over $\mathbb{C}$ is free if and only if $H^i(L_X, \mathcal{E}nd) = 0$ for all $X \in L(\mathcal{A})$ and $1 \leq i \leq \mathrm{codim}(X) - 2$.*

Proof. The "only if" part follows immediately from Theorem 4.6 (a). To show the "if" part, let us assume that $\mathcal{A}$ is not free. We show in this case the module $\mathcal{E}nd(U)$ is also not free. Assume $\mathcal{E}nd(U)$ is free. Then the sheaf $\widetilde{\mathcal{E}nd(U)} \simeq \widetilde{\mathrm{End}_S(D(\mathcal{A}))} \simeq \mathcal{E}nd_{\mathcal{O}_{\mathbf{P}(V)}}(\widetilde{D(\mathcal{A})})$ splits. Hence $H^1(\mathbf{P}(V), \mathcal{E}nd(\widetilde{D(\mathcal{A})})(d)) = 0$ for all $d \in \mathbb{Z}$ and Theorem 4.2 shows $\widetilde{D(\mathcal{A})}$ splits. Since it holds that $D(\mathcal{A}) \simeq \oplus_{d \in \mathbf{Z}} H^0(\mathbf{P}(V), \widetilde{D(\mathcal{A})}(d))$ (see Lemma 4.4 in [AY]), we can see $\mathcal{A}$ is free, which is a contradiction. So it holds that $\mathcal{E}nd(U) = \mathrm{End}_S(D(\mathcal{A}))$ is not free. Then the same argument as in the proof of Theorem 3.2, combined with Theorem 4.6 (b), completes the proof. □

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