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Topology of the Bryant-Salamon G_2-manifolds and some Ricci flat manifolds

九州大学・数理学研究院 宮岡 礼子 (Reiko Miyaoka)
Graduate School of Mathematics
Kyushu University

1 Introduction

In this article, we consider a topological aspect of the Bryant-Salamon G_2-manifolds and of some Ricci flat manifolds. We also give examples of special Lagrangian submanifolds of T^*S^n with the Stenzel metric.

Physicists as well as mathematicians are interested in Ricci-flat (Kähler) manifolds as a special case of Einstein manifolds. Ricci-flat metrics are often constructed on vector bundles over Riemannian manifolds where some group action of cohomogeneity one is effectively used. In this case, the base manifold is regarded as a degenerate orbit, while the principal orbits are of codimension one in the total space, i.e., the sphere bundles.

This reminds us of the theory of isoparametric hypersurfaces. In fact, isoparametric hypersurfaces in the sphere S^n exist in one parameter families, which laminate S^n with two other degenerate submanifolds, called the focal submanifolds. If we delete from S^n one of the focal submanifolds, then the remaining part is a disk bundle over another focal submanifold [Mü]. Two of the Bryant-Salamon G_2-manifolds fit this theory exactly. Namely, the spin bundle $\mathcal{S}$ over S^3 is associated to isoparametric hypersurfaces $S^3 \times S^3$ in S^7, and is shown to be homeomorphic to $S^7 \setminus S^3$. On the other hand, the anti-self-dual bundle $\Lambda^2_-(\mathbb{C}P^2)$ over $\mathbb{C}P^2$ is associated to the so-called Cartan hypersurfaces in S^7, and turns out to be homeomorphic to $S^7 \setminus \mathbb{C}P^2$ (Theorem 3.1).

We thus come to consider more generally, the space $\bar{M} \setminus D$, where $\bar{M}$ is a compact Riemannian manifold with positive Ricci curvature, and D is some subset. For instance, T^*S^3 can be considered as $S^6 \setminus S^3$ (§5), on which the Stenzel metric exists. Our problem to consider is when there exists a non-flat complete Ricci-flat metric on $\bar{M} \setminus D$, in the real category. In §4, we partially answer to this problem. For a complex version, see [BK],[TY].

Another aspect of Ricci flat manifolds is relations with special geometry. With respect to the Stenzel metric, T^*S^n becomes a Calabi-Yau manifold with calibrations $\Re(e^{i\theta}\Omega)$, where Ω is the global holomorphic n-form (see

§5). Then what are special Lagrangian submanifolds? Here again the theory of isoparametric hypersurfaces works well. Harvey and Lawson's result [HL], generalized by Karigiannis and Min-Oo recently [KM], tells us that the conormal bundle over an austere submanifold in S^n is a special Lagrangian submanifold of T^*S^n. We have many examples of austere submanifolds in S^n, consisting of some minimal isoparametric hypersurfaces, and of focal submanifolds of all isoparametric hypersurfaces [IKM].

In the following, we give some details, but for a full story, see [Mi2].

2 The Bryant-Salamon G_2-manifolds

By a G_2-manifold, we mean a Riemannian manifold with the holonomy group G_2. The metric is called a G_2-metric.

Denoting the exterior product of three vectors of an orthonormal coframe $e^1, \dots, e^7$ of $\mathbb{R}^7$ by $e^{ijk} = e^i \wedge e^j \wedge e^k$, define a 3-form ϕ on $\mathbb{R}^7$ by

$$\phi = e^{125} - e^{345} + e^{136} - e^{426} + e^{147} - e^{237} + e^{567}. \tag{1}$$

The automorphism group G_2 of the Cayley numbers can be also defined as the subgroup of $GL(7,\mathbb{R})$ preserving ϕ [B1]. A G_2-structure on a 7-dimensional manifold X is a reduction of the structure group of the linear frame bundle to G_2. Let $\mathcal{O}$ be the $GL(7,\mathbb{R})$-orbit of ϕ ($\cong GL(7,\mathbb{R})/G_2$), then a G_2-structure is equivalent to the existence of a global 3-form ϕ on X satisfying $\phi_x \in \mathcal{O}_x$. Since $G_2 \subset SO(7)$, a G_2-structure induces a Riemannian metric g on X. The holonomy group is contained in G_2 if and only if $d\phi = d * \phi = 0$, and is equal to G_2 if and only if there are no non-trivial parallel 1-forms on X, provided that X is simply connected and connected. Note that a G_2-metric is Ricci flat [B1].

Many G_2-manifolds, complete or compact ones, have been constructed ([BS], [J], [K]). The first examples of complete G_2 manifolds were obtained by Bryant and Salamon on $X = \mathcal{S}$, $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{C}P^2)$. Let g_b and g_f be the standard metrics on the base and the fiber space, respectively, normalized appropriately by constant multiples. Consider a hypersurface N_r of X consisting of fiber vectors of length r. On $X = \cup_{r\geq 0} N_r$, they seek a metric so that the 3-form which depends on the metric, satisfies the non-linear PDE's $d\phi = d * \phi = 0$. Restricting the metric to the warped product form $g = u(r)f_b + v(r)g_f$, where $u(r)$ and $v(r)$ are functions of r, they reduce the equations to ODE's, and obtain metrics

$$\begin{aligned} g &= (\lambda + r^2)^{2/3} g_b + (\lambda + r^2)^{-1/3} g_f \quad \text{on } \mathcal{S} \\ g &= (\lambda + r^2)^{1/2} g_b + (\lambda + r^2)^{-1/2} g_f \quad \text{on } \Lambda^2_-(M^4). \end{aligned}$$

When $\lambda > 0$, these metrics extend to complete ones on $X = \cup_{r\geq 0} N_r$. The non-existence of non-trivial parallel 1-forms is then proved, which establishes $\mathrm{Hol}(g) = G_2$. It turns out that the metric is homogeneous on N_r, where $N_r \cong S^3 \times S^3$ for $X = \mathcal{S}$, $N_r \cong \mathbb{C}P^3$ for $X = \Lambda^2_-(S^4)$, and $N_r \cong SU(3)/T^2$ for $X = \Lambda^2_-(\mathbb{C}P^2)$ (T^2 is a maximal torus). We note that, however, the metric is different from the standard one. Indeed when $X = \Lambda^2_-(S^4)$, the metric on $N_r \cong \mathbb{C}P^3$ is not the Fubini-Study metric, but a non-Kähler Einstein metric. The one on N_r of $X = \Lambda^2_-(\mathbb{C}P^2)$ is also non-Kähler Einstein.

We notice that $S^3 \times S^3$ and the flag manifold $SU(3)/T^2$ are homeomorphic to typical isoparametric hypersurfaces in S^7. In particular, the latter is the Cartan hypersurface, on which the induced metric is Kähler Einstein.

3 Homogeneous and isoparametric hypersurfaces in the sphere

Now, we give a brief review of isoparametric hypersurfaces in the sphere.

By isoparametric hypersurfaces, we mean hypersurfaces with constant principal curvatures (see [Th]). These are given as level sets $W_t = f^{-1}(t)$ of the so-called Cartan-Münzner function $f : S^n \to [-1, 1]$, for $t \in (-1, 1)$ [Mü]. The level sets $W_\pm = f^{-1}(\pm 1)$ have lower dimension and are called the focal submanifolds. Note that $W_\pm$ and W_0 are minimal submanifolds. Typical examples of isoparametric hypersurfaces are homogeneous hypersurfaces, which are given as principal orbits of the isotropy representation of rank two symmetric spaces [HsL]. All homogeneous ones are classified, but there exist infinitely many non-homogeneous isoparametric hypersurfaces with four principal curvatures [OT], [FKM].

The most important fact in our argument is that the ambient sphere is stratified as

$$S^n = \cup_{t\in[-1,1]} W_t, \tag{2}$$

by hypersurfaces W_t, $t \in (-1, 1)$ and the two focal submanifolds $W_\pm$. The number g of principal curvatures is limited to 1,2,3,4, or 6 [Mü]. Those with three principal curvatures are known as Cartan hypersurfaces, which are tubes over standardly embedded Veronese surfaces $\mathbb{F}P^2$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{C}$ in S^4, S^7, S^{13} and S^{25} ($\mathcal{C}$ is the Cayley algebra) [C].

We are concerned with the case S^7. All isoparametric hypersurfaces in S^7 are homogeneous and so are classified. We denote a k-dimensional sphere with radius a by $S^k(a)$.

(a) $g = 1 : f^{-1}(t) = S^6(\sqrt{1-t^2})$, $t \in (-1, 1)$.

(b) $g = 2$: $f^{-1}(t) = S^k(\sqrt{\frac{1-t}{2}}) \times S^{6-k}(\sqrt{\frac{1+t}{2}})$, $t \in (-1, 1)$, $1 \leq k \leq 5$, *generalized Clifford torus.*

(c) $g = 3$: isotropy orbits of the symmetric space $SU(3) \times SU(3)/SU(3)$, *the Cartan hypersurfaces* $\cong SU(3)/T^2$.

(d) $g = 4$: isotropy orbits of the symmetric space $SO(6)/SO(2) \times SO(4)$.

(e) $g = 6$: isotropy orbits of the symmetric space $G_2/SO(4)$

Theorem 3.1 *We have homeomorphisms $\mathcal{S} \cong S^7 \setminus S^3$ and $\Lambda^2_-(\mathbb{C}P^2) \cong S^7 \setminus \mathbb{C}P^2$, where S^3 and $\mathbb{C}P^2$ are embedded in S^7 in the standard way. In other words, a compactification of $\mathcal{S}$ and $\Lambda^2_-(\mathbb{C}P^2)$ is given by S^7.*

Proof: (Sketch. See [Mi2]) By Münzner's result, $S^n \setminus W_+$ ($S^n \setminus W_-$, respectively) is a disk bundle over W_- (W_+, respectively). In the case of isoparametric hypersurface $S^3 \times S^3$ in S^7, both W_+ and W_- are totally geodesic S^3, and $S^7 \setminus S^3$ is a disk bundle over S^3. Since $\mathcal{S}$ is a $\mathbb{R}^4$ bundle over S^3 with $N_r \cong S^3 \times S^3$ for $r > 0$, we have a homeomorphism $\mathcal{S} \cong S^7 \setminus S^3$. On the other hand, the Cartan hypersurfaces in S^7 is homeomorphic to the flag manifold $SU(3)/T^2$, and W_+ and W_- are both standardly embedded $\mathbb{C}P^2$. Thus $S^7 \setminus \mathbb{C}P^2$ is a disk bundle over $\mathbb{C}P^2$. Since $\Lambda^2_-(\mathbb{C}P^2)$ is a $\mathbb{R}^3$ bundle over $\mathbb{C}P^2$ with $N_r \cong SU(3)/T^2$, we can find a homeomorphism $\Lambda^2_-(\mathbb{C}P^2) \cong S^7 \setminus \mathbb{C}P^2$. □

Remark 3.2 : The Bryant-Salamon G_2-manifolds are investigated in details in [AW], but this simple fact is not referred to.

Remark 3.3 : In the case $M = S^4$, the hypersurface N_r of $\Lambda^2_-(S^4)$ is diffeomorphic with $\mathbb{C}P^3$. By the result of Cleyton and Swann, [CS], Theorem 9.3, we see that a compactification of $\Lambda^2_-(S^4)$ is given by $\mathbb{C}P^3 \times S^1$.

4 Complete Ricci flat metric on $S^n \setminus D$

Lü, Page and Pope constructed complete Ricci flat metrics on $S^m \times \mathbb{R}^{2n+2}$ ($m, n \geq 1$), modifying the construction of non-homogeneous Einstein metrics on compact manifolds [LPP]. Since $S^m \times \mathbb{R}^{2n+2} \cong \cup_{r \geq 0} S^m \times S^{2m+1}(r)$, using the isoparametric family $S^m \times S^{2n+1}$ in S^{n+2m+2} and the focal submanifolds S^m and S^{2n+1}, we see that $S^{m+2n+2} \setminus S^{2n+1}$ is a disk bundle over S^m, i.e., $S^m \times \mathbb{R}^{2n+2}$. Thus we obtain the following (part (iii) will be proved in the next section):

Theorem 4.1 *A complete non-flat Ricci flat metric exists on*

(i) $S^7 \setminus \mathbb{C}P^2$, $S^7 \setminus S^3$: *the Bryant-Salamon metric*

(ii) $S^{m+2n+2} \setminus S^{2n+1}$, $n, m \geq 1$: *the Lü-Page-Pop metric*

(iii) $S^6 \setminus S^2$, $S^{14} \setminus S^6$: *the Stenzel metric (Kähler)*

Note that $S^N \setminus S^n \cong \mathbb{R}^N \setminus \mathbb{R}^n$. As a less topologically trivial case, we may ask a question, which we will consider on another occasion.

Problem. For each isoparametric family $\{W_t\}$ in S^n, does there exist a complete Ricci-flat, non-flat metric on $S^n \setminus W_+$ or on $S^n \setminus W_-$?

5 Stenzel metric and calibrated geometry

We give a brief introduction to the Stenzel metric on T^*S^n. Identify T^*S^n with $Q^n = \{z \in \mathbb{C}^{n+1} \mid z_0^2 + \cdots + z_n^2 = 1\}$ by $T^*S^n \ni (x, \xi) \to x \cosh|\xi| + i\xi/|\xi| \sinh|\xi|$, and induce a complex structure on T^*S^n from Q^n. Then consider a holomorphic $(n, 0)$ form Ω given by

$$\Omega(T) := (dz_0 \wedge \cdots \wedge dz_n)(Z, T), \quad Z = z_0\frac{\partial}{\partial z_0} + \cdots + z_n\frac{\partial}{\partial z_n} \in \mathbb{C}^{n+1}$$

The Kähler form of the Stenzel metric is given by

$$\omega_{St} = \frac{i}{2}\sum_{j,k=1}^{n} a_{jk} dz_j \wedge d\bar{z}_k$$

where

$$a_{jk} = (\delta_{jk} + \frac{z_j\bar{z}_k}{|z_0|^2})u' + 2\Re(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k)u'',$$

and u is a function of $r = |z|$, of which details we need not here. This is a highly generalized Eguchi-Hanson metric, first constructed explicitly.

Proposition 5.1 *There exists a non-flat complete Ricci-flat Kähler metric on* $S^6 \setminus S^2$ *and* $S^{14} \setminus S^6$.

Proof: Because S^3 and S^7 are parallelizable, it follows that $T^*S^n \cong S^n \times \mathbb{R}^n \cong \cup_{r \geq 0} S^n \times S^{n-1}(r) \cong S^{2n} \setminus S^{n-1}$. $\square$

In the calibrated geometry developed by Harvey and Lawson, one way to obtain special Lagrangian submanifolds in $\mathbb{C}^{n+1}$ is to take the conormal bundle of the so-called *austere* submanifold in $\mathbb{R}^{n+1}$ (Theorem 3.17 in Chapter III [HL]). A submanifold N in $\mathbb{R}^{n+1}$ or in S^n is austere if any shape operators have eigenvalues in pairs $\pm\lambda_j$, and if the multiplicities of $\pm\lambda_j$ coincide, where $\lambda_j = 0$ is admissible. The cone over an austere submanifold of S^n is austere in $\mathbb{R}^{n+1}$. Austere surfaces are nothing but minimal surfaces. In some cases, austere submanifolds are classified [B2], [DF].

In [IKM], we found a large class of compact austere submanifolds in S^n:

Theorem 5.2 *[IKM] The focal submanifolds of any isoparametric hypersurfaces in S^n are austere. Minimal isoparametric hypersurfaces in S^n, whose principal curvatures have the same multiplicity, are austere.*

On the other hand, Karigiannis and Min-Oo proved:

Theorem 5.3 *[KM] The conormal bundle of a submanifold N in S^n is a special Lagrangian submanifold of T^*S^n with the Stenzel metric, if and only if N is austere.*

Note that the Stenzel metric is standard when restricted to S^n. From these and the detailed version of Theorem 5.2, we obtain

Theorem 5.4 *The conormal bundles of the focal submanifolds $W_\pm$ of* any *isoparametric hypersurfaces are special Lagrangian submanifold of T^*S^n equipped with the Stenzel metric. Infinitely many non-homogeneous examples are included among them. The conormal bundle of the following* minimal *isoparametric hypersurfaces in S^n are special Lagrangian submanifold of T^*S^n:*

$$\begin{array}{lll} W = S^{n-1} & & \\ W^{2d} = S^d \times S^d, & n = 2d+1 & \\ W^{3d}, & n = 3d+1 & d = 1,2,4,8 \\ W^{4d}, & n = 4d+1, & d = 1,2 \\ W^{6d}, & n = 6d+1, & d = 1,2 \end{array}$$

where W has, respectively, $1, 2, 3, 4$ *and 6 principal curvatures. The phase $e^{i\theta}$ is determined by the dimension of W or $W_\pm$.*

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Graduate School of Mathematics,
Kyushu University
Hakozaki, Fukuoka, 812-8581/JAPAN
E-mail Address: r-miyaok@math.kyushu-u.ac.jp