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BI-LAGRANGIAN AND CAUSAL STRUCTURES ON SYMMETRIC SPACES

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INTRODUCTION

This is a brief survey of my recent work on the geometry of hyperbolic (semisimple) adjoint orbits of semisimple Lie groups. In §1, we give a geometric characterization of those orbits, namely, homogeneous parakähler manifolds and their equivariant compactification. In §2, we consider a more specific object, parahermitian symmetric spaces. The automorphism groups of double foliations are considered by using the compactification. In §3, we consider much more specific one, parahermitian symmetric spaces with causal structures. We determine the causal automorphism groups by using the compactification.

1. HOMOGENEOUS PARAKÄHLER MANIFOLDS

Let us consider the two series of composition algebras (with unit) over $\mathbb{R}$:

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}, \qquad \text{(division series)}$$

$$\mathbb{C}' \to \mathbb{H}' \to \mathbb{O}'. \qquad \text{(split series)}$$

To each member of the division series there corresponds a geometric structure —complex, quaternionic, or octonionic structure on a manifold. One may expect the similar situation for the split series. *The algebra of paracomplex numbers* $\mathbb{C}'$ is the algebra $\{a1 + bj : a, b \in \mathbb{R}, j^2 = 1\}$, which is isomorphic to the sum $\mathbb{R}\oplus\mathbb{R}$. P. Libermann [13] considered the geometric structure corresponding to $\mathbb{C}'$, so-called the *paracomplex structure.*

We say that $(M, F^{\pm})$ is a *paracomplex manifold*, if $F^{\pm}$ are two n-dimensional completely integrable transversal distributions on the $2n$-dimensional smooth manifold M. In this case the tangent bundle $T(M)$ is expressed as the Whitney sum

$$T(M) = F^{+} \oplus F^{-}. \tag{1.1}$$

Let $I = (I_p)_{p\in M}$ be the $(1,1)$-tensor field defined by

$$I_p = \begin{cases} 1, & \text{on } F_p^{+}, \quad p \in M, \\ -1, & \text{on } F_p^{-}. \end{cases}$$

A paracomplex structure $F^{\pm}$ usually occurs with a symplectic structure on M.

Definition 1.1 ([7]). $(M, F^{\pm}, \omega)$ is a *parakähler manifold*, if $(M, F^{\pm})$ is a paracomplex manifold and ω is a symplectic form on M such that $F^{\pm}$ are Lagrangian distributions. In this case $F^{\pm}$ is called a *bi-Lagrangian structure.*

On a parakähler manifold $(M, F^{\pm}, \omega)$ one can define a parakähler metric g by

$$g(X, Y) = \omega(IX, Y),$$

where X, Y are vector fields on M. g is pseudo-Riemannian of signature (n,n). There are two kinds of automorphism groups on a parakähler manifold: $\mathrm{Aut}(M, F^{\pm})$ is the subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ consisting of elements leaving the bi-Lagrangian structure $F^{\pm}$ invariant, while $\mathrm{Aut}(M, F^{\pm}, \omega)$ is the subgroup of $\mathrm{Aut}(M, F^{\pm})$ consisting of symplectomorphisms. Note that the latter one is the closed subgroup of the isometry group of the parakähler metric. But the former one is not finite-dimensional in general.

Let G be a connected Lie group and H be a closed subgroup. Suppose that G acts on $M := G/H$ almost effectively. Let $(F^{\pm}, \omega)$ be a parakähler structure on M. We say that $(M = G/H, F^{\pm}, \omega)$ is a *homogeneous parakähler* manifold, if $F^{\pm}$ and ω are G-invariant. In the following we will give a brief survey on how to construct homogeneous parakähler manifolds and their compactifications ([7], [4]).

Definition 1.2. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{u}^{\pm}$ two subalgebras, and let ρ be an alternating bilinear form on $\mathfrak{g}$. We say that $(\mathfrak{u}^{\pm}, \rho)$ is a *weak dipolarization* in $\mathfrak{g}$, if the following conditions are satisfied:

(WD1) $\mathfrak{g} = \mathfrak{u}^+ + \mathfrak{u}^-$,
(WD2) $\mathfrak{u}^+ \cap \mathfrak{u}^- = \{X \in \mathfrak{g} : \rho(X, \mathfrak{g}) = 0\}$,
(WD3) $\rho(\mathfrak{u}^{\pm}, \mathfrak{u}^{\pm}) = 0$,
(WD4) ρ is a cocycle in the sense of Lie algebra cohomology.

One has a one-to-one correspondence between homogenous parakähler structures on G/H (up to covering) and weak dipolarizations $(\mathfrak{u}^{\pm}, \rho)$ in $\mathfrak{g} = \mathrm{Lie}\, G$ such that $\mathfrak{h} = \mathfrak{u}^+ \cap \mathfrak{u}^-$, where $\mathfrak{h} = \mathrm{Lie}\, H$.

Definition 1.3 ([7]). Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{u}^{\pm}$ two subalgebras, and let f be a linear form on $\mathfrak{g}$. We say that $(\mathfrak{u}^{\pm}, f)$ is a *dipolarization* in $\mathfrak{g}$, if the following conditions are satisfied:

(D1) $(\mathfrak{u}^+, f)$ and $(\mathfrak{u}^-, f)$ are polarizations in $\mathfrak{g}$.
(D2) $\mathfrak{g} = \mathfrak{u}^+ + \mathfrak{u}^-$.

It follows that $(\mathfrak{u}^{\pm}, f)$ is a dipolarization, if and only if $(\mathfrak{u}^{\pm}, df)$ is a weak dipolarization. Hence we have only to consider dipolarizations, as long as we are concerned with homogeneous parakähler structures on a coset space of a semisimple Lie group.

From now on, we assume G to be semisimple. Then, homogeneous parakähler structures on G/H (up to covering) are in one-to-one correspondence with dipolarizations $(\mathfrak{u}^{\pm}, f)$ in $\mathfrak{g} = \mathrm{Lie}\, G$ such that $\mathfrak{h} = \mathfrak{u}^+ \cap \mathfrak{u}^-$. We want to consider the relation between dipolarizations in $\mathfrak{g}$ and $\mathbb{Z}$-gradings of $\mathfrak{g}$. Let $Z_f \in \mathfrak{g}$ be the dual element of f with respect to the Killing form B of $\mathfrak{g}$, that is,

$$B(Z_f, X) = f(X), \ X \in \mathfrak{g}.$$

Z_f is called the *characteristic element* of the dipolarization $(\mathfrak{u}^{\pm}, f)$. Sometimes we use the notation $(\mathfrak{u}^{\pm}, Z_f)$, instead of $(\mathfrak{u}^{\pm}, f)$. The element Z_f is semisimple in $\mathfrak{g}$, but not hyperbolic in general (Recall that a semisimple element $X \in \mathfrak{g}$ is *hyperbolic*, if $\mathrm{ad}\, X$ has only real eigenvalues). By using a result of [14], it can be shown that $\mathfrak{u}^{\pm}$ are parabolic subalgebras. Furthermore, the intersection $\mathfrak{u}^+ \cap \mathfrak{u}^-$ coincides with the centralizer $\mathfrak{c}(Z_f)$ of Z_f in $\mathfrak{g}$. For a semisimple $\mathbb{Z}$-graded Lie algebra (shortly GLA) $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ of the ν-th kind, the unique element $Z \in \mathfrak{g}_0$ satisfying the condition $\mathrm{ad}\, Z|_{\mathfrak{g}_k} = k1$, $-\nu \leq k \leq \nu$, is called the *characteristic element* of the grading. Note that Z is hyperbolic and the centralizer $\mathfrak{c}(Z)$ coincides with $\mathfrak{g}_0$. For a semisimple Lie group G, the orbit of $\mathrm{Ad}\, G$ through a hyperbolic element in $\mathfrak{g} = \mathrm{Lie}\, G$ is called a *hyperbolic orbit*.

Theorem 1.4. *([4]) Let G be a connected semisimple Lie group and H a closed subgroup. Then the following three are equivalent:*

(i) *The coset space $M = G/H$ is homogeneous parakähler manifold,*

(ii) *H is an open subgroup of the Levi subgroup of a parabolic subgroup of G,*

(iii) *M is a G-equivariant covering manifold of a hyperbolic* $\mathrm{Ad}\, G$*-orbit.*

For the proof, choose the dipolarization $(\mathfrak{u}^{\pm}, Z_f)$ in $\mathfrak{g}$ corresponding to the homogeneous parakähler structure on M. The crucial point of the proof is to construct a grading $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ with characteristic element Z, satisfying the following two conditions:

$$\mathfrak{u}^{\pm} = \sum_{k\geq 0} \mathfrak{g}_{\pm k}, \quad \mathfrak{u}^{+} \cap \mathfrak{u}^{-} = \mathfrak{c}(Z_f) = \mathfrak{c}(Z) = \mathfrak{g}_0 . \tag{1.2}$$

As a conclusion of Theorem 1.4, we have that among adjoint orbits of a semisimple Lie group, a hyperbolic orbit can be characterized geometrically as a homogeneous parakähler manifold.

Next we will mention the compactification of homogeneous parakähler manifold. Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a semisimple GLA with characteristic element Z. Let G be a connected Lie group with $\mathfrak{g} = \mathrm{Lie}\, G$, and let G_0 be the centralizer of Z in G. Then the coset space $M = G/G_0$ is a hyperbolic G-orbit, and hence a homogeneous parakähler manifold. Let $U^{\pm}$ be the parabolic subgroups corresponding to $\mathfrak{u}^{\pm}$ in (1.2), and let us consider the flag manifolds $M^{\pm} = G/U^{\pm}$. We denote by o, $o^{\pm}$ the origins of the coset spaces M, $M^{\pm}$, respectively. Consider the product manifold $\widetilde{M} := M^{-} \times M^{+}$. By the *horizontal* (resp. *vertical*) distribution on $\widetilde{M}$, we mean the $G \times G$-invariant distribution $\widetilde{F}^{+}$ (resp. $\widetilde{F}^{-}$) obtained by transporting the tangent space $T_{o^-}(M^{-})$ (resp. $T_{o^+}(M^{+})$) to each point of $\widetilde{M}$. The leaves $F^{\pm}(o)$ of the bi-Lagrangian distribution $F^{\pm}$ on M through o are given by the orbits $U^{\pm}o$. We define the map φ of M to $\widetilde{M}$ by putting

$$\varphi(go) = (go^{-}, go^{+}), \quad g \in G. \tag{1.3}$$

Theorem 1.5 ([7]). *The map φ is a G-equivariant open dense embedding of M into $\widetilde{M}$. In particular, $\widetilde{M}$ is the compactification of M. Moreover φ sends the Lagrangian distribution F^{+} or F^{-} on M to the horizontal or the vertical distribution on $\widetilde{M}$, respectively.*

2. Parahermitian symmetric spaces

Definition 2.1 ([11]). A homogeneous parakähler manifold $(M = G/H, F^{\pm}, \omega)$ is a *parahermitian symmetric space*, if the pair (G, H) is a symmetric pair.

Let $M = G/H$ be the homogeneous parakähler manifold corresponding to a semisimple GLA $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$. Then M is parahermitian symmetric, if and only if $\nu = 1$. So one can start with a simple GLA:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 . \tag{2.1}$$

We fix the associated pair (Z, τ), where Z is the characteristic element and τ is a grade-reversing Cartan involution of $\mathfrak{g}$. Let G_0 be the centralizer of Z in the automorphism group $\mathrm{Aut}\, \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. Then $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$. G_0 acts on $\mathfrak{g}$ in grade-preserving way. Let G be the open subgroup of $\mathrm{Aut}\, \mathfrak{g}$ generated by G_0 and the inner automorphism group $\mathrm{Ad}\, \mathfrak{g}$. Consider the involution $\sigma := \mathrm{Ad} \exp \pi i Z$ of $\mathfrak{g}$. The coset space $M = G/G_0$ is a symmetric space corresponding to the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. M is also realized as the orbit $(\mathrm{Ad}\,\mathfrak{g})Z$, which is hyperbolic. Hence, by Theorem 1.4, $M = G/G_0$ is a parahermitian symmetric space. Note that G is the maximum subgroup of $\mathrm{Aut}\,\mathfrak{g}$ acting on M.

Example 2.2. (i) The space $\mathcal{H} = \mathrm{SL}(2,\mathbb{R})/\mathbb{R}^*$ is a symmetric space, where $\mathbb{R}^*$ denotes the subgroup of diagonal matrices. $\mathcal{H}$ is a hyperbolic $\mathrm{SL}(2,\mathbb{R})$-orbit, realized as the one-sheeted hyperboloid, given by the equation $x^2 + y^2 - z^2 = 1$ in $\mathbb{R}^3$. The bi-Lagrangian distribution is given by the two families of generating lines.

(ii) Let S^n be an n-sphere, and let $M = S^n \times S^n \setminus$ (diagonal set). M is expressed as the parahermitian symmetric space $\mathrm{SO}(1, n+1)/\mathrm{SO}(n)\mathbb{R}^*$, and M^- is S^n. The corresponding root space Δ defined below is of C_1-type. Note that M can be identified with the set of oriented geodesics in the $(n+1)$-dimensional Lobachevsky space.

Now let us consider the parabolic subgroups $U^\pm = G_0 \exp \mathfrak{g}_{\pm 1}$ corresponding to the subalgebras $\mathfrak{u}^\pm = \mathfrak{g}_0 + \mathfrak{g}_{\pm 1}$. The flag manifolds $M^\pm = G/U^\pm$ are *symmetric R-spaces.* Let r be the rank of $M^\pm$. Let K be the maximal compact subgroup of G corresponding to the grade-reversing Cartan involution τ. $K_0 := K \cap G_0$ is a maximal compact subgroup of G_0.

Proposition 2.3 ([1], [15]). *There is a $3r$-dimensional graded subalgebra $\mathfrak{a} = \mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1$ of the GLA $\mathfrak{g}$ in (2.1), satisfying the conditions:*

(i) *$\mathfrak{a}$ is the direct sum of the pairwise commutative r $\mathfrak{sl}(2,\mathbb{R})$-triples $< E_{-i}, \check{\beta}_i, E_i >$, $1 \le i \le r$, where $E_{-i} = -\tau(E_i)$,*

(ii) *$\mathfrak{a}_{\pm 1} = \sum_{i=1}^r \mathbb{R}E_{\pm i}$, $\mathfrak{a}_0 = \sum_{i=1}^r \mathbb{R}\check{\beta}_i$,*

(iii) *$\mathfrak{g}_{\pm 1} = K_0 \mathfrak{a}_{\pm 1}$.*

The graded subalgebra $\mathfrak{a}$ is called the *spine* of the GLA $\mathfrak{g}$. It is known ([1], [15]) that there exists a root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}_0)$ of $\mathfrak{g}$ with respect to the $\mathbb{R}$-split abelian subalgebra $\mathfrak{a}_0$. $\beta_1, \cdots, \beta_r$ are strongly orthogonal roots in Δ. Δ is either of C_r-type or BC_r-type. We need the following elements of G:

$$a_k = \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{k}(E_i - E_{-i})\Big), \ 1 \le i \le r, \quad a_0 = 1. \tag{2.2}$$

Note that a_k is the square of the partial Cayley element c_k associated to the strongly orthogonal roots $\beta_1, \cdots, \beta_r$ It follows from (1.3) that the compactification map φ sends the G-action on M to the diagonal G-action on $\widetilde{M}$. In the following, we will give the G-orbit decomposition of $\widetilde{M}$.

Theorem 2.4 ([9], [5]). *Let M_k denote the orbit $G(o^-, a_{r-k}o^+)$, $0 \le k \le r$. Then we have*

(i) *The G-orbit decomposition of $\widetilde{M}$ is given by*

$$\widetilde{M} = M_r \amalg M_{r-1} \amalg \cdots \amalg M_0. \tag{2.3}$$

(ii)

$$\dim \widetilde{M} = \dim M_r > \dim M_{r-1} > \cdots > \dim M_0. \tag{2.4}$$

(iii) *If we denote the union $\amalg_{i=0}^k M_i$ by $M_{\le k}$, and denote the closure of M_k by $\overline{M}_k$, then $\overline{M}_k = M_{\le k}$, $0 \le k \le r$.*

(iv) *$M_{\le k}$ $(0 \le k \le r-1)$ is a real analytic set in $\widetilde{M}$, and its singular locus $\mathrm{Sing}(M_{\le k})$ coincides with $M_{\le k-1}$ for $1 \le k \le r-1$.*

(v) *We have $M_r = \varphi(M)$ and $M_0 = G/U^- \cap a_r U^+ a_r^{-1}$, which is a flag manifold.*

(vi) *Suppose that* Δ *is of* C_r*-type. Then we have* $a_r U^+ a_r^{-1} = U^-$*, in which case* $M_0 = G/U^-$.

The decomposition (2.3) of $\widetilde{M}$ having the property (iii) is called a *stratification* of $\widetilde{M}$.

Remark 2.5. Let us mention some consequences obtained from Theorem 2.4(vi). From the condition $a_r U^+ a_r^{-1} = U^-$, it follows that if we choose the point $a_r o^+ \in M^+$ as the new origin, then we have $M^+ = G/a_r U^+ a_r^{-1} = G/U^- = M^-$ and $a_r o^+ = o^-$, and hence $\widetilde{M}$ is expressed as $M^- \times M^-$. Since the point $(o^-, a_r o^+) \in M^- \times M^+ = \widetilde{M}$ is expressed as $(o^-, o^-) \in M^- \times M^- = \widetilde{M}$, we have that $M_0 = G(o^-, a_r o^+) = G(o^-, o^-)$, which is the diagonal set of $M^- \times M^-$.

The following proposition follows from Theorem 2.4(iv).

Proposition 2.6 ([9]). *Let* f *be a smooth diffeomorphism of* $\widetilde{M}$. *If* $f(M_r) = M_r$*, then* $f(M_i) = M_i$ *for* $0 \le i \le r-1$.

For a GLA $\mathfrak{g}$ in (2.1), the union L of singular G_0-orbits in $\mathfrak{g}_1$ is $\mathbb{R}^*$-invariant and is called a *generalized light cone*. For the case where Δ is of C_r-type, $\mathfrak{g}_1$ is a simple Jordan algebra, and L is defined as the set of zeroes of the generic norm of the Jordan algebra. For example, let $\mathfrak{g} = \mathfrak{so}(2,n)$. Then we have $\mathfrak{g}_0 = \mathfrak{so}(1, n-1) + \mathbb{R}$, the Lie algebra of the conformal group of the quadratic form with signature $(1, n-1)$, and $\mathfrak{g}_1 = \mathrm{M}_{1,n-1}(\mathbb{R})$. In this case, L is the usual Lorentz light cone $x_1^2 - x_2^2 - \cdots - x_n^2 = 0$.

By using the generalized light cone $L \subset \mathfrak{g}_1$, one can introduce a *generalized conformal structure* $\mathcal{K}$ (cf. [2]) on the symmetric R-space $M^- = G/U^-$. We identify $\mathfrak{g}_1$ with the tangent space $T_{o^-}(M^-)$ at the origin o^-. The cone L sits in $T_{o^-}(M^-)$. Let p be an arbitrary point of M^-, and write it as $p = go^-$, $g \in G$. Then $L_p := g_{*o^-} L$ is a (well-defined) light cone in $T_p(M^-)$. $\mathcal{K}$ is defined to be the field $\{L_p\}_{p \in M^-}$ of generalized light cones on M^-. We denote the group of smooth diffeomorphisms of $M^-$by $\mathrm{Diff}(M^-)$. We say that $f \in \mathrm{Diff}(M^-)$ leaves $\mathcal{K}$ invariant (and denote it by $f_*\mathcal{K} = \mathcal{K}$), if f satisfies the condition $f_* L_p = L_{f(p)}$ for each $p \in M^-$. Clearly $\mathcal{K}$ is G-invariant. We define the automorphism group of the generalized conformal structure $\mathcal{K}$ by

$$\mathrm{Aut}(M^-, \mathcal{K}) = \{f \in \mathrm{Diff}(M^-) : f_*\mathcal{K} = \mathcal{K}.\}$$

Theorem 2.7 ([2]). *Let* $M^- = G/U^-$ *be the symmetric R-space associated to a simple GLA* $\mathfrak{g}$ *in (2.1). Then we have*

$$\mathrm{Aut}(M^-, \mathcal{K}) = \begin{cases} \mathrm{Diff}(M^-), & \textit{if } \Delta \textit{ is of } C_1\textit{-type,} \\ G, & \textit{otherwise.} \end{cases} \tag{2.5}$$

Note that the symmetric R-space M^- of C_1-type is a sphere, as was mentioned in Example 2.2(ii).

The final goal of this section is

Theorem 2.8 ([9]). *Let* $(M = G/G_0, F^\pm)$ *be a* $2n$*-dimensional parahermitian symmetric space (realized as the hyperbolic orbit) associated to a simple GLA* $\mathfrak{g}$ *in (2.1). Then the automorphism group* $\mathrm{Aut}(M, F^\pm)$ *of the bi-Lagrangian structure* $F^\pm$ *is given by*

$$\mathrm{Aut}(M, F^\pm) = \begin{cases} \mathrm{Diff}(S^n), & \textit{if } \Delta \textit{ is of } C_1\textit{-type,} \\ G, & \textit{otherwise.} \end{cases} \tag{2.6}$$

Proof. (Sketch) We identify M with the open dense G-orbit M_r in $\widetilde{M}$. We denote by $\mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M)$ (resp. $\mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M_0)$) the group of diffeomorphisms of $\widetilde{M}$ leaving the product structure $\widetilde{F}^{\pm}$ invariant, and leaving M (resp. M_0) stable. Consider the double fibration

$$M^- \xleftarrow{\pi^-} M \xrightarrow{\pi^+} M^+, \tag{2.7}$$

where $\pi^{\pm}$ are the natural projections between the coset spaces. The fibers of $\pi^{\pm}$ are the leaves of $F^{\pm}$.

Now let $f \in \mathrm{Aut}(M, F^{\pm})$. Then f is fiber-preserving, and hence it induces the diffeomorphisms $f^{\pm} \in \mathrm{Diff}(M^{\pm})$ such that $\pi^{\pm} \cdot f = f^{\pm} \cdot \pi^{\pm}$. It follows that the correspondence $f \mapsto \widetilde{f} := f^- \times f^+$ gives an isomorphism

$$\mathrm{Aut}(M, F^{\pm}) \simeq \mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M). \tag{2.8}$$

By using Proposition 2.6, we have the inclusion:

$$\mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M) \hookrightarrow \mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M_0). \tag{2.9}$$

Consider first the case where Δ is of BC_r-type. The product foliation $\widetilde{F}^{\pm}$ induces the non-trivial foliation $F_0^{\pm}$ on M_0. We have the isomorphism

$$\mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M_0) \simeq \mathrm{Aut}(M_0, F_0^{\pm}). \tag{2.10}$$

The latter group was described by Tanaka [17] in connection with the 5-grading of $\mathfrak{g}$. But our group G is related to the 3-grading of $\mathfrak{g}$. One can show that both groups are identical. It follows that

$$\mathrm{Aut}(M_0, F_0^{\pm}) = G. \tag{2.11}$$

By using the G-invariance of $F^{\pm}$ and (2.8)–(2.11), we conclude that $\mathrm{Aut}(M, F^{\pm}) = G$.

Next consider the case where Δ is of C_r-type. We use the method of changing of the origin given in Remark 2.5. We decompose $\widetilde{f}$ as $\widetilde{f} = f_1 \times f_2$ corresponding to the expression $\widetilde{M} = M^- \times M^-$. By Proposition 2.6, $\widetilde{f}$ leaves M_0 invariant, which is the diagonal set of $M^- \times M^-$. So we have $f_1 = f_2$, and hence $\widetilde{f} = f_1 \times f_1$. By using the relation $\widetilde{f}(M_{\leq r-1}) = M_{\leq r-1}$, we can prove that $f_1 \in \mathrm{Aut}(M^-, \mathcal{K})$. The correspondence $\widetilde{f} \mapsto f_1$ yields the inclusion

$$\mathrm{Aut}(\widetilde{M}, \widetilde{F}^{\pm} : M_0) \hookrightarrow \mathrm{Aut}(M^-, \mathcal{K}). \tag{2.12}$$

By Theorem 2.7, we have the conclusion. Note that the inclusion in (2.9) is the equality, provided that $r = 1$.

□

Theorem 2.8 was also obtained by Tanaka [17] under the assumption that $\mathfrak{g}$ is classical simple. Our setting is Lie-theoretic and more general.

3. Symmetric space of Cayley type

We will begin with causal structures.

Definition 3.1. A subset C in $\mathbb{R}^n$ is a *causal cone* (with vertex at 0), if C is a closed convex cone, whose interior is not empty, satisfying the condition $C \cap (-C) = (0)$.

Let C be a causal cone in $\mathbb{R}^n$. The group

$$\operatorname{Aut} C = \{g \in \mathrm{GL}(\mathbb{R}^n) : gC = C\} \tag{3.1}$$

is called the *automorphism group* of C. The following definition is due to Hilgert-Ólafsson [3].

Definition 3.2. Let C be a causal cone in $\mathbb{R}^n$, and let M be an n-dimensional smooth manifold. Let $\mathcal{C} = \{C_p\}_{p\in M}$ be a family of causal cones C_p, where C_p is in the tangent space $T_p(M)$ at $p \in M$. We say that $\mathcal{C}$ is a *causal structure with model cone* C on M, if the following conditions are satisfied: there exists an open covering $\{U_i\}_{i\in I}$ of M and, for each $i \in I$, there exists a local trivialization φ_i on U_i of the tangent bundle $T(M)$, that is, φ_i is a diffeomorphism of $U_i \times \mathbb{R}^n$ onto $T(M)|_{U_i}$ such that $\varphi_i(p, C) = C_p$ for $p \in U_i$.

Thus a causal structure $\mathcal{C}$ on M is a conical subbundle of $T(M)$. Let $(M, \mathcal{C})$ be a causal manifold, and let $\mathcal{C} = \{C_p\}_{p\in M}$. A diffeomorphism f of M is a *causal automorphism*, if f leaves $\mathcal{C}$ invariant, that is, $f_*C_p = C_{f(p)}$ holds for each $p \in M$. The group of causal automorphisms is denoted by $\operatorname{Aut}(M, \mathcal{C})$.

One can interpret a causal structure as a G-structure in a usual sense.

Proposition 3.3. *Let C be a causal cone in $\mathbb{R}^n$, and let M be an n-dimensional smooth manifold. Then M has a causal structure $\mathcal{C}$ with a model cone C, if and only if there exists an* $\operatorname{Aut} C$*-structure on M.*

The following lemma is easy, but useful.

Lemma 3.4 ([6]). *Let G be a Lie group and H be a closed subgroup of G. Let o denote the origin of the coset space $M = G/H$. Let C be a causal cone in the tangent space $T_o(M)$. Suppose that the group* $\operatorname{Aut} C$ *contains the linear isotropy representation of H as a subgroup. Then there exists a G-invariant causal structure with C as a model cone.*

Let D be an irreducible bounded symmetric domain of tube type, and let $G(D)$ be the full holomorphic automorphism group of D. The Lie algebra $\mathfrak{g} := \operatorname{Lie} G(D)$ is simple of Hermitian type. $\mathfrak{g}$ can be expressed as a GLA in (2.1). By a theorem of E. Cartan, $\operatorname{Aut}\mathfrak{g}$ is isomorphic to the isometry group $I(D)$ with respect to the Bergman metric of D. We identify the both groups. It can be shown that the group G constructed in §2 coincides with the full group $\operatorname{Aut}\mathfrak{g}$. $G(D)$ is a normal subgroup of G with index 2. Now let $G_0(D) = G_0 \cap G(D)$ and $U^\pm(D) = U^\pm \cap G(D)$.

The parahermitian symmetric space $M = G/G_0$ associated to the GLA $\mathfrak{g}$ of Hermitian type in (2.1) is called a *symmetric space of Cayley type.* Note that $\dim M = \dim_{\mathbb{R}} D$. For a Cayley type symmetric space, the root system Δ is always of C_r-type. M and $M^\pm$ are expressed as

$$M = G(D)/G_0(D), \;\; M^\pm = G(D)/U^\pm(D). \tag{3.2}$$

M^+ or M^- is the Shilov boundary of D, depending on the choice of the complex structure (KW[12]).

Let us introduce the causal structures on $M^\pm$. Let $E_\pm = \sum_{i=1}^r E_{\pm i} \in \mathfrak{g}_{\pm 1}$. Then the orbits $V^\pm = G_0(D)E_\pm$ are so-called selfdual open convex cones, and the closures $C^\pm := \overline{V^\pm}$ are causal cones in $\mathfrak{g}_{\pm 1}$. We have that $\operatorname{Aut} C^\pm$ coincide with $G_0(D)$, which is the linear isotropy groups of $U^\mp(D)$ at $o^\mp \in M^\mp$. Hence, by Lemma 3.4, there exist the $G(D)$-invariant causal structures $\mathcal{C}^\pm$ on $M^\mp$ with the model cones $C^\pm$. We need another causal structure $-\mathcal{C}^-$ on M^+ with the model cone $-C^-$. Thus we have three causal manifolds:

$(M^-, \mathcal{C}^+), (M^+, \mathcal{C}^-), (M^+, -\mathcal{C}^-)$. The following theorem can be proved by using Proposition 3.3 and TA[16].

Theorem 3.5 (KA[6]). *The action of the holomorphic automorphism group $G(D)$ extends to the Shilov boundary $(M^-, \mathcal{C}^+)$, and $G(D)$ acts on it effectively as causal automorphisms. Furthermore we have*

$$\mathrm{Aut}(M^-, \mathcal{C}^+) = \begin{cases} \mathrm{Diff}^+(S^1), & \text{if } \dim_{\mathbb{C}} D = 1, \\ G(D), & \text{if } \dim_{\mathbb{C}} D \geq 2, \end{cases} \tag{3.3}$$

where $\mathrm{Diff}^+(S^1)$ *denotes the group of orientation-preserving diffeomorphisms of the unit circle S^1. The same equality holds for* $\mathrm{Aut}(M^+, \mathcal{C}^-)$ *and* $\mathrm{Aut}(M^+, -\mathcal{C}^-)$.

The following is a list of the model cones and the corresponding $G(D)$-invariant (resp. $G(D) \times G(D)$-invariant) causal structures and low-dimensional cone fields on M (resp. $\widetilde{M} = M^- \times M^+$). $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ is identified with the tangent spaces $T_o(M)$ and $T_{(o^-,o^+)}(\widetilde{M}) = T_{o^-}(M^-) \oplus T_{o^+}(M^+)$.

model cone	causal str. or cone field on M	causal str. or cone field on $\widetilde{M}$
$C = C^+ \oplus C^-$	$\mathcal{C}$	$\widetilde{\mathcal{C}}$
$C' = C^+ \oplus (-C^-)$	$\mathcal{C}'$	$\widetilde{\mathcal{C}}'$
$C^+ \oplus (0)$	$\mathcal{C}_M^+$	$\widetilde{\mathcal{C}}^+$
$(0) \oplus C^-$	$\mathcal{C}_M^-$	$\widetilde{\mathcal{C}}^-$
$(0) \oplus (-C^-)$	$-\mathcal{C}_M^-$	$-\widetilde{\mathcal{C}}^-$

The causal structure $\mathcal{C}$ (resp. $\mathcal{C}'$) on M is *noncompactly causal* (resp.*compactly causal*) in the sense of HÓ [3], that is, there are no nontrivial closed $\mathcal{C}$-causal curves on M, while there are nontrivial closed $\mathcal{C}'$-causal curves on M. $\mathcal{C}_M^\pm$ (resp. $\widetilde{\mathcal{C}}^\pm$) are conical subbundles of $F^\pm$ (resp. $\widetilde{F}^\pm$). Corresponding to the Whitney sums $T(M) = F^+ \oplus F^-$ and $T(\widetilde{M}) = \widetilde{F}^+ \oplus \widetilde{F}^-$, we have the Whitney sums of the conical subbundles:

$$\begin{aligned} \mathcal{C} = \mathcal{C}_M^+ \oplus \mathcal{C}_M^-, \quad & \mathcal{C}' = \mathcal{C}_M^+ \oplus (-\mathcal{C}_M^-), \\ \widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}^+ \oplus \widetilde{\mathcal{C}}^-, \quad & \widetilde{\mathcal{C}}' = \widetilde{\mathcal{C}}^+ \oplus (-\widetilde{\mathcal{C}}^-). \end{aligned} \tag{3.4}$$

Definition 3.6. Let $(X, \mathcal{C}), (\widetilde{X}, \widetilde{\mathcal{C}})$ be causal manifolds, $\widetilde{X}$ being compact. Suppose that a Lie group G acts on X and $\widetilde{X}$ as causal automorphisms. We say that $(\widetilde{X}, \widetilde{\mathcal{C}})$ is a *causal compactification* of $(X, \mathcal{C})$, if

(i) $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ have the same model cone,
(ii) there exists a G-equivariant open dense causal embedding of $(X, \mathcal{C})$ into $(\widetilde{X}, \widetilde{\mathcal{C}})$.

Lemma 3.7. *Let $(M, \mathcal{C})$ be a causal symmetric space of Cayley type given above, and let $\varphi : M \to \widetilde{M}$ be the compactification map as in (1.3), which is $G(D)$-equivariant. Then $(\widetilde{M}, \widetilde{\mathcal{C}})$ is a causal compactification of $(M, \mathcal{C})$.*

Let $\varpi^\pm$ be the natural projections of $\widetilde{M}$ onto $M^\pm$, respectively. Then one has

$$\pi^\pm = \varpi^\pm \cdot \varphi. \tag{3.5}$$

Lemma 3.8. *$\widetilde{\mathcal{C}}^{\pm}$, $-\widetilde{\mathcal{C}}^{-}$, $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}'$ are φ-related to $\mathcal{C}_M^{\pm}$, $-\mathcal{C}_M^{-}$, $\mathcal{C}$, $\mathcal{C}'$, respectively. In particular, if we identify M with $\varphi(M)$, then the restrictions of $\widetilde{\mathcal{C}}^{\pm}$, $-\widetilde{\mathcal{C}}^{-}$, $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}'$ to M are equal to $\mathcal{C}_M^{\pm}$, $-\mathcal{C}_M^{-}$, $\mathcal{C}$, $\mathcal{C}'$, respectively. Also we have*

$$\pi_*^{\mp}\mathcal{C}_M^{\pm} = \varpi_*^{\mp}\widetilde{\mathcal{C}}^{\pm} = \mathcal{C}^{\pm} \tag{3.6}$$

Lemma 3.9. *We have the following expressions:*

$$(\widetilde{M},\ \widetilde{\mathcal{C}}) = (M^-, \mathcal{C}^+) \times (M^+, \mathcal{C}^-),$$

$$(\widetilde{M},\ \widetilde{\mathcal{C}}') = (M^-, \mathcal{C}^+) \times (M^+, -\mathcal{C}^-),$$

We define the diffeomorphisms ϑ and θ of $\widetilde{M}$ by

$$\vartheta(g_1o^-, g_2o^+) = (a_r^{-1}g_2a_ro^-,\ a_r^{-1}g_1a_ro^+), \qquad g_1, g_2 \in G(D), \tag{3.7}$$

$$\theta(g_1o^-, g_2o^+) = (a_r^{-1}\widetilde{\sigma}(g_2)a_ro^-,\ a_r^{-1}\widetilde{\sigma}(g_1)a_ro^+), \qquad g_1, g_2 \in G(D), \tag{3.8}$$

where $\widetilde{\sigma}$ is an involutive automorphism of $G(D)$ defined by $\widetilde{\sigma}(a) = \sigma a \sigma$, $a \in G(D)$.

Lemma 3.10. *ϑ lies in the causal automorphism group $\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}')$. ϑ is involutive and interchanges $\widetilde{\mathcal{C}}^+$ with $-\widetilde{\mathcal{C}}^-$. Under the identification of M with $\varphi(M)$, ϑ leaves M stable. Let $\vartheta_M = \vartheta|_M$. Then $\vartheta|_M \in \mathrm{Aut}(M, \mathcal{C}')$. $\vartheta|_M$ interchanges $\mathcal{C}_M^+$ with $-\mathcal{C}_M^-$.*

Similarly for the involutive diffeomorphism θ we have

Lemma 3.11 ([10]). *$\theta \in \mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}})$ holds. θ interchanges $\widetilde{\mathcal{C}}^+$ with $\widetilde{\mathcal{C}}^-$. The restriction $\theta_M := \theta|_M$ lies in $\mathrm{Aut}(M, \mathcal{C})$, and interchanges $\mathcal{C}_M^+$ with $\mathcal{C}_M^-$.*

By using Lemmas 3.8—3.11, we have

Lemma 3.12.

$$\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}') = \big(\mathrm{Aut}(M^-, \mathcal{C}^+) \times \mathrm{Aut}(M^+, -\mathcal{C}^-)\big) \ltimes < \vartheta >,$$

$$\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}) = \big(\mathrm{Aut}(M^-, \mathcal{C}^+) \times \mathrm{Aut}(M^+, \mathcal{C}^-)\big) \ltimes < \theta >,$$

where $< \vartheta >$ and $< \theta >$ are the cyclic groups of order 2 generated by ϑ and θ, respectively. Note that $\mathrm{Aut}(M^+,\ -\mathcal{C}^-) = \mathrm{Aut}(M^+,\ \mathcal{C}^-)$.

We denote by $\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}^{\pm})$ (resp. $\mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^{\pm})$) the group of diffeomorphisms of $\widetilde{M}$ leaving the two cone fields $\widetilde{\mathcal{C}}^+$ and $\widetilde{\mathcal{C}}^-$ (resp. $\widetilde{\mathcal{C}}^+$ and $-\widetilde{\mathcal{C}}^-$) invariant. Clearly we have $\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}^{\pm}) = \mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^{\pm})$. Let $\tilde{f} \in \mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}^{\pm})$. Then there exist $\tilde{f}^{\pm} \in \mathrm{Diff}(M^{\pm})$ such that $\varpi^{\pm} \cdot \tilde{f} = \tilde{f}^{\pm} \cdot \varpi^{\pm}$. It follows from (3.6) that $\tilde{f}^{\pm} \in \mathrm{Aut}(M^{\pm}, \mathcal{C}^{\mp})$. By using the expression $\tilde{f} = \tilde{f}^- \times \tilde{f}^+$, we have the following lemma.

Lemma 3.13.

$$\begin{aligned} \mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}^{\pm}) &= \mathrm{Aut}(M^-, \mathcal{C}^+) \times \mathrm{Aut}(M^+, \mathcal{C}^-), \\ \mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^{\pm}) &= \mathrm{Aut}(M^-, \mathcal{C}^+) \times \mathrm{Aut}(M^+, -\mathcal{C}^-). \end{aligned} \tag{3.9}$$

The final goal of this section is the following theorem. A part of the results has been published in KA[10].

Theorem 3.14. *Let D be the bounded symmetric domain associated with a simple GLA $\mathfrak{g}$ of Hermitian type in (2.1), and let $G(D)$ be the full holomorphic automorphism group of D. Let $M = G(D)/G_0(D)$ be a symmetric space of Cayley type associated with the GLA $\mathfrak{g}$. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the noncompactly (resp. compactly) causal structure of M. Let $\mathcal{C} = \mathcal{C}_M^+ \oplus \mathcal{C}_M^-$ and $\mathcal{C}' = \mathcal{C}_M^+ \oplus (-\mathcal{C}_M^-)$ be the splittings of $\mathcal{C}$ and $\mathcal{C}'$ into the low-dimensional cone fields, respectively (cf. (3.4)). Then we have*

$$\mathrm{Aut}(M, \mathcal{C}) = \mathrm{Aut}(M, \mathcal{C}_M^\pm) \ltimes < \theta_M >, \tag{3.10}$$

$$\mathrm{Aut}(M, \mathcal{C}') = \mathrm{Aut}(M, \pm\mathcal{C}_M^\pm) \ltimes < \vartheta_M >, \tag{3.11}$$

$$\begin{aligned} &\mathrm{Aut}(M, \mathcal{C}_M^\pm) = \mathrm{Aut}(M, \pm\mathcal{C}_M^\pm) \\ &\simeq \mathrm{Aut}(M^-, \mathcal{C}^+) = \begin{cases} G(D), & \textit{if } \dim_{\mathbb{C}} D \geq 2, \\ \mathrm{Diff}^+(S^1), & \textit{if } \dim_{\mathbb{C}} D = 1. \end{cases} \end{aligned} \tag{3.12}$$

Proof. (Sketch) (3.11) and (3.10) are the restriction of the two equalities in Lemma 3.12 to M, in view of (3.9). Let A be a subset of $\widetilde{M}$. We denote by $\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}' : A)$ the subgroup of $\mathrm{Aut}(\widetilde{M}, \widetilde{\mathcal{C}}')$ consisting of elements g satisfying the condition $g(A) = A$. To prove (3.12), we will use the causal compactification $(\widetilde{M}, \widetilde{\mathcal{C}}')$ and take a similar way as in the proof of Theorem 2.8. First note that

$$\mathrm{Aut}(M, \pm\mathcal{C}_M^\pm) = \mathrm{Aut}(M, \mathcal{C}') \cap \mathrm{Aut}(M, F^\pm) \tag{3.13}$$

Analogous to (2.8) and (2.9), it can be proved that

$$\mathrm{Aut}(M, \pm\mathcal{C}_M^\pm) \simeq \mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^\pm : M) \hookrightarrow \mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^\pm : M_0). \tag{3.14}$$

We apply the method of changing of the origin of $\widetilde{M}$ from (o^-, o^+) to $(o^-, a_r o^+)$, given in Remark 2.5. Then the right-hand side of the second equality in Lemma 3.9 is converted into $(M^-, \mathcal{C}^+) \times (M^-, \mathcal{C}^+)$, and simultaneously M_0 changes to the diagonal set of $M^- \times M^-$, which is isomorphic to the causal submanifold $(M^-, \mathcal{C}^+)$. In fact, the model cone of $-\mathcal{C}^-$ is $-C^-$ at o^+. Hence the cone at $a_r o^+$ belonging to $-\mathcal{C}^-$ is seen to be $a_{r*}(-C^-) = -(\mathrm{Ad}\, a_r)C^- = C^+$, which is the model cone of $\mathcal{C}^+$ at o^+. Thus $(M^+, -\mathcal{C}^-)$ is converted into $(M^-, \mathcal{C}^+)$, and hence it follows from (3.9) that

$$\mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^\pm) = \mathrm{Aut}(M^-, \mathcal{C}^+) \times \mathrm{Aut}(M^-, \mathcal{C}^+).$$

Since M_0 is the diagonal set of $M^- \times M^-$, we have

$$\mathrm{Aut}(\widetilde{M}, \pm\widetilde{\mathcal{C}}^\pm : M_0) = \mathrm{diag}(\mathrm{Aut}(M^-, \mathcal{C}^+) \times \mathrm{Aut}(M^-, \mathcal{C}^+)) \simeq \mathrm{Aut}(M^-, \mathcal{C}^+) \tag{3.15}$$

The cone fields $\pm\mathcal{C}_M^\pm$ are $G(D)$-invariant, and for $\dim_{\mathbb{C}} D = 1$ the inclusion in (3.14) is an equality. Consequently, (3.12) follows from (3.14), (3.15) and Theorem 3.5. □

Remark 3.15. The above procedure of the proof indicates that the $G(D)$-action on M can be reconstructed geometrically from that on D. M_0 is the Shilov boundary of M in $\widetilde{M}$ (only in the sense that it is the minimal boundary orbit). The $G(D)$-action on D extends to the compact dual of D holomorphically. The extended one can be restricted to the Shilov boundary M^- as $\mathcal{C}^+$-causal automorphism group. The $\mathcal{C}^+$-causal action of $G(D)$ on M^-(= the diagonal set M_0 in $\widetilde{M} = M^- \times M^-$) extends to the $\widetilde{\mathcal{C}}'$-causal action on $\widetilde{M}$, which can be restricted to the $\mathcal{C}'$-causal action on M. $G(D)$ acts on M as $\mathrm{Aut}(M, \mathcal{C}_M^\pm) = \mathrm{Aut}(M, \mathcal{C}) \cap \mathrm{Aut}(M, \mathcal{C}')$.

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