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A CATEGORICAL CONSTRUCTION OF ROOT SYSTEMS TYPE ADE CASE

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(JOINT WORK WITH HIROSHIGE KAJIURA AND ATSUSHI TAKAHASHI)

There is a celebrated correspondence, which is nowadays often called the McKay correspondence, between simple singularities and simple (or classical) root systems. For this reason, simple singularities are also called ADE-singularities. In fact, in [Bs], Brieskorn has described the universal deformation and the simultaneous resolution of a simple singularity by the corresponding simple Lie algebra. Then the primitive form [Sa2] for the simple singularity is described by the Kostant-Kirirov form of the corresponding simple Lie algebra [Ya]. The flat structure (Frobenius mfd structure) on the deformation space is described by corresponding simple Weyl group [Sa6]. As the next case to the simple singularities, the simple elliptic singularities [Sa1] correspond to elliptic root systems [Sa4]. In fact, the flat structure is describe by the elliptic root systems [Sa4]. The constructions of the deformation and the primitive form for the elliptic singularities in terms of the elliptic root systems are in progress.

Inspired by these studies and in order to construct the primitive forms for a wider class of singularities, the author introduced a concept of a regular weight systems [Sa3] and asked to construct a suitable Lie theory by generalizing the concept of root systems by abstracting the structure of vanishing cycles of the singularity ([Sa5], Problem in p.124 in English version, see also [Sa8] sections 6 and 7 for the relation with the primitive form).

In [T2], based on the mirror symmetry for the Landau-Ginzburg orbifolds and also the duality theory of the weight systems [Sa7],[T1], A. Takahashi proposed a new approach to the root systems, answering to the above problem. Namely, he, by introducing a triangulated category of graded matrix factorizations (which is equivalent to the one introduced by Orlov [O2] independently) for a weighted homogeneous polynomial f attached to a regular weight system, showed that the category for a polynomial of type A_l is equivalent to the bounded derived category of modules over the path algebra of the Dynkin quiver of type A_l.

The goal of the talk is to showed the same type equivalences for all simple polynomials of type ADE (joint with H. Kajiura and A. Takahashi [KTS]). We denote by $HMF_R^{gr}(f)$ the triangulated category of graded matrix factorizations over the graded ring $R := \mathbf{C}[x, y, z]$.

Theorem 0.1 ([KTS]). *For a polynomial f of type ADE, $HMF_R^{gr}(f)$ is equivalent, as a triangulated category, to the derived category $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta})$ of finite modules over the path* *algebra* $\mathbb{C}\vec{\Delta}$ *of any Dynkin quiver* $\vec{\Delta}$ *of the type of* f. *In particular, the set of indecomposable objects in the* K*-group of the category* $HMF_R^{gr}(f)$ *is the root system of the type of* f.

Here, by a Dynkin quiver, we mean an oriented Dynkin diagram of type ADE.

The matrix factorizations were introduced and studied by Eisenbud [E] in the study of the maximal Cohen-Macaulay modules (see [K], [Yo] and their references). For the proof of Theorem 0.1, we first describe the Auslander-Reiten quiver for the triangulated category $HMF_{\hat{\mathcal{O}}}(f)$ of matrix factorizations over the local rings $\mathcal{O}$ and $\hat{\mathcal{O}}$ due to [E], [AR2] and [A]. Then, by "lifting" the results to the graded category, in [KTS], we list up all graded matrix factorizations for f explicitly, and give a complete list of indecomposable objects and irreducible morphisms in $HMF_R^{gr}(f)$. We also show the Serre duality holds in the category. From these data, we can show an existence of a collection indecomposable objects in $HMF_R^{gr}(f)$ corresponding to any given Dynkin quiver $\vec{\Delta}$ of ADE-type of the polynomial f. Then, applying a theorem by Bondal-Kapranov ([BK] Theorem 1) we see that $D^b(\text{mod-}\mathbb{C}\vec{\Delta})$ is a full triangulated subcategory of $HMF_R^{gr}(f)$. On the other hand, we observe the equalities:

The number of indecomposable objects of $HMF_R^{gr}(f)$ up to the shift functor T
$=$ the number of the positive roots for the root system of type ADE ($= \frac{l \cdot h}{2}$)
$=$ the number of indecomposable objects of $D^b(\text{mod-}\mathbb{C}\vec{\Delta})$ up to the shift functor T
(Gabriel's theorem [Ga]). This proves our main theorem $D^b(\text{mod-}\mathbb{C}\vec{\Delta}) \simeq HMF_R^{gr}(f)$.

One of the advantage of our formulation is that we can further define a stability condition, the notion of which is introduced by Bridgeland [Bd], for the triangulated category $HMF_R^{gr}(f)$. A stability condition can be naturally given by the grading of matrix factorizations. In fact, the abelian category assocated to the stability condition (as a full subcategory of $HMF_R^{gr}(f)$) is equivalent to the category $\text{mod-}\mathbb{C}\vec{\Delta}$ of finite modules over the path algebra of a Dynkin quiver $\vec{\Delta}$, whose orientation is the principal orientation introduced in [Sa9].

References

[A] M. Auslander, *Rational singularities and almost split sequences*, Trans. Amer. Math. Soc. 293 (1986) 511–531.

[AR2] M. Auslander and I. Reiten, *Almost split sequences for rational double points*, Trans. AMS., 302 (1987) 87-99.

[Bd] T. Bridgeland, *Stability conditions on triangulated category*, math.AG/0212237.

[Bs] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 279–284. Gauthier-Villars, Paris, 1971.

[BK] A. Bondal and M. Kapranov, *Enhanced triangulated categories*, Math. USSR Sbornik, Vol.70, (1991) No.1, 93–107.

[E] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. AMS., 260 (1980) 35–64.

[Ga] P. Gabriel, *Unzerleghare Derstellungen I*, Manuscripta Math., **6** (1972) 71–163.

[K] H. Knörrer, *Cohen-Macauley modules on hypersurface singularities I*, Invent. Math., **88** (1987) 153-164.

[KTS] H. Kajiura, K. Saito and A. Takahashi, "Matrix Factorizations and Representations of Quivers II: type ADE case," math.ag/0511155.

[O2] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, math.AG/0503632.

[Sa1] K. Saito, *Einfach elliptische Singularitäten*, Invent. Math. **23** (1974) 289-325,

[Sa2] K. Saito, *Period Mapping Associated to a Primitive Form*, Publ. RIMS, Kyoto Univ. **19**(1983) 1231-1264.

[Sa3] K. Saito, *Regular system of weights and associated singularities*, Complex analytic singularities, 479–526, Adv. Stud. Pure Math., 8, North-Holland, Amsterdam, 1987.

[Sa4] K. Saito, *Extended affine root systems. I-IV*, Publ. Res. Inst. Math. Sci. 21 (1985), 75–179; 26 (1990)15–78; 33 (1997)301–329; 36 (2000) 385–421.

[Sa5] K. Saito, *Around the Theory of the Generalized Weight System: Relations with Singularity Theory, the Generalized Weyl Group and Its Invariant Theory, Etc.*, (Japanese) Sugaku 38 (1986), 97–115, 202–217; (Translation in English), Amer. Math. Soc. Transl. (2) Vol.183 (1998) 101- 143.

[Sa6] K. Saito, *On a linear Structure of the Quotient Variety by a finite Reflection Group*, Publ. RIMS, Kyoto Univ. **29**(1993) 535-579.

[Sa7] K. Saito, *Duality for Regular Systems of Weights*, Asian. J. Math. **2** no.4 (1998) 983-1048.

[Sa8] K. Saito, *Primitive Automorphic Forms*, Mathematics Unlimited - 2001 and Beyond, edited by Engquist & Schmidt, Springer Verlag, (2001) 1003-1018.

[Sa9] K. Saito, *Principal Γ-cone for a tree* Γ, preprint, RIMS-1507, June 2005, math.CO/0510623.

[T1] A. Takahashi, *K. Saito's Duality for Regular Weight Systems and Duality for Orbifoldized Poincaré Polynomials*, Commun. Math. Phys. **205** (1999) 571-586.

[T2] A. Takahashi, *Matrix Factorizations and Representations of Quivers I*, math.AG/0506347.

[Ya] H. Yamada, *Lie group theoretic construction of period mappings*, Math. Z. **220** (1995) 231-255.

[Yo] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990. viii+177 pp.

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