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ON INVERSES FOR DIFFERENTIAL OPERATORS

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ABSTRACT. We propose geometrically invariant formulas for fundamental solutions and heat kernels of subelliptic partial differential operators.

1. The fundamental solution

The differential operator

$$\Delta_\lambda = \frac{1}{2}(X_1^2 + X_2^2) - \frac{1}{2}i\lambda[X_1, X_2], \tag{1.1}$$

with

$$\begin{cases} X_1 = \frac{\partial}{\partial x_1} + 2kx_2|x|^{2k-2}\frac{\partial}{\partial u}, \\ X_2 = \frac{\partial}{\partial x_2} - 2kx_1|x|^{2k-2}\frac{\partial}{\partial u}, \end{cases} \tag{1.2}$$

and $|x|^2 = x_1^2 + x_2^2$, $[X_1, X_2] = X_1X_2 - X_2X_1$ is not elliptic since it is the sum of squares of only 2 linearly independent vector fields in 3 dimensions, but consecutive Lie brackets of X_1 and X_2 do generate the full tangent space at every point of $\mathbb{R}^3$, so Δ_λ is subelliptic. The fundamental solution K_λ is the distribution solution of

$$\Delta_\lambda K_\lambda(x, u; x^{(0)}, u^{(0)}) = \delta(x - x^{(0)})\delta(u - u^{(0)}), \tag{1.3}$$

parametrized by $(x^{(0)}, u^{(0)}) \in \mathbb{R}^3 \times \mathbb{R}^3$. We set

$$K_\lambda(x, u; x^{(0)}, u^{(0)}) = \int_{\mathbb{R}} \frac{v_\lambda(x, u; x^{(0)}, u^{(0)}, \tau)}{g(x, u; x^{(0)}, u^{(0)}, \tau)} E(x, u; x^{(0)}, u^{(0)}, \tau)d\tau, \tag{1.4}$$

where g is a solution of the Hamilton-Jacobi equation

$$\frac{\partial g}{\partial \tau} + \frac{1}{2}(X_1g)^2 + \frac{1}{2}(X_2g)^2 = 0. \tag{1.5}$$

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g is given by a modified action integral of a complex Hamiltonian problem. The associated energy

$$E = -\frac{\partial g}{\partial \tau} \tag{1.6}$$

is the first invariant of the motion, and the volume element v_λ is the solution of a second order transport equation. Let

$$H = \frac{1}{2}(\xi_1 + 2kx_2|x|^{2k-2}\theta)^2 + \frac{1}{2}(\xi_2 - 2kx_1|x|^{2k-2}\theta)^2 \tag{1.7}$$

denote the Hamiltonian where ξ_1, ξ_2 and θ represent the momenta of x_1, x_2, and u, respectively. The complex bicharacteristics are solutions of the Hamiltonian system of differential equations

$$\begin{cases} \dot{x}_j = H_{\xi_j}, & \dot{\xi}_j = -H_{x_j}, \quad j = 1, 2, \\ \dot{u} = H_\theta, & \dot{\theta} = -H_u \end{cases} \tag{1.8}$$

with the somewhat unusual boundary condition

$$\begin{cases} x_1(0) = x_1^{(0)}, & x_1(\tau) = x_1, \\ x_2(0) = x_2^{(0)}, & x_2(\tau) = x_2, \end{cases} \tag{1.9}$$

$$u(\tau) = u, \tag{1.10}$$

$$\theta(0) = -i. \tag{1.11}$$

Then the energy E is

$$E = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2,$$

and the modified action g is given by

$$g = -iu(0) + \int_0^\tau \Big[\sum_{j=1}^{2} \xi_j(s)\dot{x}_j(s) + \theta(s)\dot{u}(s) - H\big(x(s), u(s), \xi(s), \theta(s)\big)\Big] ds.$$

We note that the "missing direction" u must be treated separately.

The volume element v_λ is the solution of the following second order "transport" equation:

$$(T + \Delta_\lambda g)\frac{\partial v_\lambda}{\partial \tau} + E\Delta_\lambda v_\lambda = 0, \tag{1.12}$$

see [1], p. 92, where

$$T = \frac{\partial}{\partial \tau} + \sum_{j=1}^{2}(X_j g)X_j \tag{1.13}$$

is differentiation along the bicharacteristic. Formula (1.4) has a simple geometric interpretation. The operator Δ_λ has a characteristic variety in $T^*\mathbb{R}_3$ given by $H = 0$. Over every point $(x, u) \in \mathbb{R}_3$, $H = 0$ is a line parametrized by $\theta \in (-\infty, \infty)$,

$$\xi_1 = -2kx_2|x|^{2k-2}\theta, \quad \xi_2 = 2kx_1|x|^{2k-2}\theta. \tag{1.14}$$

Consequently, K_λ may be thought of as the $(\text{action})^{-1}$ summed over the characteristic variety with measure Ev_λ. We note that when Δ_λ is elliptic, its characteristic variety is the zero section, so we do get simply $(\text{action})^{-1}$, the Newton potential, as expected; τg behaves like the square of a distance function, even though it is complex.

K_λ has been worked out explicitly in [1]. Let $(x^{(0)}, u^{(0)})$ and (x, u) denote 2 arbitrary points of $\mathbb{R}^3$. We obtain 2 invariants of the motion, the energy E and the angular momentum Ω. Then

$$\begin{aligned} g = -i(u - u^{(0)}) &+ \Big(1 - \frac{1}{k}\Big)E\tau \\ &+ \frac{1}{2k}\operatorname{sgn}\tau\Big[\sqrt{2E|x|^2 + W\big(|x|^2\big)^2} - \sqrt{2E|x^{(0)}|^2 + W(|x^{(0)}|^2)^2}\Big], \end{aligned}$$

where we use the principal branch of the square roots, and

$$W(u) = 2ku^k - \Omega, \quad \Omega = \Omega(x, u; x^{(0)}, u^{(0)}, \tau). \tag{1.15}$$

Theorem 1.1. *The fundamental solution $K_\lambda(x, u; x^{(0)}, u^{(0)})$ has the following invariant representation:*

$$K_\lambda = \int_{\mathbb{R}} v_\lambda \frac{E d\tau}{g} = -\int_{g_-}^{g_+} v_\lambda \frac{dg}{g}, \tag{1.16}$$

where the second order transport equation for v_λ may be reduced to an Euler-Poisson-Darboux equation and solved explicitly as a function of E and Ω. Namely,

$$v_\lambda = \frac{c_\lambda}{k}(A_+ - g)^{-\frac{1-\lambda}{2}}(A_- + g)^{-\frac{1+\lambda}{2}} F_\lambda(q_+, q_-),$$

with

$$c_\lambda = \frac{-e^{-i\pi\frac{1-\lambda}{2}}}{4\pi^2\Gamma(\frac{1-\lambda}{2})\Gamma(\frac{1+\lambda}{2})},$$

$$A_\pm = \frac{1}{k}\Omega_\pm + g_\pm, \quad \Omega_\pm = \lim_{\tau\to\pm\infty}\Omega, \quad g_\pm = \lim_{\tau\to\pm\infty} g,$$

and

$$q_\pm = \frac{2^{1/k}(x_1 \pm ix_2)(x_1^{(0)} \mp ix_2^{(0)})}{(A_\pm \mp g)^{1/k}},$$

and $F_\lambda(q_+, q_-)$ *is a hypergeometric function of 2 variables*

$$F_\lambda(q_+,q_-) = \frac{1}{\Gamma(\frac{1-\lambda}{2})\Gamma(\frac{1+\lambda}{2})}\int_0^1\int_0^1 \frac{ds_+ds_-}{s_+s_-}\Big\{\Big(\frac{s_+}{1-s_+}\Big)^{\frac{1-\lambda}{2}}\Big(\frac{s_-}{1-s_-}\Big)^{\frac{1+\lambda}{2}} \cdot \frac{1-q_+q_-(s_+s_-)^{1/k}}{(1-q_+s_+^{1/k})(1-q_-s_-^{1/k})\big(1-(q_+q_-)^k s_+s_-\big)}\Big\}.$$

2. The heat kernel

$P_t(x,u;x^{(0)},u^{(0)})$, $t > 0$, the heat kernel, is the solution of the following initial value problem:

$$\frac{\partial P_t}{\partial t} - \Delta_\lambda P_t = 0, \quad t > 0, \tag{2.1}$$

$$\lim_{t\to 0} P_t(x,u;x^{(0)},u^{(0)}) = \delta(x-x^{(0)})\delta(u-u^{(0)}). \tag{2.2}$$

We expect P_t to have the form

$$P_t = \int_{\mathbb{R}} \frac{e^{-f/t}}{t^2} w_\lambda(-f_\tau d\tau), \tag{2.3}$$

where $f = \tau g$ and $f_\tau = \partial f/\partial\tau$. So far (2.3) has been justified and worked out explicitly only when $k = 1$; in that case

$$f = -iu\tau + |x|^2\tau\coth(2\tau),$$

$$w_\lambda = \frac{-2\tau e^{-2\tau\lambda}}{4\pi^2 f_\tau \sinh(2\tau)},$$

see [2]. We do expect (2.3) to hold in general and to that end state and prove

Theorem 2.1. *(2.3) is a solution of (2.1) if* w_λ *is a solution of*

$$\tau(T+\Delta_\lambda g)\frac{\partial w_\lambda}{\partial\tau} - f_\tau\Delta_\lambda w_\lambda = 0. \tag{2.4}$$

Furthermore, f_τ *is a constant of motion.*

Proof. To work out

$$\Big(\Delta_\lambda - \frac{\partial}{\partial t}\Big)P_t,$$

we start with

$$\Delta_\lambda e^{-f/t} = \Big(\frac{H(\nabla f)}{t^2} - \frac{\Delta_\lambda f}{t}\Big)e^{-f/t}.$$

Also,

$$\begin{aligned}\Delta_\lambda(e^{-f/t}W) &= \Delta_\lambda(e^{-f/t})W + \sum_{j=1}^{2} X_j(e^{-f/t})X_jW + e^{-f/t}\Delta_\lambda W\\ &= \Delta_\lambda(e^{-f/t})W - \frac{e^{-f/t}}{t}\sum_{j=1}^{2}(X_jf)(X_jW) + e^{-f/t}\Delta W,\end{aligned}$$

and

$$\frac{\partial}{\partial t}\frac{e^{-f/t}}{t^{\alpha+1}} = \frac{e^{-f/t}}{t^{\alpha+1}}\Big(\frac{f}{t^2} - \frac{\alpha+1}{t}\Big).$$

Consequently,

$$\begin{aligned}\Big(\Delta_\lambda - \frac{\partial}{\partial t}\Big)\frac{e^{-f/t}W}{t^{\alpha+1}} = \frac{e^{-f/t}}{t^{\alpha+1}}\Big[&\frac{H(\nabla f) - f}{t^2}W\\ &+ \frac{(\alpha+1)W - (\Delta_\lambda f)W - \tau(Xg)\cdot(XW)}{t} + \Delta W\Big].\end{aligned}$$

Using

$$\tau\frac{\partial f}{\partial \tau} + H(\nabla f) = f, \tag{2.5}$$

one has

$$\begin{aligned}e^{-f/t}[H(\nabla f) - f]W &= -e^{-f/t}\tau\frac{\partial f}{\partial\tau}W\\ &= t\tau\frac{\partial}{\partial\tau}(e^{-f/t})W,\end{aligned} \tag{2.6}$$

and integrating by parts we obtain

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)P_t = \int_{\mathbb{R}}\frac{e^{-f/t}}{t^{\alpha+1}}\Big[&\frac{-\frac{\partial}{\partial\tau}(\tau W)}{t}\\ &+ \frac{(\alpha+1)W - (\Delta_\lambda f)W - \tau(Xg)\cdot(XW)}{t}\\ &+ \Delta_\lambda W\Big]d\tau\\ = \int_{\mathbb{R}}\frac{e^{-f/t}}{t^{\alpha+1}}\Big[&\frac{-\tau TW + (\alpha - \Delta_\lambda f)W}{t} + \Delta_\lambda W\Big]d\tau.\end{aligned} \tag{2.7}$$

Next,

$$Tg = \frac{\partial g}{\partial \tau} + (Xg) \cdot (Xg) = -E + 2E = E, \tag{2.8}$$

so from (2.5) and from $TE = 0$,

$$\begin{aligned} f_\tau + \tau H(\nabla g) &= g, \\ f_\tau + \tau E &= g \\ Tf_\tau + E &= E, \\ Tf_\tau &= 0, \end{aligned} \tag{2.9}$$

and f_τ is an invariant of the motion. Also,

$$Tf = g + \tau E. \tag{2.10}$$

Since $W = -f_\tau w_\lambda$, one has

$$\begin{aligned} &\Big(\Delta_\lambda - \frac{\partial}{\partial t}\Big) P_t \\ &= \int_{\mathbb{R}} \frac{e^{-f/\tau}}{t^{\alpha+1}} \Big[\frac{f_\tau\big(\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda\big)}{t} + \Delta W\Big] d\tau \\ &= \int_{\mathbb{R}} \frac{d\tau}{t^{\alpha+1}} \Big[-\frac{\partial}{\partial \tau}(e^{-f/t})\big(\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda\big) + e^{-f/t} \Delta_\lambda W\Big] \\ &= \int_{\mathbb{R}} \frac{e^{-f/t}}{t^{\alpha+1}} \Big[\frac{\partial}{\partial \tau}\big(\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda\big) + \Delta_\lambda(-f_\tau w_\lambda)\Big] d\tau \end{aligned}$$

after a second integration by parts. Next

$$\Delta_\lambda(-f_\tau w_\lambda) = -(\Delta_\lambda f_\tau) w_\lambda - (Xf) \cdot (X w_\lambda) - f_\tau \Delta_\lambda w_\lambda,$$

so,

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\big(\tau T w_\lambda + (\Delta_\lambda f - \alpha)w_\lambda\big) - \Delta_\lambda(f_\tau w_\lambda) \\
&= T w_\lambda + \tau\frac{\partial}{\partial\tau}(T w_\lambda) + (\Delta_\lambda f_\tau)w_\lambda + (\Delta_\lambda f - \alpha)\frac{\partial w_\lambda}{\partial\tau} \\
&\quad - (\Delta_\lambda f_\tau)w_\lambda - (X f_\tau)\cdot(X w_\lambda) - f_\tau\Delta_\lambda w_\lambda \\
&= \frac{\partial w_\lambda}{\partial\tau} + (Xg)\cdot(Xw_\lambda) + \tau T\frac{\partial w_\lambda}{\partial\tau} + \tau(Xg_\tau)\cdot(Xw_\lambda) \\
&\quad + (\Delta_\lambda f - \alpha)\frac{\partial w_\lambda}{\partial\tau} - (Xf_\tau)\cdot(Xw_\lambda) - f_\tau\Delta_\lambda w_\lambda \\
&= \tau T\frac{\partial w_\lambda}{\partial\tau}[\tau X g_\tau + Xg - Xf_\tau]\cdot(Xw_\lambda) \\
&\quad + (\Delta_\lambda f + 1 - \alpha)\frac{\partial w_\lambda}{\partial\tau} - f_\tau\Delta_\lambda w_\lambda \\
&= \tau T\frac{\partial w_\lambda}{\partial\tau} + (\Delta_\lambda f + 1 - \alpha)\frac{\partial w_\lambda}{\partial\tau} - f_\tau\Delta_\lambda w_\lambda.
\end{aligned}$$

Setting $\alpha = 1$, the integrand in

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)P_t$$

vanishes if

$$\tau(T + \Delta_\lambda f)\frac{\partial w_\lambda}{\partial\tau} - f_\tau\Delta_\lambda w_\lambda = 0$$

which yields (2.4) and we have derived Theorem 2.1. □

In the proof of Theorem 2.1 we assumed that the non-integrated terms after the integrations-by-parts vanish at $\tau = \pm\infty$. These should be considered boundary conditions which may fix the required solution w_λ of (2.4) uniquely.

We note that (2.4) may be written in the following form:

$$\tau\Big[(T + \Delta_\lambda g)\frac{\partial w_\lambda}{\partial\tau} + E\Delta_\lambda w_\lambda\Big] = g\Delta_\lambda w_\lambda. \tag{2.11}$$

In view of (1.13) one may try to find w_λ as a power series expansion in g with first term v_λ.

We expect that formulas (1.4) and (2.3) apply to rather general subelliptic differential operators. Note that for the operators discussed here $\theta(s) =$ constant, which is not expected in general.

References

[1] R. Beals, B. Gaveau, P.C. Greiner: *On a geometric formula for the fundamental solution of subelliptic Laplacians,* Math. Nachr. **181** (1996) 81–163.

[2] R. Beals, B. Gaveau, P.C. Greiner: *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups,* J. Math. Pures Appl. **79** (2000) 633–689.

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