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CR immersions from S^{2n+1} to S^{4n+1}

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Introduction

Elie Cartan's contribution to CR geometry is well known. In his fundamental papers, he solved the problem of equivalence for a piece of real hypersurface in $\mathbb{C}^2$ up to biholomorphism [Ca1]. It was Chern and Moser, among others, who vastly generalized this work to higher dimensions [ChM].

Recently, Huang and Ji gave a classification of the proper holomorphic maps from the unit ball $\mathbf{B}^{n+1}\subset\mathbb{C}^{n+1}$ to $\mathbf{B}^{2n+1}$, $n\geq 2$, [HJ]. We wish to show in this note that the corresponding local CR analogue is true. Let us denote $\Sigma^n=\partial\mathbf{B}^{n+1}=S^{2n+1}$.

Theorem. *Let $f:\Sigma^n\hookrightarrow\Sigma^{2n}$ be a C^3, local CR immersion, $n\geq 2$. Then up to automorphisms of the spheres, either f is linear, or f is locally equivalent to the boundary of Whitney map.*

When the codimension is small, CR Gauß equation puts the second fundamental form of a CR immersion into a simple normal form. Our idea is to explore the successive derivatives of this relation.

The computation involved is reminiscent of Cartan's local isometric embedding of Hyperbolic space $\mathbb{H}^n$ in Euclidean space $\mathbb{E}^{2n-1}$ via exteriorly orthogonal symmetric bilinear forms [Ca2]. Overdetermined nature of CR geometry forces the structure equation to close up rather than become involutive.

Local CR immersions $\Sigma^1\hookrightarrow\Sigma^2$ have been classified by Faran [Fa]. In contrast to $n\geq 2$ cases, there exist four inequivalent such immersions.

Corollary [HJ]. *Let* $F : B^{n+1} \to B^{2n+1}$ *be a proper holomorphic map which is* C^3 *up to the boundary,* $n \geq 2$. *Then up to automorphisms of the unit balls, either* F *is linear, or* F *is equivalent to Whitney map.*

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1 CR immersion

We first set up the basic structure equations for CR immersions in spheres. For general reference in CR geometry, [ChM][EHZ].

Let $\mathbb{C}^{N+1,1}$ be the complex vector space with coordinates $z = (z^0, z^A, z^{N+1})$, $1 \leq A \leq N$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^A \bar{z}^A + \mathrm{i}\,(z^0 \bar{z}^{N+1} - z^{N+1} \bar{z}^0).$$

Let Σ^N be the set of equivalence classes up to scale of null vectors with respect to this product. Let $\mathrm{SU}(N+1,1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. Then $\mathrm{SU}(N+1,1)$ acts transitively on Σ^N, and

$$p : \mathrm{SU}(N+1,1) \to \Sigma^N = \mathrm{SU}(N+1,1)/P$$

for an appropriate subgroup P [ChM].

Explicitly, consider an element $Z = (Z_0, Z_A, Z_{N+1}) \in \mathrm{SU}(N+1,1)$ as an ordered set of $(N+2)$-column vectors in $\mathbb{C}^{N+1,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_A, \bar{Z}_B \rangle = \delta_{AB}, \quad \langle Z_0, \bar{Z}_{N+1} \rangle = -\langle Z_{N+1}, \bar{Z}_0 \rangle = \mathrm{i}, \tag{1}$$

while all other scalar products are zero. We define $p(Z) = [Z_0]$, where $[Z_0]$ is the equivalence class of null vectors represented by Z_0. The left invariant Maurer-Cartan form π of $\mathrm{SU}(N+1,1)$ is defined by the equation

$$dZ = Z\,\pi,$$

which is in coordinates

$$d(\,Z_0,\,Z_A,\,Z_{N+1}\,) = (\,Z_0,\,Z_B,\,Z_{N+1}\,)\begin{pmatrix} \pi^0_0 & \pi^0_A & \pi^0_{N+1} \\ \pi^B_0 & \pi^B_A & \pi^B_{N+1} \\ \pi^{N+1}_0 & \pi^{N+1}_A & \pi^{N+1}_{N+1} \end{pmatrix}. \tag{2}$$

Coefficients of π are subject to the relations obtained from differentiating (1) which are

$$\begin{aligned} \pi^0_0 + \bar{\pi}^{N+1}_{N+1} &= 0 \\ \pi^{N+1}_0 &= \bar{\pi}^{N+1}_0, \quad \pi^0_{N+1} = \bar{\pi}^0_{N+1} \\ \pi^{N+1}_A &= -\,\mathrm{i}\,\bar{\pi}^A_0, \quad \pi^A_{N+1} = \mathrm{i}\,\bar{\pi}^0_A \\ \pi^A_B + \bar{\pi}^B_A &= 0 \\ \operatorname{tr}\pi &= 0, \end{aligned}$$

and π satisfies the structure equation

$$-\,d\pi = \pi \wedge \pi. \tag{3}$$

It is well known that the SU$(N+1,1)$-invariant CR structure on $\Sigma^N \subset \mathbb{C}P^{N+1}$ as a real hypersurface is biholomorphically equivalent to the standard CR structure on $S^{2N+1} = \partial\mathbf{B}^{N+1}$, where $\mathbf{B}^{N+1} \subset \mathbb{C}^{N+1}$ is the unit ball. The structure equation (2) shows that for any local section $s : \Sigma^N \to \mathrm{SU}(N+1,1)$, this CR structure is defined by the hyperplane fields $(s^*\pi^{N+1}_0)^{\perp} = \mathcal{H}$ and the set of (1,0)-forms $\{\, s^*\pi^A_0 \,\}$.

Definition. Let M be a CR manifold of hypersurface type with CR hyperplane fields $\mathcal{H}^M$ equipped with a complex structure. An immersion $f : M \hookrightarrow \Sigma^N$ is a *CR immersion* if $f_* : \mathcal{H}^M \to \mathcal{H}$ is complex linear.

Consider $f^*\mathrm{SU}(N+1,1) \to M$. From the definition, we may arrange so that $\pi^a_0 = 0$ for $n+1 \le a \le m = N-n$ on this bundle. Differentiating this, we get

$$\pi^a_i \wedge \pi^i_0 + \pi^a_{N+1} \wedge \pi^{N+1}_0 = 0.$$

By Cartan's lemma,

$$\pi_i^a \equiv h_{ij}^a \pi_0^j \qquad \mathrm{mod}\ \pi_0^{N+1}, \tag{4}$$

for coefficients $h_{ij}^a = h_{ji}^a$. h_{ij}^a represents the second fundamental form of f [EHZ].

Example [Whitney immersion]. Let $\xi = (\xi^0, \xi^i, \xi^{n+1})$, $\mu = (\mu^0, \mu^i, \mu^{n+i}\, \mu^{2n+1})$, $1 \le i \le n$, be the coordinates of $\mathbb{C}^{n+1,1}$, and $\mathbb{C}^{2n+1,1}$ respectively. *Whitney immersion* $\Gamma_n : \Sigma^n \to \Sigma^{2n}$ is induced from the quadratic map $\widehat{\Gamma}_n : \mathbb{C}^{n+1,1} \to \mathbb{C}^{2n+1,1}$ defined as

$$\begin{aligned} \mu^0 &= 2\,\xi^0\,\xi^{n+1}, \\ \mu^{2n+1} &= (\xi^{n+1})^2 - (\xi^0)^2, \\ \mu^i &= \xi^i\,(\mathrm{i}\,\xi^0 + \xi^{n+1}), \\ \mu^{n+i} &= \xi^i\,(-\mathrm{i}\,\xi^0 + \xi^{n+1}). \end{aligned}$$

$\widehat{\Gamma}_n^* \langle \mu, \bar{\mu} \rangle = 2\,(\xi^0 \bar{\xi}^0 + \xi^{n+1} \bar{\xi}^{n+1})\, \langle \xi, \bar{\xi} \rangle$, and the induced map Γ_n is well defined.

It is easy to check Γ_n is CR-equivalent to the boundary map $\partial\mathcal{W}_n : S^{2n+1} \to S^{4n+1}$ of the following *Whitney map* $\mathcal{W}_n : \mathbf{B}^{n+1} \to \mathbf{B}^{2n+1}$. Here (z^0, z^i) is a coordinate of $\mathbb{C}^{n+1}$.

$$\mathcal{W}_n(z^0, z^i) = (\,(z^0)^2,\, z^0 z^i,\, z^i\,). \tag{5}$$

This equivalence is via the isomorphism $\Sigma^n \simeq S^{2n+1}$ given in coordinates

$$\begin{aligned} z^0 &= \frac{\mathrm{i}\,\xi^0 + \xi^{n+1}}{-\mathrm{i}\,\xi^0 + \xi^{n+1}} \\ z^i &= \frac{\sqrt{2}\,\xi^i}{-\mathrm{i}\,\xi^0 + \xi^{n+1}} \end{aligned}$$

Set $\Sigma_0^n = \{\,[\xi] \in \Sigma^n \mid \xi^i = 0,\ \forall i\,\}$ and $\Sigma_s^n = \{\,[\xi] \in \Sigma^n \mid \mathrm{i}\,\xi^0 + \xi^{n+1} = 0\,\}$. Then Γ_n is an immersion which is 1 to 1 on $\Sigma^n - \Sigma_0^n$, 2 to 1 on Σ_0^n, and the second fundamental form vanishes along Σ_s^n.

2 Proof of theorem

Our proof of **Theorem** is based on the following algebraic Lemma due to Iwatani on the asymptotic subspace of the second fundamental form of a Bochner-Kähler submanifold [Iw][Br]. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^n$ with the standard Hermitian scalar product. Let $\{ z^i \}$ be a unitary (1,0)-basis for V^*, and $\{ w_a \}$ be a unitary basis for W. Let $S^{p,q}$ denote the space of polynomials of type (p, q) on V.

Lemma [Iw]. *Suppose* $H = h^a_{ij}\, z^i z^j \otimes w_a \in S^{2,0} \otimes W$ *satisfies*

$$\gamma(H, H) = h^a_{ij}\, \bar{h}^a_{kl}\, z^i z^j \otimes \bar{z}^k \bar{z}^l = (z^k \bar{z}^k)\, h \in S^{2,2}, \quad h \in S^{1,1},$$

or simply $\gamma(H, H)$ *is Bochner-flat [Br]. Then the asymptotic vectors* $\{ v \in V \mid H(v, v) = 0 \}$ *form a subspace of* V *of codimension at most one.*

Up to a unitary transformation on V, we may thus arrange

$$h^a_{ij} = h^a_i\, \delta_{jn} + h^a_j\, \delta_{in},$$

for coefficients h^a_i. Set $\nu_i = h^a_i\, w_a \in W$. A computation shows $\gamma(H, H)$ is Bochner-flat when $\langle \nu_i, \nu_j \rangle = 0$ for $i \neq j$, and $\langle \nu_i, \nu_i \rangle = \langle \nu_j, \nu_j \rangle$ for all i, j. Up to a unitary transformation of W, we may set

$$\nu_i = r\, w_i,$$

for some $r \geq 0$.

Let $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ be a local CR immersion. Let $\mathcal{H}$ be the CR hyperplane fields on Σ^{2n}. Since Σ^n is CR flat, after identifying $V = f_* T\Sigma^n \cap \mathcal{H} \simeq \mathbb{C}^n$ and $W = V^\perp \simeq \mathbb{C}^n \subset \mathcal{H}$, the second fundamental form of f is Bochner-flat [EHZ]. From (4) and the argument above, we may write

$$\pi^{n+i}_q \equiv r\, \delta_{iq}\, \pi^n_0 \qquad \text{mod } \pi^{2n+1}_0, \quad \text{for } q < n,$$
$$\pi^{n+i}_n \equiv r(1 + \delta_{in})\, \pi^i_0 \qquad \text{mod } \pi^{2n+1}_0.$$

Assume f is not linear, $H \not\equiv 0$, and we may scale $r = 1$ using the group action by $\mathrm{Re}\,\pi_0^0$. We obtain the following normalized structure equation for a nonlinear local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{2n}$;

$$\pi_q^{n+i} = \delta_{iq}\,\omega^n + h_q^i \pi_0^{2n+1} \quad \text{for } q < n, \tag{6}$$
$$\pi_n^{n+i} = (1 + \delta_{in})\,\omega^i + h_n^i \pi_0^{2n+1},$$

for coefficients h_j^i.

Theorem is now obtained by successive application of Maurer-Cartan equation (3) to this structure equation. We assume $n \geq 3$ for simplicity for the rest of this section, as $n = 2$ case can be treated in a similar way. We shall agree on the index range $1 \leq p, q, s, t \leq n-1$, and denote $p' = n + p$, $n' = n + n$. We denote $\theta = \pi_0^{2n+1}$, $-d\theta \equiv \mathrm{i}\,\pi_0^k \wedge \bar{\pi}_0^k = \mathrm{i}\,\varpi \mod \theta$, and $\pi_0^i = \omega^i$ for the sake of notation.

Step 1. Differentiating $\pi_s^{n'} = h_s^n\,\theta \mod \theta$, we get

$$\mathrm{i}\,h_s^n\,\varpi \equiv (\pi_{s'}^{n'} - 2\pi_s^n) \wedge \omega^n + \pi_{2n+1}^{n'} \wedge (-\mathrm{i}\,\bar{\omega}^s) \qquad \mod \theta.$$

Since $n - 1 \geq 2$, this implies $h_s^n = 0$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{2n+1}^{n'} \end{pmatrix} \equiv \begin{pmatrix} 2c_s & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ -\mathrm{i}\,\bar{\omega}^s \end{pmatrix} \qquad \mod \theta$$

for coefficients c_s, u.

Step 2. Differentiating $\pi_s^{t'} = h_s^t\,\theta \mod \theta$ for $t \neq s$, we get

$$\mathrm{i}\,h_s^t\,\varpi \equiv (\pi_{s'}^{t'} - \pi_s^t) \wedge \omega^n - \pi_s^n \wedge \omega^t + \pi_{2n+1}^{t'} \wedge (-\mathrm{i}\,\bar{\omega}^s) \qquad \mod \theta.$$

Since $n - 1 \geq 3$, this implies $h_s^t = 0$ for $t \neq s$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{t'} - \pi_s^t \\ -\pi_s^n \\ \pi_{2n+1}^{t'} \end{pmatrix} \equiv \begin{pmatrix} 0 & b_s & -\mathrm{i}\,\bar{b}_t \\ b_s & 0 & e \\ -\mathrm{i}\,\bar{b}_t & e & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ \omega^t \\ -\mathrm{i}\,\bar{\omega}^s \end{pmatrix} \qquad \mod \theta$$

for coefficients b_s, e. Since $\pi_{s'}^{t'} - \pi_s^t$ is skew Hermitian, it cannot have any ω^n-term.

Step 3. Differentiating $\pi_t^{t'} = \omega^n + \mathrm{i}\, h_t^t \theta \mod \theta$ and collecting terms, we get

$$h_t^t \varpi \equiv (\pi_{t'}^{t'} - \pi_t^t + \pi_0^0 - \pi_n^n) \wedge \omega^n + (b_p\, \omega^n - \mathrm{i}\, e\, \bar{\omega}^p) \wedge \omega^p + (-b_t\, \omega^t + \bar{b}_t\, \bar{\omega}^t) \wedge \omega^n$$

mod θ. Since $n - 1 \geq 2$, this implies $h_t^t = e$, and

$$\begin{aligned} \Delta_t &= \pi_{t'}^{t'} - \pi_t^t + \pi_0^0 - \pi_n^n \\ &= a_t\, \omega^n - \mathrm{i}\, e\, \bar{\omega}^n + (b_t\, \omega^t - \bar{b}_t\, \bar{\omega}^t) + \sum_p b_p\, \omega^p - A_t\, \theta \end{aligned}$$

for coefficients a_t, A_t.

Step 4. From *Step 2*, we may use the group action by π_{2n+1}^n to translate $e = 0$, which we assume from now on. We also translate $h_n^t = 0$ by π_{2n+1}^t. Differentiating $\pi_n^{t'} = \omega^t \mod \theta$ with these relations, we get

$$0 \equiv \bar{b}_t \,(\sum_p \omega^p \wedge \bar{\omega}^p) + (2\bar{c}_t - 3\bar{b}_t\,)\omega^n \wedge \bar{\omega}^n + (a_t + 2\mathrm{i}\, \bar{u})\omega^n \wedge \omega^t \qquad \mod \theta.$$

Thus $b_t = c_t = 0$, $a_t = -2\, \mathrm{i}\, \bar{u}$.

Step 5. Differentiating $\pi_n^{n'} = 2\, \omega^n + h_n^n \theta \mod \theta$ and collecting terms, we get

$$\mathrm{i}\, h_n^n \varpi \equiv 2\, (\pi_{n'}^{n'} - \pi_n^n + \pi_0^0 - \pi_n^n) \wedge \omega^n + \mathrm{i}\, u\, \omega^p \wedge \bar{\omega}^p - \mathrm{i}\, u\, \omega^n \wedge \bar{\omega}^n \qquad \mod \theta.$$

This implies $h_n^n = u$, and

$$\begin{aligned} \Delta_n &= \pi_{n'}^{n'} - \pi_n^n + \pi_0^0 - \pi_n^n \\ &= a_n\, \omega^n - \mathrm{i}\, u\, \bar{\omega}^n - A_n\, \theta \end{aligned}$$

for coefficients a_n, A_n. But $\Delta_t - \Delta_n$ is purely imaginary, and comparing with *Step 3*, $a_n = -3\, \mathrm{i}\, \bar{u}$.

Step 6. Now by considering θ-terms in *Step 1, 2, 3, 4, 5* and the fact $\pi_i^a \wedge \omega^i + \pi_{2n+1}^a \wedge \theta = 0$, we obtain the following simple structure equations. We omit the details of computations.

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{2n+1}^{n'} \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ -\mathrm{i}\,\bar\omega^s \end{pmatrix},$$

$$\pi_{s'}^{t'} = \pi_s^t, \quad t \neq s,$$

$$\Delta_t = -2\,\mathrm{i}\,\bar u\,\omega^n - A\,\theta,$$

$$\begin{pmatrix} -\pi_s^n \\ \pi_{2n+1}^{t'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\pi_{2n+1}^t = (A - \mathrm{i}\,u\,\bar u)\,\omega^t + B_t\,\theta$$

$$\pi_{2n+1}^n = A\,\omega^n + B_n\,\theta$$

Step 7. Differentiating $\pi_{s'}^{t'} - \pi_s^t = 0$, $t \neq s$, $\pi_s^n = 0$, we get first $B_s = 0$, $B = 0$, and

$$A - \bar A = \mathrm{i}\,(u\,\bar u - 1). \tag{7}$$

Step 8. Differentiating $\pi_n^{n'} = 2\,\omega^n + u\,\theta$ and $\pi_{2n+1}^{n'} = u\,\omega^n$,

$$du = u\,(\pi_{2n+1}^{2n+1} - \pi_0^0 + \pi_n^n - \pi_{n'}^{n'}) + 2(A - A_n)\,\omega^n - 2uA\,\theta. \tag{8}$$

Step 9. Differentiating $\pi_{s'}^{n'} = -\mathrm{i}\,u\,\bar\omega^s$ using (8) and collecting terms in $\theta \wedge \bar\omega^s$,

$$A_n = 2\,A - \bar A.$$

Step 10. Differentiating $\pi_{2n+1}^t = (\bar A - \mathrm{i})\,\omega^t$, $\pi_{2n+1}^n = A\,\omega^n$, we get

$$dA = A\,(\pi_{2n+1}^{2n+1} - \pi_0^0) + \pi_{2n+1}^0 + 2(u\,\bar\omega^n - \bar u\,\omega^n) + (u\bar u - A^2)\,\theta. \tag{9}$$

We normalize $A = \mathrm{i}\,\alpha$ for a real number α using group action by π_{2n+1}^0. Since $\pi_i^i + \bar\pi_i^i = 0$, $\Delta_t + \bar\Delta_t = \pi_0^0 + \bar\pi_0^0$, and (9) is now reduced to

$$d\alpha = 2\,\mathrm{i}\,(\alpha + 1)\,(\bar u\,\omega^n - u\,\bar\omega^n) \tag{10}$$

$$\pi_{2n+1}^0 = -(\alpha + 1)^2\,\theta.$$

When $u \neq 0$, we may also rotate u to be a positive number, in which case it is determined by (7)

$$2\,\alpha + 1 = u\,\bar{u}. \tag{11}$$

At this stage, note that the only independent coefficients in the structure equations are α, u, and that the expression for their derivatives does not involve any new coefficients. The structure equations for local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ thus *close up* as follows.

$$\begin{pmatrix} \pi^{p'}_q & \pi^{p'}_n \\ \pi^{n'}_q & \pi^{n'}_n \end{pmatrix} = \begin{pmatrix} \delta_{pq}\,\omega^n & \omega^p \\ 0 & 2\,\omega^n + u\,\theta \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} \pi^{n'}_{s'} - 2\pi^n_s \\ \pi^{n'}_{2n+1} \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ -\mathrm{i}\,\bar{\omega}^s \end{pmatrix}$$

$$\pi^{t'}_{s'} = \pi^t_s, \quad t \neq s,$$

$$\begin{pmatrix} -\pi^n_s \\ \pi^{t'}_{2n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\pi^t_{2n+1} = (A - \mathrm{i}\,u\,\bar{u})\,\omega^t$$

$$\pi^n_{2n+1} = A\,\omega^n$$

$$\Delta_t = -2\,\mathrm{i}\,\bar{u}\,\omega^n - A\,\theta$$

$$\Delta_n = -3\,\mathrm{i}\,\bar{u}\,\omega^n - \mathrm{i}\,u\,\bar{\omega^n} - A_n\,\theta$$

$$du = u\,(\pi^{2n+1}_{2n+1} - \pi^0_0 + \pi^n_n - \pi^{n'}_{n'}) + 2(A - A_n)\,\omega^n - 2uA\,\theta,$$

$$d\alpha = 2\,\mathrm{i}\,(\alpha + 1)\,(\bar{u}\,\omega^n - u\,\bar{\omega}^n)$$

$$\pi^0_{2n+1} = -(\alpha + 1)^2\,\theta$$

where $A = \mathrm{i}\,\alpha$, $A_n = 3\,\mathrm{i}\,\alpha$, and α, u satisfy the relation (11). Moreover, a long but direct computation shows that this structure equation is compatible, i.e., $d^2 = 0$ is a formal identity of the structure equation.

Remark. The structure equation closes up at order 3. This implies a C^3, nonlinear CR immersion $f : \Sigma^n \to \Sigma^{2n}$ is real analytic.

The remaining step in the proof of **Theorem** consists of identifying this as the structure equation of Whitney immersion. We make one useful observation. Let $\Sigma^* = \Sigma^n - \Sigma^n_s$, which is a connected set. Note also the structure equation (12) implies that the set of points where $u = 0$, or equivalently $\alpha = -\frac{1}{2}$, cannot have any interior on Σ^*. We claim the invariant α takes any value $> -\frac{1}{2}$ on Σ^*.

Suppose $\alpha_+ = \sup_{\Sigma^*} \alpha > -\frac{1}{2}$ is finite. Applying the existence part of Cartan's generalization of Lie's third fundamental theorem on closed structure equations, [Br3], there exists for any $p_0 \in \Sigma^n$ a neighborhood $U \subset \Sigma^n$ of p_0 and a CR immersion $g : U \to \Sigma^{2n}$ with invariant $\alpha|_{p_0} = \alpha_+$, hence necessarily $u_{p_0} = \sqrt{2\,\alpha_0 + 1}$. From (12), $d\alpha|_{p_0} \neq 0$ and let $p_- \in U$ be a point with $-\frac{1}{2} < \alpha|_{p_-} < \alpha_+$. Then by uniqueness part of Cartan's theorem, there exists a neighborhood $U' \subset U$ of p_- on which g agrees with Whitney immersion Γ_n up to automorphisms of Σ^n and Σ^{2n}. Since g and Γ_n satisfy the closed set of structure equations, they are real analytic. Thus g is a part of Γ_n. But $d\alpha|_{p_0} \neq 0$, and there exists a point $p_+ \in U \subset \Sigma^*$ such that $\alpha|_{p_+} > \alpha_+$, a contradiction. By similar argument, $\inf_{\Sigma^*} \alpha = -\frac{1}{2}$, and the claim follows for Σ^* is connected.

Proof of Theorem. Since the set of points $\alpha = -\frac{1}{2}$ cannot have any interior, let $p \in \Sigma$ be a point with $\alpha|_p > -\frac{1}{2}$. From the results above and the uniqueness part of aforementioned Cartan's theorem, $f = \Gamma_n$ on a neighborhood U of p up to automorphisms of the spheres. The theorem follows for both f and Γ_n are real analytic. □

Proof of Corollary. By the regularity theorem [HJX], F is real analytic up to ∂F. Since the CR structure on $S^{2n+1} = \Sigma^n$ is definite, the set of points where ∂F has holomorphic rank n is a dense open subset. Assume F is not linear. There exists a point $p \in \Sigma^n$ where the second fundamental form does not vanish either. By **Theorem**, ∂F agrees with Whitney immersion Γ_n in a neighborhood of p up to automorphisms of the spheres. The real analyticity then implies $\partial F = \Gamma_n$ on Σ^n, and hence $F = \mathcal{W}_n$ on $\mathbf{B}^{n+1}$. □

We may apply the existence part of Cartan's generalization of Lie's third fundamental theorem and show that Whitney immersion gives an example of a deformable CR-submanifold. Take a point $p \in \Sigma^n$, and an analytic one parameter family of real numbers $\alpha_t > -\frac{1}{2}$, and set $u_t = \sqrt{2\,\alpha_t + 1}$. Then by the existence part of Cartan's theorem, there exists a neighborhood U of p and a one parameter family of CR immersions $f_t : U \to \Sigma^{2n}$ with the induced structure equations (12) such that the invariants α, u have the prescribed values α_t, u_t at p. This deformation of course is *tangential* and does not actually deform the submanifold. It is due to an intrinsic CR symmetry of Σ^n that cannot be extended to a symmetry of the ambient Σ^{2n} along Whitney immersion.

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