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Some asymptotic boundary behavior of a proper harmonic map between Carnot spaces

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§1. Introduction.

Let (M,g) and (M',h) be Riemannian manifolds, and $u: M \to M'$ a C^2 map. For the differential map $d_xu : T_xM \to T_{u(x)}M'$ of u at $x \in M$, we denote by $|d_xu|$ the Hilbert-Schmidt norm of d_xu. For a relatively compact domain $D \subset M$ we define the total energy of a map u on D by

$$E_D(u) = \frac{1}{2}\int_D |d_xu|^2 dv_g,$$

where dv_g is the volume form induced from the Riemannian metric g. Then a map u is a harmonic map if it is a critical point of E_D for any relatively compact domain $D \subset M$.

In terms of local coordinates $(x^1, x^2, \ldots, x^m)$ on M and $(y^1, y^2, \ldots, y^n)$ on M', where $m = \dim M$ and $n = \dim M'$, respectively, we express the Riemannian metric locally by

$$g = \sum_{i,j=1}^{m} g_{ij}dx^i dx^j, \qquad h = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}dy^\alpha dy^\beta,$$

and a map u in the following way:

$$u(x) = (u^1(x^1, \ldots, x^m), \ldots, u^n(x^1, \ldots, x^m)).$$

Then the Euler-Lagrange equation of E_D is given by the following system of the second order semi-linear elliptic partial differential equations:

$$\tau(u)^\alpha := \Delta_M u^\alpha(x) + \sum_{i,j}\sum_{\beta,\gamma} g^{ij}(x)^N\Gamma^\alpha_{\beta\gamma}(u(x))\frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\gamma}{\partial x^j}(x) = 0 \qquad (\alpha = 1, \ldots, n),$$

where Δ_M is the Laplace-Beltrami operator of (M,g), $(g^{ij}) = (g_{ij})^{-1}$, and ${}^N\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol of (M',h).

Let (M,g) be a Hadamard manifold, that is, it is complete, connected, simply connected Riemannian manifold of nonpositive sectional curvature, and $\gamma_1, \gamma_2 : [0,\infty) \to M$ unit speed geodesic rays. Then γ_1 and γ_2 are asymptotic if there exists a positive constant C such that $d(\gamma_1(t), \gamma_2(t)) \leq C$ holds for any $t \geq 0$, where d is the distance function on M induced from the Riemannian metric g. Then the asymptotic relation defines equivalence classes on the set of unit speed geodesic rays, and we denote the set of equivalence classes by $M(\infty)$, which is called the ideal boundary of M. If we set $\overline{M} = M \cup M(\infty)$, then with respect to a suitable topology, what is called the cone topology, $M(\infty)$ is homeomorphic to the $(m-1)$-dimensional unit sphere S^{m-1}, and $\overline{M}$ is homeomorphic to the m-dimensional closed unit ball $\overline{B^m}$. $\overline{M}$ is called the Eberlein-O'Neill compactification of M.

Let (M, g) and (M', h) be Hadamard manifolds, $\overline{M} = M \cup M(\infty)$ and $\overline{M'} = M' \cup M'(\infty)$ their Eberlein-O'Neill compactifications. Then $\overline{M}$ and $\overline{M'}$ can be regarded as the manifolds with the boundary. Thus we can consider the following Dirichlet problem for harmonic maps at infinity.

Dirichlet problem for harmonic maps at infinity:
Given a map $f \in C^0(M(\infty), M'(\infty))$, find a harmonic map $u \in C^2(M, M') \cap C^0(\overline{M}, \overline{M'})$ which assumes f as the boundary value.

We note that the Dirichlet problem for harmonic maps between compact Riemannian manifolds with the boundary has been considered around 1975 by Hamilton, who proved the existence of a harmonic map assuming any continuous boundary map. However, in our problem, manifolds are not compact and Riemannian metrics can not be extended to the ideal boundaries.

For example, we investigate the case of real hyperbolic spaces. If we take the ball model, $\mathbf{D}^m$, of the m-dimensional real hyperbolic space which is given by

$$\mathbf{D}^m = \left(\{x \in \mathbf{R}^m \mid |x| < 1\}, \frac{1}{(1-|x|^2)^2} \sum_{i=1}^{m} (dx^i)^2 \right),$$

where $|x|^2 = \sum_{i=1}^m (x^i)^2, x = (x^1, x^2, \dots, x^m)$. Then the Eberlein-O'Neill compactification $\overline{\mathbf{D}^m}$ is nothing but the closure of $\mathbf{D}^m$ with respect to the standard topology of the m-dimensional real Euclidean space $\mathbf{R}^m$, and the ideal boundary is the $(m-1)$-dimensional unit sphere S^{m-1}.

For a map $u \in C^2(\mathbf{D}^m, \mathbf{D}^n)$, the Euler-Lagrange equation of E_D has the form

$$\tau(u)^\alpha(x) = (1-|x|^2)^2 \Delta_0 u^\alpha + (\text{lower derivative terms}),$$

where Δ_0 is the Laplace-Beltrami operator of the real Euclidean space $\mathbf{R}^m$. Since the principal part of the above equation vanishes at the ideal boundary, we have some difficulties in the analysis of the Euler-Lagrange equation.

Our primary object is to deduce a necessary condition for the existence of a harmonic map. In other words, if there exists a proper harmonic map u between Hadamard manifolds which assumes a map f as a boundary value, then what is the condition u or f satisfies at the ideal boundary. Here a map $u : M \to M'$ between Hadamard manifolds is proper if for any sequence $\{x_i\}$ in M which tends to the ideal boundary $M(\infty)$ as $i \to \infty$, then the sequence $\{u(x_i)\}$ also tends to the ideal boundary $M'(\infty)$ as $i \to \infty$. Assume that $u \in C^0(\overline{M}, \overline{M'})$. Then the properness means that u maps the ideal boundary into the ideal boundary.

The following is the first result in this direction.

Fact. (Li-Tam [3]) Let $u \in C^2(\mathbf{D}^m, \mathbf{D}^n) \cap C^1(\overline{\mathbf{D}^m}, \overline{\mathbf{D}^n})$ be a proper harmonic map, and $(r, \theta^1, \ldots, \theta^{m-1})$ and $(\rho, \eta^1, \ldots, \eta^{n-1})$ the polar coordinate on $\mathbf{D}^m$ and $\mathbf{D}^n$, respectively. Then, at the ideal boundary $\mathbf{D}(\infty)$, we have

$$\begin{cases} (m-1)\left(\dfrac{\partial \rho}{\partial r}\right)^2 = e(f), \\ \dfrac{\partial \eta^\alpha}{\partial r} = \dfrac{\partial \rho}{\partial \theta^i} = 0 \quad (1 \le i \le m-1, 1 \le \alpha \le n-1), \end{cases}$$

where $f = u_{|\mathbf{D}^m(\infty)}$ and $e(f)(x) = (1/2)|d_x f|^2$ is the energy density of f at x with respect to the standard Riemannian metrics on the unit spheres.

Thus, if a proper harmonic map $u : \mathbf{D}^m \to \mathbf{D}^n$ has a sufficient regularity up to the ideal boundary, then its boundary behavior should be restricted. We have to remark that, using these conditions, Li and Tam proved the existence and uniqueness of proper harmonic map assuming any C^1 map $f : S^{m-1} \to S^{n-1}$ as a boundary value whose energy density is nowhere vanishing.

We shall extend their investigations and results on the boundary behavior of a proper harmonic map to the case of other homogeneous Riemannian manifolds, say, Carnot spaces.

§2. Carnot spaces.

We firstly review some geometric and algebraic structure of complex hyperbolic spaces.

Let M be the ball model of 2-dimensional complex hyperbolic space, that is, $M = (\mathbf{B}^2, g_B)$, where $\mathbf{B}^2 = \{z \in \mathbf{C}^2 \mid |z| < 1\}$ and g_B is the Bergman metric. For $J = \operatorname{diag}[-1, 1, 1]$, let $SU(1,2) = \{g \in M(3; \mathbf{C}) \mid g^{-1}Jg = J, \det g = 1\}$, and $SU_0(1,2)$ the identity component of $SU(1,2)$. Then $SU_0(1,2)$ acts on M transitively and isometrically as a linear fractional transformation. On the other hand, we have the Iwasawa decomposition $SU_0(1,2) = N \cdot A \cdot K$, where N is the Heisenberg group, A is the 1-dimensional Lie group, and K is the isotropic subgroup. We can easily verify that the semi-direct product $S = N \cdot A$ of N and A is solvable, and acts on M simply transitively. Since N is the Heisenberg group, the corresponding Lie algebra $\mathfrak{n}$ of N satisfies $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = \{0\}$. If we take $\mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}]$ and the orthogonal complement $\mathfrak{n}_1$ of $\mathfrak{n}_2$ in $\mathfrak{n}$, then the Lie algebra $\mathfrak{s}$ of S is decomposed into

$$\mathfrak{s} = \mathfrak{n}_1 + \mathfrak{n}_2 + \mathbf{R}\{H\},$$

where H is a generator of $\mathfrak{a}$. Moreover, $\mathfrak{n}_1 + \mathfrak{n}_2$ is a graded Lie algebra. Indeed, let ad be the adjoint representation of $\mathfrak{su}(1,2)$, then

$$\mathfrak{n}_i = \{X \in \mathfrak{n} \mid \operatorname{ad} H(X) = iX\} \qquad (i = 1, 2)$$

and $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, where $\mathfrak{n}_k = \{0\}$ for $k \ge 3$.

Using orthonormal frame fields $\{U, V\}$ and $\{T\}$ of $\mathfrak{n}_1$ and $\mathfrak{n}_2$, respectively, we define a map $\Psi : \mathbf{R}^3 \times \mathbf{R}_+ \to (\mathbf{B}^2, g_B)$ by the following way.

$$\mathbf{R}^3 \times \mathbf{R}_+ \ni ((u, v, t), y) \mapsto \exp(uU + vV + tT) \cdot \exp((\log y)H) \cdot o \in \mathbf{B}^2,$$

where o is the origin of $\mathbf{B}^2$, and "$\cdot$" stands for the action of the element of $SU(1,2)$ on $\mathbf{B}^2$ as a linear fractional transformation, and exp means the exponential map on Lie algebra. Then the map Ψ is a diffeomorphism and the pull-back metric $\Psi^* g_B$ of the Bergman metric g_B on $\mathbf{B}^2$ via Ψ is

$$\Psi^* g_B = \frac{1}{y^2} dy^2 + \frac{1}{y^2} g_1 + \frac{2}{y^4} g_2,$$

where

$$g_1 = du^2 + dv^2 \quad \text{and} \quad g_2 = (dt + udv - vdu)^2$$

are left invariant metrics on the Lie group N. Since $\exp : \mathfrak{n} \to N$ is a diffeomorphism, the 2-dimensional complex hyperbolic space is realized as the upper half space $N \times \mathbf{R}_+$ equipped with the 2-ply warped product metric

$$\left(N \times \mathbf{R}_+, \frac{1}{y^2} dy^2 + \frac{1}{y^2} g_1 + \frac{2}{y^4} g_2 \right).$$

Moreover, the ideal boundary of Eberlein-O'Neill compactification is given by

$$\Big(N \times \{y = 0\} \Big) \cup \{\infty\}.$$

Following this investigation, we introduce the notion of a Carnot space.

Definition. Let S be a simply connected, connected solvable Lie group and g_S a left-invariant metric on S. Then the pair (S, g_S) is a k-term Carnot space if the following conditions hold.

(1) S is a semi-direct product $N \times \mathbf{R}_+$ of nilpotent Lie group N and the positive real line $\mathbf{R}_+$.

(2) Let $\mathfrak{n} = \mathrm{Lie}(N)$ and $\mathfrak{s} = \mathrm{Lie}(S) = \mathfrak{n} + \mathbf{R}\{H\}$. Then $\mathfrak{n}$ has a decomposition

$$\mathfrak{n} = \sum_{i=1}^{k} \mathfrak{n}_i,$$

where

$$\mathfrak{n}_i = \{X \in \mathfrak{n} \mid \operatorname{ad} H(X) = iX\} \quad (1 \le i \le k).$$

(3) g_S is a left invariant metric on S whose sectional curvature is negative and

$$g_S(H, H) = 1 \quad \text{and} \quad H \perp \mathfrak{n}.$$

Note. $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$, where $\mathfrak{n}_l = \{0\}$ $(l > k)$.

For a k-term Carnot space, define a map $\Psi : N \times \mathbf{R}_+ \to S$ by

$$N \times \mathbf{R}_+ \ni (n, y) \mapsto n \cdot \exp((\log y)H) \in S.$$

Ψ is called the generalized Cayley transformation. Then we can prove the following

Fact. (1) Ψ is a diffeomorphism.

(2) The pull-back metric $\Psi^* g_S$ is given by

$$\Psi^* g_S = \frac{dy^2}{y^2} + \frac{1}{y^2} g_1 + \cdots + \frac{1}{y^{2k}} g_k,$$

where $g_1 + g_2 + \cdots + g_k$ is a left invariant metric on N.

(3) For any fixed $n \in N$, the curve $y \mapsto (n, y)$ defines an asymptotic geodesic. Moreover, $M(\infty) - \{\infty\} \simeq N \times \{0\}$, where ∞ denotes the point at infinity at where these asymptotic curves meet.

Example. (1) The m-dimensional real hyperbolic space is a 1-term Carnot space. In fact, $\mathfrak{n}$ is abelian and (N, g_1) is nothing but the $(m-1)$-dimensional real Euclidean space $\mathbf{R}^{m-1}$ equipped with the standard metric on $\mathbf{R}^{m-1}$.

(2) One of a typical example of 2-term Carnot space is the 2-dimensional complex hyperbolic space as seen in the beginning of this section. In general, the m-dimensional complex hyperbolic space is a 2-term Carnot space if $m \geq 2$, and N is the $(2m-1)$-dimensional Heisenberg Lie group.

(3) Let $\mathfrak{gl}(k+1; \mathbf{R})$ be the general linear Lie algebra consisting of real $(k+1) \times (k+1)$-matrices with the natural Lie bracket, and $E_{ij} \in \mathfrak{gl}(k+1; \mathbf{R})$ the matrix unit, that is, whose (i,j)-entry is 1 and otherwise entries are 0. Let $H = (\{k - 2(i-1)\}\delta_{ij}) \in \mathfrak{gl}(k+1; \mathbf{R})$. Since

$$\operatorname{ad} H(E_{ij}) = [H, E_{ij}] = (j - i)E_{ij},$$

if we define Lie algebras by

$$\mathfrak{n}_i = \mathbf{R}\{E_{1i+1}\} \ (1 \leq i \leq k), \quad \mathfrak{n} = \sum_{i=1}^{k} \mathfrak{n}_i, \qquad \mathfrak{s} = \mathbf{R}\{H\} + \mathfrak{n},$$

then $\mathfrak{n}_i$ is the eigenspace of $\operatorname{ad} H$ with the eigenvalue i and $\mathfrak{n}$ is abelian. If we take the inner product on $\mathfrak{s}$ by

$$\langle H, H \rangle = 1, \ \langle H, E_{1j} \rangle = 0, \quad \langle E_{1i}, E_{1j} \rangle = \delta_{ij},$$

then the left invariant extension g_S on S of $\langle , \rangle$ is given by

$$g_S = \frac{dy^2}{y^2} + \frac{1}{y^2} e_2^* \otimes e_2^* + \cdots + \frac{1}{y^{2k}} e_{k+1}^* \otimes e_{k+1}^*,$$

where e_i is the left invariant extension of E_{1i} on S and e_i^* is its dual form. Thus (S, g_S) is a k-term Carnot space.

§3. Harmonic maps between Carnot spaces

Let (S, g_S) and $(S', g_{S'})$ be k-term Carnot spaces and diffeomorphic to $N \times \mathbf{R}_+$ and $N' \times \mathbf{R}_+$, respectively, where N and N' are nilpotent Lie groups. Following the decompositions of the corresponding Lie algebras

$$\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2 + \cdots + \mathfrak{n}_k, \quad \mathfrak{n}' = \mathfrak{n}'_1 + \mathfrak{n}'_2 + \cdots + \mathfrak{n}'_k$$

we can also decompose the tangent spaces of $N \times \{0\}$ and $N' \times \{0\}$ as

$$T_pN = (\mathfrak{n}_1)_p + (\mathfrak{n}_2)_p + \cdots + (\mathfrak{n}_k)_p,$$

$$T_qN' = (\mathfrak{n}'_1)_q + (\mathfrak{n}'_2)_q + \cdots + (\mathfrak{n}'_k)_q$$

for $p \in N \times \{0\}$ and $q \in N' \times \{0\}$.

Definition. Let $u \in C^1(\overline{S}, \overline{S'})$ be a proper map and $f := u_{|S(\infty)}$. Then u is nondegenerate at $N \times \{0\}$ if

$$d_pf((\mathfrak{n}_k)_p) \not\subset \sum_{j=1}^{k-1} (\mathfrak{n}'_j)_{f(p)}$$

holds for any $p \in N \times \{0\}$. In other words, u is nondegenerate if

$$\Big(d_pf((\mathfrak{n}_k)_p)\Big) \cap (\mathfrak{n}'_k)_{f(p)} \neq \{0\}$$

holds for any $p \in N \times \{0\}$.

For example, if (S, g_S) and $(S', g_{S'})$ are real hyperbolic spaces, that is, 1-term Carnot spaces, then a proper map $u \in C^1(\overline{S}, \overline{S'})$ is said to be nondegenerate if

$$d_pf((\mathfrak{n}_1)_p) \not\subset (\mathfrak{n}'_0)_{f(p)} = \{0\}.$$

Hence u is nondegenerate if and only if $|d_pf| \neq 0$ for any $p \in S(\infty)$.

Theorem. Let (S, g_S) and $(S', g_{S'})$ be k-term Carnot spaces and $u \in C^k(\overline{S}, \overline{S'})$ a proper harmonic map and $f = u_{|S(\infty)}$. Assume u is nondegenerate. Then for $1 \leq i \leq k$ and any $p \in N \times \{0\}$, we have

$$d_pf\left(\sum_{j=1}^{i} (\mathfrak{n}_j)_p\right) \subset \sum_{j=1}^{i} (\mathfrak{n}'_j)_{f(p)}.$$

Namely, f is a filtration preserving map.

We shall investigate some geometric meaning of this statement in the case of complex hyperbolic spaces.

Example. (Donnelly [2]) Let (S, g_S) and $(S', g_{S'})$ be complex hyperbolic spaces of dimension $m \geq 2$ and $n \geq 2$, respectively. Then their ideal boundaries are identified with the S^{2m-1} and S^{2n-1}, respectively. If we consider the Hopf fibration $S^{2m-1} \to \mathbf{CP}^m$, then $\mathfrak{n}_1$ and $\mathfrak{n}_2$ part correspond to the horizontal distribution $\mathcal{H}$, and the vertical distribution $\mathcal{V}$ of the fibration, respectively. Therefore we have the following correspondence.

$$u \text{ is nondegenerate} \Longleftrightarrow df(\mathcal{V}) \not\subset \mathcal{H}',$$
$$f \text{ is a filtration preserving map} \Longleftrightarrow df(\mathcal{H}) \subset \mathcal{H}',$$

where $f = u_{|S(\infty)}$. On the other hand, we can define a natural contact structure, or contact form on the odd dimensional unit sphere. Since $\mathfrak{n}_1$ is the null space of the contact form on the unit sphere S^{2m-1}, the property of the filtration preserving means that it maps the contact distribution on S^{2m-1} into one of S^{2n-1}. Namely, the boundary value of a proper harmonic map preserves the contact structure on the boundaries.

We briefly review the notations to prove the Theorem. Let (S, g_S) and $(S', g_{S'})$ be k-term Carnot spaces, and express them as

$$(S, g_S) \simeq \left(N \times \mathbf{R}_+, \frac{dy^2}{y^2} + \frac{1}{y^2}g_1 + \cdots + \frac{1}{y^{2k}}g_k\right),$$
$$(S', g_{S'}) \simeq \left(N' \times \mathbf{R}_+, \frac{dY^2}{Y^2} + \frac{1}{Y^2}g_1' + \cdots + \frac{1}{Y^{2k}}g_k'\right),$$

where $(n, y) \in N \times \mathbf{R}_+, (n', Y) \in N' \times \mathbf{R}_+$, $g_1 + \cdots + g_k$ and $g_1' + \cdots + g_k'$ are left invariant metrics on N and N', respectively. We shall decompose the Lie algebras $\mathfrak{n} = \mathrm{Lie}(N)$ and $\mathfrak{n}' = \mathrm{Lie}(N')$ into k spaces as the following way

$$\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2 + \cdots + \mathfrak{n}_k, \quad \mathfrak{n}' = \mathfrak{n}_1' + \mathfrak{n}_2' + \cdots + \mathfrak{n}_k',$$

and set $n_A = \dim \mathfrak{n}_A, n_P' = \dim \mathfrak{n}_P'$, and $n = n_1 + \cdots + n_k$, $n' = n_1' + \cdots + n_k'$.

We take adapted frame fields $\{e_i\}$ on $N \times \mathbf{R}_+$ and $\{f_\alpha\}$ on $N' \times \mathbf{R}_+$ in the following way. For $1 \leq A \leq k$, let $\{e_{A_i}\}$ be an orthonormal basis of $(\mathfrak{n}_A, g_A)$ and denote their left invariant extension on N by the same letters. Then

$$\left\{e_0 = \frac{\partial}{\partial y}\right\} \cup \left\{\{e_{A_i}\}_{i=1}^{n_A}\right\}_{A=1}^{k}$$

is an adapted frame field on $N \times \mathbf{R}_+$. For the target manifold $N' \times \mathbf{R}_+$, we define an adapted frame field

$$\left\{f_0 = \frac{\partial}{\partial Y}\right\} \cup \left\{\{f_{P_\beta}\}_{\beta=1}^{n_P'}\right\}_{P=1}^{k}$$

as the same manner.

Let $u \in C^2(S, S')$. In terms of these adapted frame fields, we write the differential of u as follows.

$$du = \sum_{i=0}^{n} \sum_{\alpha=0}^{n'} u_i^\alpha e_i^* \otimes f_\alpha,$$

where e_i^* denotes the dual frame of e_i. Since

$$u_{k_i}^{k_\beta} = f_{k_\beta}^*(du(e_{k_i})),$$

we note that u is nondegenerate if and only if

$$\sum_{i=1}^{n_k} \sum_{\beta=1}^{n'_k} (u_{k_i}^{k_\beta})^2 \neq 0$$

holds at the ideal boundary. Using these adapted frame fields, we can calculate the first component $\tau(u)^0$ of the Euler-Lagrange equation as the following:

$$\begin{aligned}\tau(u)^0 = &\sum_{i=0}^{n} g^{ii}(e_i \cdot u_i^0) + (1 - \sum_{A=1}^{k} n_A A) y u_0^0 \\ &- Y(u)^{-1} y^2 (u_0^0)^2 - Y(u)^{-1} \sum_{A=1}^{k} y^{2A} \sum_{i=1}^{n_A} (u_{A_i}^0)^2 \\ &+ y^2 \sum_{P=1}^{k} P Y(u)^{-2P+1} \sum_{\beta=1}^{n'_P} (u_0^{P_\beta})^2 + \sum_{A=1}^{k} y^{2A} \sum_{P=1}^{k} P Y(u)^{-2P+1} \sum_{i=1}^{n_A} \sum_{\beta=1}^{n'_P} (u_{A_i}^{P_\beta})^2.\end{aligned}$$

Note. Since u is a proper map, $Y(u) \to 0$ as $y \to 0$, which yields

$$\lim_{y \to 0} y^{-1} Y(u) = u_0^0.$$

Lemma 1. Let $u \in C^1(\overline{S}, \overline{S'})$ be a proper map and $1 \leq B \leq k$. Then we have

(1) The sum of the first four terms in the equation $\tau(u)^0 \times Y(u)^{2k-1} \times y^{-2B}$ tends to

$$\begin{cases} -\left(\sum_{A=1}^{k} n_A A\right) (u_0^0)^{2k} & (B = k), \\ 0 & (B < k). \end{cases}$$

(2) The fifth term in the equation $\tau(u)^0 \times Y(u)^{2k-1} \times y^{-2B}$ is given by

$$\sum_{P=1}^{k} P \sum_{\beta=1}^{n'_P} \left[(Y(u) y^{-1})^{k-P} y^{k-P-B+1} u_0^{P_\beta} \right]^2 + o(1)$$

as $y \to 0$.

(3) The sixth term in the equation $\tau(u)^0 \times Y(u)^{2k-1} \times y^{-2B}$ is given by

$$\sum_{A=1}^{k}\sum_{P=1}^{k} P \sum_{i=1}^{n_A}\sum_{\beta=1}^{n'_P}\left[(Y(u)y^{-1})^{k-P}y^{k-P+A-B}u_{A_i}^{P_\beta}\right]^2 + o(1)$$

as $y \to 0$. In particular, if $B < A$, the sixth term vanishes at the ideal boundary.

Especially, if u is a proper harmonic map, then using the Lemma 1 with $B = 1$, we have the following

Corollary 1. Let $u \in C^1(\overline{S}, \overline{S'})$ be a proper harmonic map. Then we have the following identity at the ideal boundary.

(1) When $k = 1$,

$$n_1(u_0^0)^2 = \sum_{i=0}^{n_1}\sum_{\beta=1}^{n'_1}(u_i^{1_\beta})^2.$$

(2) When $k > 1$,

$$u_0^{k_\beta} = u_{1_i}^{k_\beta} = 0.$$

Let $f^*_{P_\beta}$ be the dual frame of f_{P_β}. Then

$$u_{1_i}^{k_\beta} = f^*_{k_\beta}(du(e_{1_i})).$$

Therefore, the second statement in Corollary 1 implies that

$$d_pu\Big((\mathfrak{n}_1)_p\Big) \cap (\mathfrak{n}'_k)_{u(p)} = \{0\}$$

holds for any $p \in N \times \{0\}$. In other words, we have

$$d_pu\Big((\mathfrak{n}_1)_p\Big) \subset \sum_{j=1}^{k-1}(\mathfrak{n}'_j)_{u(p)}.$$

If a proper harmonic map u can be extended to the ideal boundary with C^k regularity, then higher order derivatives of u should satisfy more identities at the ideal boundary. Indeed, applying Lemma 1 inductively, we can prove the following

Lemma 2. Let $k \geq 2$ and $1 \leq r \leq k-1$. Then any proper harmonic map $u \in C^r(\overline{S}, \overline{S'})$ satisfies the following identities at $N \times \{0\}$.

(1) For any P $(k - r + 1 \leq P \leq k)$,

$$e_0^C \cdot \Big((Y(u)y^{-1})^{k-P}u_0^{P_\beta}\Big) = 0 \quad (0 \leq C \leq r-1).$$

(2) For any P $(k-r+1 \le P \le k)$ and A $(1 \le A \le P-k+r)$,

$$e_0^C \cdot \left((Y(u)y^{-1})^{k-P} u_{A_i}^{P_\beta}\right) = 0 \quad (0 \le C \le r-1).$$

Lemma 3. Let $k \ge 2$ and $1 \le r \le k-1$. Assume that $u \in C^r(\overline{S}, \overline{S'})$ is a proper harmonic map and satisfies $u_0^0 \ne 0$ at $N \times \{0\}$. Then the following holds at $N \times \{0\}$.
(1) For any P $(k-r+1 \le P \le k)$,

$$e_0^s \cdot u_0^{P_\beta} = 0 \quad (0 \le s \le P-k+r-1).$$

(2) For any P $(k-r+1 \le P \le k)$ and A $(1 \le A \le P-k+r)$,

$$e_0^s \cdot u_{A_i}^{P_\beta} = 0 \quad (0 \le s \le P-A+r-1).$$

Moreover, if $u \in C^{k-1}(\overline{S}, \overline{S'})$, that is, $r = k-1$ in Lemma 3, we have the following

Proposition 1. Let $k \ge 2$ and $u \in C^{k-1}(\overline{S}, \overline{S'})$ a proper harmonic map which satisfies $u_0^0 \ne 0$ at $N \times \{0\}$. Then, at $N \times \{0\}$, we have
(1) For any P $(2 \le P \le k)$,

$$e_0^s \cdot u_0^{P_\beta} = 0 \quad (0 \le s \le P-2).$$

(2) For any P $(2 \le P \le k)$ and A $(1 \le A \le P-1)$,

$$e_0^s \cdot u_{A_i}^{P_\beta} = 0 \quad (0 \le s \le P-A+k-2).$$

In particular, applying the result in Proposition 1 (2) for the case $s = 0$, we can easily verify that a proper harmonic map $u \in C^{k-1}(\overline{S}, \overline{S'})$ should satisfy

$$u_{A_i}^{P_\beta} = 0 \quad (2 \le P \le k, 1 \le A \le P-1),$$

that is,

$$u_{A_i}^{P_\beta} = 0 \quad (1 \le A \le k-1, A+1 \le P \le k)$$

at $N \times \{0\}$. Therefore for each A $(1 \le A \le k-1)$, it holds that

$$d_p u((\mathfrak{n}_1)_p + \cdots + (\mathfrak{n}_A)_p) \cap \left((\mathfrak{n}'_{A+1})_{u(p)} + \cdots + (\mathfrak{n}'_k)_{u(p)}\right) = \{0\}$$

for any $p \in N \times \{0\}$. Thus we have

Corollary 2. Let $k \ge 2$. Assume that $u \in C^{k-1}(\overline{S}, \overline{S'})$ is a proper harmonic map and satisfies $u_0^0 \ne 0$ at $N \times \{0\}$. Then, for any $p \in N \times \{0\}$, it holds that

$$d_p u\left(\sum_{j=1}^{A} (\mathfrak{n}_j)_p\right) \subset \sum_{j=1}^{A} (\mathfrak{n}'_j)_{u(p)}.$$

Finally, we investigate a relation between the nondegeneracy of a proper harmonic map and the condition $u_0^0 \neq 0$ at the ideal boundary.

Applying Lemma 1 with $B = k$ and by virtue of Lemma 2, we have

Lemma 4. Let $u \in C^k(\overline{S}, \overline{S'})$ be a proper harmonic map. Then the following identity holds at $N \times \{0\}$.

$$\left(\sum_{A=1}^{k} n_A A\right)(u_0^0)^{2k} - \sum_{P=1}^{k} P \sum_{\beta=1}^{n'_P} \{(k-1)!\}^{-2}\left(e_0^{k-1} \cdot (Y(u)^{k-P} u_0^{P_\beta})\right)^2$$
$$- \sum_{P=1}^{k}\sum_{A=1}^{P} P \sum_{i=1}^{n_A}\sum_{\beta=1}^{n'_P} \{(k-A)!\}^{-2}\left(e_0^{k-A} \cdot (Y(u)^{k-P} u_{A_i}^{P_\beta})\right)^2 = 0. \qquad (*)$$

Proof of Theorem. If we separate the sum of the third term of $(*)$ into two parts; one is for $P = A = k$ and otherwise, then $(*)$ is rewritten as

$$\left(\sum_{A=1}^{k} n_A A\right)(u_0^0)^{2k} + (\text{nonpositive term}) - k \sum_{i=1}^{n_k}\sum_{\beta=1}^{n'_k} (u_{k_i}^{k_\beta})^2 = 0. \qquad (**)$$

Assume that u is nondegenerate. Then it implies that

$$\sum_{i=1}^{n_k}\sum_{\beta=1}^{n'_k} (u_{k_i}^{k_\beta})^2 \neq 0$$

holds at the ideal boundary. Hence, from the equation $(**)$, we have

$$u_0^0 \neq 0$$

at the ideal boundary. Combining this result with Corollary 2, we obtain the conclusion. □

Note. If we investigate the asymptotic behavior of the component $\tau(u)^{P_\alpha}$, then we can prove the following

Theorem. Let $u \in C^k(\overline{S}, \overline{S'})$ be a proper harmonic map, which is nondegenerate at the ideal boundary. Then

(1) For any $2 \leq P \leq k$ and $1 \leq A \leq P - 1$, we have

$$e_0^r \cdot u_{A_i}^{P_\beta} = 0 \qquad (0 \leq r \leq P - A)$$

at the ideal boundary.

(2) The derivative u_0^0 satisfies the following algebraic equation at the ideal boundary:

$$\left(\sum_{A=1}^{k} n_A A\right)(u_0^0)^{2k} - \sum_{P=1}^{k}\left\{\sum_{i=1}^{n_P}\sum_{\beta=1}^{n'_P} (u_{P_i}^{P_\beta})^2\right\}(u_0^0)^{2(k-P)} = 0.$$

Especially, the boundary value of u_0^0 is completely determined by the derivatives of the boundary map.

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