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Some twistor spaces of 6-dimensional submanifolds in the octonions.

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1 Introduction

In this paper, we shall consider the twistor space of 6-dimensional submanifolds in the octonions.

First we recall the induced almost Hermitian structure of such 6-dimensional submanifolds. In ([Br1]), R.L.Bryant showed that any oriented 6-dimensional submanifold $\varphi : M^6 \to \mathfrak{C}$ of the octonions admits the almost complex (Hermitian) structure as follows

$$\varphi_*(JX) = \varphi_*(X)(\eta \times \xi)$$

where ξ, η is the oriented orthonormal frame field of the normal bundle of φ, which is defined locally, but $\eta \times \xi$ is a global $S^6(\subset \mathrm{Im}\mathfrak{C})$-valued function on M^6. We obtain the almost complex structure whole on M^6. Therefore there exist a principal $U(3)$-bundle structure on M^6, and obtained the associated fibre bundle over M^6 with fibre $P^2(\mathbf{C})$. This fibre bundle is called the twistor space of M^6. In this paper, we consider the integrability conditions on some almost complex structures on this twistor space. Usually the twistor space is defined as a fibre bundle (over an even dimensional Riemmnian manifold) whose fibre (at each point) consists of all almost complex structures on the tangent space compatible with the metric and the orientation. The fibre is isomorphic to the rank one Hermitian symmetric space $SO(2n)/U(n)$. We note that $SO(4)/U(2) \simeq P^1(\mathbf{C})$ for $n = 2$ and $SO(6)/U(3) \simeq P^3(\mathbf{C})$ for $n = 3$ (see P. Wong [Wo]). The twistor space which is treated in this paper is different from the usual one.

We note that the induced almost complex (Hermitian) structure is a Spin(7) invariant in the following sense.

Let $\varphi_1, \varphi_2 : M^6 \to \mathfrak{C}$ be two isometric immersions from the same source manifold to the octonions. If there exist an element $g \in \mathrm{Spin}(7)$ such that $g \circ \varphi_1 = \varphi_2$ (up to the parallel translation), then the two maps are said to be Spin(7)-congruent. If two maps are Spin(7)-congruent, then the induced almost complex structures coincide.

2 Preliminaries

Let $\mathbf{H}$ be the skew field of all quaternions with canonical basis $\{1, i, j, k\}$, which satisfy

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

The octonions (or Cayley algebra) $\mathfrak{C}$ over $\mathbf{R}$ can be considered as a direct sum $\mathbf{H} \oplus \mathbf{H} = \mathfrak{C}$ with the following multiplication

$$(a + b\varepsilon)(c + d\varepsilon) = ac - \bar{d}b + (da + b\bar{c})\varepsilon,$$

where $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$ and $a, b, c, d \in \mathbf{H}$, the symbol "$^-$" denote the conjugation of the quaternion. For any $x, y \in \mathfrak{C}$, we have

$$< xy, xy > = < x, x >< y, y >$$

which is called "normed algebra" in ([H-L]). The octonions is a non-commutative, non-associative, alternative, division algebra. The group of automrphisms of the octonions is the exceptional simple Lie Group

$$G_2 = \{g \in SO(8) \mid g(uv) = g(u)g(v) \text{ for any } u, v \in \mathfrak{C}\}.$$

In this paper, we shall concern the group of spinors $Spin(7)$ which is defined as follows

$$Spin(7) = \{g \in SO(8) \mid g(uv) = g(u)\chi_g(v) \text{ for any } u, v \in \mathfrak{C}\}.$$

where $\chi_g(v) = g(g^{-1}(1)v)$. We note that G_2 is a Lie subgroup of $Spin(7)$; $G_2 = \{g \in Spin(7) \mid g(1) = 1\}$. The map χ defines the double covering map from $Spin(7)$ to $SO(7)$, which satisfy the following equivariance

$$g(u) \times g(v) = \chi_g(u \times v)$$

for any $u, v \in \mathfrak{C}$ and $u \times v = (1/2)(\bar{v}u - \bar{u}v)$ (which is called the "exterior product") where $\bar{v} = 2 < v, 1 > -v$ is the conjugation of $v \in \mathfrak{C}$. We note that $u \times v$ is an element of pure-imaginary part of $\mathfrak{C}$.

2.1 Spin(7)-structure equations

In this section, we shall recall the structure equation of Spin(7) which is established by R.Bryant ([Br1]). To construct this, we fix a basis of the complexification of the octonions $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C}$ as follows

$$N = (1/2)(1 - \sqrt{-1}\varepsilon),\ \bar{N} = (1/2)(1 + \sqrt{-1}\varepsilon)$$

$$E_1 = iN,\ E_2 = jN,\ E_3 = -kN,\ \bar{E}_1 = i\bar{N},\ \bar{E}_2 = j\bar{N},\ \bar{E}_3 = -k\bar{N}.$$

We extend the multiplication of the octonions complex linearly. Then we have the following multiplication table;

$A\backslash B$	N	E_1	E_2	E_3	$\bar{N}$	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$
N	N	0	0	0	0	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$
E_1	E_1	0	$-\bar{E}_3$	$\bar{E}_2$	0	$-\bar{N}$	0	0
E_2	E_2	$\bar{E}_3$	0	$-\bar{E}_1$	0	0	$-\bar{N}$	0
E_3	E_3	$-\bar{E}_2$	$\bar{E}_1$	0	0	0	0	$-\bar{N}$
$\bar{N}$	0	E_1	E_2	E_3	$\bar{N}$	0	0	0
$\bar{E}_1$	0	$-N$	0	0	$\bar{E}_1$	0	$-E_3$	E_2
$\bar{E}_2$	0	0	$-N$	0	$\bar{E}_2$	E_3	0	$-E_1$
$\bar{E}_3$	0	0	0	$-N$	$\bar{E}_3$	$-E_2$	E_1	0

We define a $\mathfrak{C} \rtimes Spin(7)$ (semi-direct product) admissible frame field as follows. Let o be the origin of the octonions. The Lie Group $\mathfrak{C} \rtimes Spin(7)$ acts on $\mathfrak{C} \oplus End(\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{C})$ such that

$$
\begin{aligned}
(x,g)(o;N,E,\bar{N},\bar{E}) &= (g\cdot o + x, g(N), g(E), g(\bar{N}), g(\bar{E})) \\
&= (x, g(N), g(E), g(\bar{N}), g(\bar{E})) \\
&= (o;N,E,\bar{N},\bar{E}) \begin{pmatrix} 1 & 0_{1\times 8} \\ \rho(x) & \rho(g) \end{pmatrix}
\end{aligned}
$$

where $(x,g) \in \mathfrak{C} \rtimes Spin(7)$ and $\begin{pmatrix} 1 & 0_{1\times 8} \\ \rho(x) & \rho(g) \end{pmatrix}$ is an its matrix representation. A frame $(x;n,f,\bar{n},\bar{f})$ is said to be a $\mathfrak{C}\rtimes Spin(7)$ admissible one if there exists a $(x,g) \in \mathfrak{C}\rtimes Spin(7)$ such that

$$(x;n,f,\bar{n},\bar{f}) = (x,g)(o;N,E,\bar{N},\bar{E}).$$

Proposition 2.1 ([Br1]) *The Maurer-Cartan form of* $\mathfrak{C} \rtimes Spin(7)$ *is given by*

$$
\begin{aligned}
d(x;n,f,\bar{n},\bar{f}) &= (x;n,f,\bar{n},\bar{f}) \left(\begin{array}{c|cc|cc} 0 & 0 & 0_{1\times 3} & 0 & 0_{1\times 3} \\ \hline \nu & \sqrt{-1}\rho & -{}^t\mathfrak{h} & 0 & -{}^t\theta \\ \omega & \mathfrak{h} & \kappa & \theta & [\bar{\theta}] \\ \hline \bar{\nu} & 0 & -{}^t\bar{\theta} & -\sqrt{-1}\rho & -{}^t\bar{\mathfrak{h}} \\ \bar{\omega} & \bar{\theta} & [\theta] & \bar{\mathfrak{h}} & \bar{\kappa} \end{array}\right) \\
&= (x;n,f,\bar{n},\bar{f})\psi
\end{aligned}
$$

where ψ *is the* $spin(7) \oplus \mathfrak{C}(\subset M_{9\times 9}(\boldsymbol{C}))$ *-valued 1-form,* ρ *is a real-valued 1-form,* ν *is a complex valued 1-form,* $\omega, \mathfrak{h}, \theta$ *are* $M_{3\times 1}$*-valued 1-form,* κ *is a* $\mathfrak{u}(3)$*-valued 1-form which satisfy* $\sqrt{-1}\rho + tr\kappa = 0$*, and*

$$[\theta] = \begin{pmatrix} 0 & \theta^3 & -\theta^2 \\ -\theta^3 & 0 & \theta^1 \\ \theta^2 & -\theta^1 & 0 \end{pmatrix}$$

for $\theta = {}^t(\theta^1, \theta^2, \theta^3)$*. The* ψ *satisfy the following integrability condition* $d\psi + \psi \wedge \psi = 0$*. More precisely*

$$\begin{aligned} dx &= (n, f, \bar{n}, \bar{f}) \begin{pmatrix} \nu \\ \omega \\ \bar{\nu} \\ \bar{\omega} \end{pmatrix}, \\ dn &= n\sqrt{-1}\rho + f\mathfrak{h} + \bar{f}\bar{\theta}, \\ df &= n(-{}^t\bar{\mathfrak{h}}) + f\kappa + n(-{}^t\bar{\theta}) + \bar{f}[\theta], \end{aligned}$$

and the integrability conditions are given by

$$\begin{aligned} d\nu &= \sqrt{-1}\rho \wedge \nu +^t \bar{\mathfrak{h}} \wedge \omega +^t \bar{\theta} \wedge \bar{\omega}, \\ d\omega &= -\mathfrak{h} \wedge \nu - \kappa \wedge \omega - \theta \wedge \bar{\nu} - [\theta] \wedge \bar{\omega}, \\ d(\sqrt{-1}\rho) &= {}^t\bar{\mathfrak{h}} \wedge \mathfrak{h} +^t \theta \wedge \bar{\theta}, \\ d\mathfrak{h} &= -\mathfrak{h} \wedge \sqrt{-1}\rho - \kappa \wedge \mathfrak{h} - [\theta] \wedge \bar{\theta}, \\ d\theta &= \theta \wedge \sqrt{-1}\rho - \kappa \wedge \theta - [\bar{\theta}] \wedge \bar{\mathfrak{h}}, \\ d\kappa &= \mathfrak{h} \wedge^t \bar{\mathfrak{h}} - \kappa \wedge \kappa + \theta \wedge^t \bar{\theta} - [\bar{\theta}] \wedge [\theta]. \end{aligned}$$

3 Gram-Schmidt process of Spin(7)

To construct the Spin(7)-frame field, we recall the Gram-Schmidt process of G_2-frame. Let $\mathfrak{C}_0 = \{u \in \mathfrak{C} | < u, 1 >= 0\}$ be the subspce of purely imginary octonions.

Lemma 3.1 *For a pair of mutually orthogonal unit vectors* e_1*,* e_4 *in* $\mathfrak{C}_0$ *put* $e_5 = e_1e_4$*. Take a unit vector* e_2*, which is perpendicular to* e_1*,* e_4 *and* e_5*. If we put* $e_3 = e_1e_2$*,* $e_6 = e_2e_4$ *and* $e_7 = e_3e_4$ *then the matrix*

$$g = [e_1, e_2, e_3, e_4, e_5, e_6, e_7] \in SO(7)$$

is an element of G_2*.*

3.1 A method of construction

By Lemma 3.1, we can take $e_4 = \eta \times \xi$, we can get the G_2-frame field as follows. We set

$$\begin{aligned} N^* &= (1/2)(1-\sqrt{-1}e_4), \quad \bar{N}^* = (1/2)(1+\sqrt{-1}e_4), \\ E_1^* &= (1/2)(e_1-\sqrt{-1}e_5), \quad \bar{E}_1^* = (1/2)(e_1+\sqrt{-1}e_5), \\ E_2^* &= (1/2)(e_2-\sqrt{-1}e_6), \quad \bar{E}_2^* = (1/2)(e_2+\sqrt{-1}e_6), \\ E_3^* &= -(1/2)(e_3-\sqrt{-1}e_7), \quad \bar{E}_3^* = -(1/2)(e_3+\sqrt{-1}e_7). \end{aligned}$$

Then $span_{\mathbf{C}}\{N^*, E_1^*, E_2^*, E_3^*\}$ is a $\sqrt{-1}$-eigen space $T_p^{(1,0)}\mathfrak{C}(\subset \mathfrak{C}\otimes\mathbf{C})$ with respect to the almost complex structure $J = R_{\eta\times\xi}$ at $p \in \mathfrak{C}$. On the other hand, $n = (1/2)(\xi - \sqrt{-1}\eta)$ is a local orthonormal frame field of the complexified normal bundle $T^{\perp(1,0)}M$. Since $T^{\perp(1,0)}_{\varphi(m)}M \subset T^{(1,0)}_{\varphi(m)}\mathfrak{C}$, there exists a $M_{4\times1}(\mathbf{C})$-valued function $a_1 = {}^t(a_{11}, a_{21}, a_{31}, a_{41})$, such that

$$n = (1/2)(\xi - \sqrt{-1}\eta) = (N^*, E_1^*, E_2^*, E_3^*)a_1.$$

By the Gram-Schmidt orthonormalization with respect to the Hermitian inner product of $T^{(1,0)}_{\varphi(m)}\mathfrak{C}$, there exist three $M_{4\times1}(\mathbf{C})$-valued functions $\{a_2, a_3, a_4\}$ such that $\{a_1, a_2, a_3, a_4\}$ is a special unitary frame. We set

$$f_i = (N^*, E_1^*, E_2^*, E_3^*)a_{i+1}$$

for $i = 1, 2, 3$, then

$$(n, f, \bar{n}, \bar{f}) = (n, f_1, f_2, f_3, \bar{n}, \bar{f}_1, \bar{f}_2, \bar{f}_3)$$

is a (local) Spin(7)-frame field on M.

Remark 3.1 *This procedure comes from the following relation*

$$Spin(7)/Spin(6) = Spin(7)/SU(4) = S^6 \cong G_2/SU(3).$$

4 Invariants of Spin(7)

We shall recall the invariants of Spin(7)-congruence classes. By Proposition 2.1, we have

Proposition 4.1 ([Br1]) *Let $\varphi : M^6 \to \mathfrak{C}$ be an isometric immersion from an oriented 6-dimensional manifold to the octonions. Then*

$$\begin{aligned} d\varphi &= f\omega + \bar{f}\bar{\omega}, && (4.1) \\ \nu &= 0, && (4.2) \\ dn &= n\sqrt{-1}\rho + f\mathfrak{h} + \bar{f}\bar{\theta}, && (4.3) \\ df &= n(-{}^t\bar{\mathfrak{h}}) + f\kappa + n(-{}^t\bar{\theta}) + \bar{f}[\theta], && (4.4) \end{aligned}$$

and the integrability conditions imply that

$$\begin{aligned}
d\omega &= -\kappa \wedge \omega - [\theta] \wedge \bar{\omega}, && (4.5)\\
d(\sqrt{-1}\rho) &= {}^t\bar{\mathfrak{h}} \wedge \mathfrak{h} + {}^t\theta \wedge \bar{\theta}, && (4.6)\\
d\mathfrak{h} &= -\mathfrak{h} \wedge \sqrt{-1}\rho - \kappa \wedge \mathfrak{h} - [\bar{\theta}] \wedge \bar{\theta}, && (4.7)\\
d\theta &= \theta \wedge \sqrt{-1}\rho - \kappa \wedge \theta - [\bar{\theta}] \wedge \bar{\mathfrak{h}}, && (4.8)\\
d\kappa &= \mathfrak{h} \wedge {}^t\bar{\mathfrak{h}} - \kappa \wedge \kappa + \theta \wedge {}^t\bar{\theta} - [\bar{\theta}] \wedge [\theta]. && (4.9)
\end{aligned}$$

The second fundamental form II is given by

$$\mathrm{II} = -2Re\{({}^t\bar{\mathfrak{h}} \circ \omega + {}^t\bar{\theta} \circ \bar{\omega}) \otimes n\}$$

where the symbol "$\circ$" is the symmetric tensor product. By Cartan's Lemma (since $\nu = 0$), there exist $M_{3\times 3}$-valued matrices A, B, C such that

$$\begin{pmatrix} \mathfrak{h} \\ \theta \end{pmatrix} = \begin{pmatrix} \overline{B} & \overline{A} \\ {}^tB & \overline{C} \end{pmatrix} \begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix} \qquad (4.10)$$

where ${}^tA = A$ and ${}^tC = C$. We have the following decomposition

$$\begin{aligned}
\mathrm{II}^{(2,0)} &= (-{}^t\omega \circ A\omega) \otimes n\\
\mathrm{II}^{(1,1)} &= (-{}^t\bar{\omega} \circ {}^t B\omega - {}^t\omega \circ B\bar{\omega}) \otimes n\\
\mathrm{II}^{(0,2)} &= (-{}^t\bar{\omega} \circ \bar{C}\bar{\omega}) \otimes n.
\end{aligned}$$

We shall write each elements more explicitly. There exists a unitary frame $\{e_i, Je_i\}$ for i=1,2,3, such that

$$n = (1/2)(\xi - \sqrt{-1}\eta), f_i = (1/2)(e_i - \sqrt{-1}Je_i).$$

Thus elements of second fundamental form are given by

$$\begin{aligned}
A_{ij} &= -2 < \mathrm{II}(f_i, f_j), \bar{n} >,\\
B_{ij} &= -2 < \mathrm{II}(f_i, \bar{f}_j), \bar{n} >,\\
C_{ij} &= -2 < \mathrm{II}(\bar{f}_i, \bar{f}_j), \bar{n} > .
\end{aligned}$$

We shall recall the relation of Ricci $*$-curvature ρ^* and $*$-scalar curvature τ^* which are fundamental invariants on almost Hermitian manifolds. Generically, these curvatures of an almost Hermitian manifold $M = (M, J, <,>)$ with even dimension 2n, are defined by

$$\rho^*(x, y) = (1/2)\sum_{i=1}^{2n} < R(e_i, Je_i)Jy, x >$$

and

$$\tau^* = \sum_{i=1}^{2n} \rho^*(e_i, e_i),$$

respectively. We note that Ricci $*$-curvature is neither symmetric nor skew-symmetric tensor.

Proposition 4.2 ([H2]) *The Ricci $*$-curvature and $*$-scalar curvature of oriented 6-dimensional submanifolds in $\mathfrak{C}$ are given by*

$$\begin{aligned} \rho^*(x,y) &= {}^t\alpha(A\bar{B} - BC -^t (A\bar{B} - BC))\beta \\ &- {}^t\alpha(A\bar{A} - B^t\bar{B} -^t \bar{B}B + C\bar{C})\bar{\beta} + \textit{its conjugation} \\ \tau^* &= -4(trA\bar{A} - 2tr^t\bar{B}B + trC\bar{C}), \end{aligned}$$

where $x = f\alpha + \bar{f}\bar{\alpha}, y = f\beta + \bar{f}\bar{\beta}$ and $\alpha, \beta \in M_{3\times1}(\boldsymbol{C})$.

4.1 Spin(7)-congruence theorem

In this section, we shall give the equivalent condition for Spin(7)-congruence. We shall prove the following.

Proposition 4.3 *Let M^6 be a connected 6-dimensional manifold and $\varphi_1, \varphi_2 : M^6 \to \mathfrak{C}$ be two isometric immersions with same induced metrics and almost complex structures. Let $II^{(2,0)}_{\varphi_1}, II^{(2,0)}_{\varphi_2}$ be the corresponding (2,0) part of the 2nd fundamental forms. Then there exists an element $g \in Spin(7)$ such that $g \circ \varphi_1 = \varphi_2$ if and only if $II^{(2,0)}_{\varphi_1} \cong II^{(2,0)}_{\varphi_2}$*

Proof. By (4.1) of Proposition 4.1, $\omega, \bar{\omega}$ are determined by the induced Hermitian Structure. We may check that $\rho, \mathfrak{h}$, and θ depend on $\omega, \bar{\omega}$ and $\mathrm{II}^{(2,0)}$. By (4.4), κ and θ depend only on the unitary frame $f, \bar{f}, df$ and $d\bar{f}$. Hence they depend only on the induced Hermitian Structure. By (4.10), B and C are also. If we fix $\mathrm{II}^{(2,0)}$, we get the desired complete information of the immersion. q.e.d

5 A spinor frame field on $S^2 \times S^4$

In this section, we give the $*$-scalar curvature $*\tau$ of the immersion

$$\varphi_{\alpha_0}(p, y_0, y) = \cos(\alpha_0)p + \sin(\alpha_0)(y_0 \cdot 1 + y\varepsilon),$$

where $\alpha_0 \in (0, \pi/2)$ is a constant, $p \in S^2 \subset \mathrm{Im}\mathbf{H}$ $(|p| = 1)$, and $y_0 \cdot 1 + y\varepsilon \in S^4 \subset \mathbf{R} \oplus \mathbf{H}\varepsilon$ $(y_0^2 + |y|^2 = 1)$. Then the oriented orthonormal basis $\{\xi, \eta\}$ of the normal bundle $T^\perp M$ is given by $\xi = y_0 \cdot 1 + y\varepsilon, \eta = p$. The almost complex structure is given by the right multiplication of the vector field $u = \eta \times \xi = y_0\, p + (y\, p)\varepsilon$. Let $\{e_1, e_2\}$ be the oriented orthonormal basis at p (i.e., $e_2 = e_1 p$), then $\{e_1, e_2, p\}$ is an associated plane in $\mathrm{Im}\mathfrak{C}$. We construct G_2-frame field from the vector field u as follows:

Let $e_1 \in T_pS^2, e_4 = u$, then $e_5 = e_1e_4 = y_0e_2 - (ye_2)\varepsilon$. We set

$$\begin{aligned}
\tilde{e_2} &= \frac{(ye_1)}{|y|}\varepsilon, \\
e_3 &= e_1\tilde{e_2} = -\frac{y}{|y|}\varepsilon, \\
e_6 &= \tilde{e_2}e_4 = -|y|e_2 - \frac{y_0(ye_2)}{|y|}\varepsilon, \\
e_7 &= e_3e_4 = -|y|p - \frac{y_0(yp)}{|y|}\varepsilon.
\end{aligned}$$

Then $\{e_1, e_2, \cdots, e_7\}$ is the G_2-adapted frame at $p + y_0 \cdot 1 + y\varepsilon \in S^2 \times S^4$. The complexified G_2-adapted frame is given by as follows;

$$\begin{aligned}
N^* &= \frac{1}{2}\Big(1 - \sqrt{-1}(y_0p + (yp)\varepsilon)\Big), \\
E_1^* &= \frac{1}{2}\Big(e_1 - \sqrt{-1}(y_0e_2 - (ye_2)\varepsilon)\Big), \\
E_2^* &= \frac{1}{2}\Big(\frac{ye_1}{|y|}\varepsilon + \sqrt{-1}(|y|e_2 + \frac{y_0}{|y|}(ye_2)\varepsilon)\Big), \\
E_3^* &= \frac{1}{2}\Big(\frac{y}{|y|}\varepsilon - \sqrt{-1}(|y|p - \frac{y_0}{|y|}(yp)\varepsilon)\Big).
\end{aligned}$$

By straightforward calculations, we get the local Spin(7) frame field along φ_{α_0} as follows

$$\begin{aligned}
n &= \frac{1}{2}\Big((y_0p + (yp)\varepsilon) - \sqrt{-1}p\Big), \\
f_1 &= E_1^* = \frac{1}{2}\Big(e_1 - \sqrt{-1}(y_0e_2 - (ye_2)\varepsilon)\Big), \\
f_2 &= E_2^* = \frac{1}{2}\Big(\frac{ye_1}{|y|}\varepsilon + \sqrt{-1}(|y|e_2 + \frac{y_0}{|y|}(ye_2)\varepsilon)\Big), \\
f_3 &= \frac{1}{2}\Big(-|y|1 + \frac{y_0y}{|y|}\varepsilon + \sqrt{-1}(\frac{1}{|y|}(yp)\varepsilon)\Big).
\end{aligned}$$

To calculate the Spin(7) invariants, we need the representation of co-frame as follows;

$$d\varphi_{\alpha_0} = \cos(\alpha_0)dp + \sin(\alpha_0)\Big(dy_0 + (dy)\varepsilon\Big) = \sum_{i=1}^{3} f_i\omega^i + \overline{f_i\omega^i}.$$

Then we have

$$\begin{aligned}
\omega^1 &= 2 < d\varphi_{\alpha_0}, \overline{f_1} >= \cos(\alpha_0)(< dp, e_1 > +\sqrt{-1} < dp, e_2 >) - \sqrt{-1}\sin(\alpha_0) < \overline{y}dy, e_2 >, \\
\omega^2 &= 2 < d\varphi_{\alpha_0}, \overline{f_2} >= -\sqrt{-1}\cos(\alpha_0)|y|(< dp, e_2 >) + \frac{\sin(\alpha_0)}{|y|}(< \overline{y}dy, e_1 > -\sqrt{-1} < \overline{y}dy, e_2 >), \\
\omega^3 &= 2 < d\varphi_{\alpha_0}, \overline{f_3} >= \sin(\alpha_0)(-|y|dy_0 + \frac{y_0}{|y|} < \overline{y}dy, 1 > -\frac{\sqrt{-1}}{|y|} < \overline{y}dy, p >).
\end{aligned}$$

By (4.10), we have

$$A = \overline{C} = \frac{1}{4}\left(\frac{1}{\sin(\alpha_0)} - \frac{\sqrt{-1}}{\cos(\alpha_0)}\right)\begin{pmatrix} y_0^2 - 1 & -y_0|y| & 0 \\ -y_0|y| & 1 - y_0^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B = \frac{1}{4}\begin{pmatrix} \frac{1-y_0^2}{\sin(\alpha_0)} + \frac{1+y_0^2}{\cos(\alpha_0)} & y_0|y|\left(\frac{1}{\sin(\alpha_0)} - \frac{\sqrt{-1}}{\cos(\alpha_0)}\right) & 0 \\ y_0|y|\left(\frac{1}{\sin(\alpha_0)} - \frac{\sqrt{-1}}{\cos(\alpha_0)}\right) & \frac{1+y_0^2}{\sin(\alpha_0)} + \frac{1-y_0^2}{\cos(\alpha_0)} & 0 \\ 0 & 0 & \frac{2}{\sin(\alpha_0)} \end{pmatrix}.$$

Hence the $*$-scalar curvature τ^* of φ_{α_0} is given by

$$\tau^* = \frac{2(\cos^2(\alpha_0) + y_0^2)}{\sin^2(\alpha_0)\cos^2(\alpha_0)}.$$

Therefore the induced almost complex structure of φ_{α_0} is not homogeneous.

6 The projective bundle over M^6

Let $T_m^{(1,0)}M^6$ be the $\sqrt{-1}$-eigen space of the induced almost complex structure at $m \in M^6$, which is a subspace of the complexified tangent space $\mathbf{C} \otimes T_m M^6$. We note that $T_m^{(1,0)}M^6$ is isomorphic to $\mathbf{C}^3$. Let $\pi : \mathfrak{F} \to M^6$ be the principal $U(3)$ bundle over M^6.

We set the projective space

$$P(T_m^{(1,0)}M^6) = \{\mathrm{span}_{\mathbf{C}}\{\mathfrak{f}_1\} \subset T_m^{(1,0)}M^6 | \mathfrak{f}_1 = \mathfrak{f}_1(m, U) = (f_1, f_2, f_3)u_1\}$$

where (f_1, f_2, f_3) is a local section of $\mathfrak{F}$, U is a 3×3 unitary matrix and $U = (u_1, u_2, u_3)$ for each u_i is a 3×1 matrix. We set

$$P(T^{(1,0)}M^6) = \cup_{m \in M^6} P(T_m^{(1,0)}M^6).$$

Then, $P(T^{(1,0)}M^6)$ is the projective bundle over M^6 with the fibre $P^2(\mathbf{C})$. We call this bundle twistor space. Then the projective bundle can be considered as a associated bundle of $\mathfrak{F}$. By (4.4), we can define the $U(3)$ connection $\tilde{\nabla}^{(1,0)}$ on $\mathfrak{F}$ as follows

$$\tilde{\nabla}^{(1,0)}(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3) = (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)\kappa,$$

where $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$ is a $U(3)$-valued function on $\mathfrak{F}$. (This connection is well-defined on $\mathfrak{F}$.) Then we have the following splitting

$$T_{\mathfrak{f}}\mathfrak{F} = H_{\mathfrak{f}}\mathfrak{F} \oplus V_{\mathfrak{f}},$$

where $H_{\mathfrak{f}}\mathfrak{F}$, (resp. $V_{\mathfrak{f}}$) is a horizontal subspace with respect to $U(3)$ connection, (resp. vertical subspace which is isomorphic to $\mathfrak{u}(3)$) at $\mathfrak{f} \in \mathfrak{F}$. From this, we get

$$T_{\mathfrak{f}_1}\left(P(T^{(1,0)}M^6)\right) = \mathcal{H}_{\mathfrak{f}_1} \oplus \mathcal{V}_{\mathfrak{f}_1}$$

where $\mathcal{V}_{\mathfrak{f}_1}$ is isomorphic to the holomorphic tangent space of the $P^2(\mathbf{C})$. Then we can define the 4-types almost complex structures on $P(T^{(1,0)}M^6)$ as follws;

1. $$\{\omega^1, \omega^2, \omega^3, \kappa_1^2, \kappa_1^3, \}$$

2. $$\{\omega^1, \omega^2, \omega^3, \overline{\kappa_1^2}, \overline{\kappa_1^3}, \}$$

3. $$\{\overline{\omega^1}, \omega^2, \omega^3, \kappa_1^2, \kappa_1^3, \}$$

4. $$\{\overline{\omega^1}, \omega^2, \omega^3, \overline{\kappa_1^2}, \overline{\kappa_1^3}, \}$$

The type 3 is very important. This construction comes from $\pi : P(T^{(1,0)}S^6)(\cong Q^5) \to S^6$. If type 1 and 2 is inteagrable, then the induced almost complex structure of M^6 is integarable, that is, M^6 is a complex manifold. If type 3 is integrable, then M^6 is isomorphic to a 6-dimensional sphere. The type 4 in never integrable.

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