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INFINITE DIMENSIONAL LIE ALGEBRAS, VERTEX ALGEBRAS AND W-ALGEBRAS

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1. Introduction

1.1. One of the distinguished features of infinite dimensional Lie algebras is the modular invariance of the characters of certain representations. There are two celebrated examples for this phenomena: One is the integrable highest weight representations of an affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a simple Lie algebra $\mathfrak{g}$ at a fixed level [KP], and the other is the minimal series representations [FFu] of the Virasoro algebra *Vir* with a fixed central charge.

However there is a relevant difference in these two examples: The Virasoro algebra is a single Lie algebra, while affine Lie algebras constitute a family of Lie algebras. Therefore it is natural to consider a generalization of the Virasoro algebra.

The *W-algebras* can be regarded as such a generalization of the Virasoro algebra. Some people say that this is the reason why they are called the "W-algebras" (because the letter "W" comes right after "V" alphabetically). The first example of a W-algebra was discovered by Zamalodchikov [Za] in his study of classification of conformal field theory (see [BS] and reference therein.).

1.2. In general, there is the W-algebra $\mathcal{W}(\mathfrak{g})$ associated with any simple Lie algebra $\mathfrak{g}$ ([FF2]). The simplest W-algebra is the W-algebra $\mathcal{W}(\mathfrak{sl}_2)$ associated with $\mathfrak{sl}_2$. This is nothing but the Virasoro algebra (or more precisely, the corresponding vertex algebra). The Virasoro algebra *Vir* is the Lie algebra with the following generators and the relations:

$$\begin{aligned} &\text{generators:} && L_n \ (n \in \mathbb{Z}),\ \mathbf{c} \\ &\text{relations:} && [L_n, \mathbf{c}] = 0 \\ & && [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m+n,0}\mathbf{c}. \end{aligned}$$

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The next simplest W-algebra is the one associated with $\mathfrak{sl}_3$; $\mathcal{W}(\mathfrak{sl}_3)$ is defined by the following generators and relations:

$$\begin{aligned}
\text{generators:}\quad & \mathbf{c}, L_n\ (n \in \mathbb{Z}),\ W_n\ (n \in \mathbb{Z}),\\
\text{relations:}\quad & [\mathbf{c}, \mathcal{W}(\mathfrak{sl}_3)] = 0,\\
& [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c},\\
& [L_m, W_n] = (2m-n)W_{m+n},\\
& [W_m, W_n]\\
& = (m-n)\left\{\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right\}L_{m+n}\\
& + \frac{16}{22+5\mathbf{c}}(m-n)\Lambda_{m+n} + \frac{\mathbf{c}}{360}m(m^2-1)(m^2-4)\delta_{m+n,0},
\end{aligned}$$

where

$$\Lambda_n = \sum_{k<0} L_k L_{n-k} + \sum_{k\geq 0} L_{n-k}L_k - \frac{3}{10}(n+2)(n+3)L_n. \tag{1}$$

In the above formula, the pole at $\mathbf{c} = -22/5$ can be removed if we multiply W_n by $22 + 5\mathbf{c}$, and therefore it is inessential. More serious is the existence of the infinite sum of the quadratic term of the form $L_{n-k}L_k$. This means that the above *does not define* a Lie algebra in the usual sense. In general, W-algebras are no more Lie algebras and one should understand them as *vertex algebras* (see [K2, FB, BD] for the definition of vertex algebras).

1.3. As we have seen in the above, $\mathcal{W}(\mathfrak{g})$ has a complicated algebraic structure except for the case that $\mathfrak{g} = \mathfrak{sl}_2$. In fact, even the defining relations of the generators are not known for a general $\mathcal{W}(\mathfrak{g})$! Thus, instead of defining it by generators and relations, W-algebras are usually defined by a cohomological method. This method is called the *quantized Drinfeld-Sokolov reduction*, or simply the *quantum reduction*, and was discovered by Feigin and Frenkel [FF2]. This is a powerful method, in the sense that it not only gives a uniform definition of $\mathcal{W}(\mathfrak{g})$, but also defines a functor form a suitable category (the category $\mathcal{O}$) of $\widehat{\mathfrak{g}}$-modules to the category of $\mathcal{W}(\mathfrak{g})$-modules. Frenkel, Kac and Wakimoto [FKW] conjectured that one can obtain a family of modular invariant representations of $\mathcal{W}(\mathfrak{g})$ from the modular invariant representations (admissible representations) of $\widehat{\mathfrak{g}}$ via this functor. If this is true then one can surely say that $\mathcal{W}(\mathfrak{g})$ is a generalization of *Vir*, for it inherits our favorite property of the Virasoro algebra.

1.4. The propose of this note to describe the representation theory of $\mathcal{W}(\mathfrak{g})$ via quantum reduction. In particular, we explain how the conjecture of Frenkel, Kac and Wakimoto follows from our general results.

2. Finite dimensional case

2.1. Recall that $\widehat{\mathfrak{g}}$ is an affinization (or a *chiralization*) of the finite dimensional Lie algebra $\mathfrak{g}$. In this sense, the Virasoro algebra *Vir* is a chiralization of its zero mode, "$\mathbb{C}L_0$". And because L_0 corresponds to the Casimir operator (via the Sugawara construction), one can think of $Vir = \mathcal{W}(\mathfrak{sl}_2)$ as a chiralization of the center $\mathcal{Z}(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$. This is true in general:

$\mathcal{W}(\mathfrak{g})$ is a chiralization of the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$.

2.2. Kostant's Theorem. Let e be a principal nilpotent element of $\mathfrak{g}$. For instance, if $\mathfrak{g} = \mathfrak{sl}_n$, then e has the form

$$e = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By the Jacobson-Morozov theorem there exists a corresponding $\mathfrak{sl}_2$-triple $\{e, h_0, f\}$:

$$[h_0, e] = 2e, \quad [h_0, f] = -2f, \quad [e, f] = h_0$$

Then we have the eigenspace decomposition of $\mathfrak{g}$ with respect to the adjoint action of $\rho^\vee := h_0/2$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbf{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g}; [\rho^\vee, x] = jx\}.$$

Because e is principal, this gives a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where

$$\mathfrak{n}_+ = \sum_{j>0} \mathfrak{g}_j, \quad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}_- = \sum_{j<0} \mathfrak{g}_j.$$

Let $\Delta_+ \subset \mathfrak{h}^*$ be the corresponding set of positive roots, $\Delta_- = -\Delta_+$, $\Delta = \Delta_+ \sqcup \Delta_-$.

Define $p \in \mathfrak{n}_-^*$ by

$$p(x) = (x, e).$$

Here $(\ , \ )$ is the normalized invariant inner product of $\mathfrak{g}$. Then $p([\mathfrak{n}_-, \mathfrak{n}_-]) = 0$ and p defines a character of $\mathfrak{n}_-$.

Let Cl be the Clifford algebra associated with the space $\mathfrak{n}_- \oplus \mathfrak{n}_-^*$ and the natural bilinear form on it. Then Cl has the following generators and relations:

$$\text{generators: } \psi_\alpha, \psi_\alpha^* \quad (\alpha \in \Delta_-),$$
$$\text{relations: } \{\psi_\alpha, \psi_\beta^*\} = \delta_{\alpha,\beta}, \ \{\psi_\alpha, \psi_\beta\} = \{\psi_\alpha^*, \psi_\beta^*\} = 0.$$

We shall regard

$$U(\mathfrak{g}) \otimes Cl$$

as a superalgebra with even generators $\mathfrak{g} \ni x = x \otimes 1$ and odd generators $\psi_\alpha = 1 \otimes \psi_\alpha$, $\psi_\alpha^* = 1 \otimes \psi_\alpha^*$.

Define an odd element $Q^{\mathrm{st}} \in U(\mathfrak{g}) \otimes Cl$ by

$$Q^{\mathrm{st}} = \sum_{\alpha \in \Delta_-} x_\alpha \psi_\alpha^* - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \Delta_-} c_{\alpha,\beta}^\gamma \psi_\alpha^* \psi_\beta^* \psi_\gamma.$$

Here x_α is a (fixed) root vector of root α and $c_{\alpha,\beta}^\gamma$ is the structure constant. Then by direct calculation one can check that $[Q^{\mathrm{st}}, Q^{\mathrm{st}}] = 0$, or equivalently,

$$(Q^{\mathrm{st}})^2 = 0.$$

We remark that the "st" suffix stands for "standard", because Q^{st} is the differential of the standard Lie algebra cohomology or homology.

Set

$$Q := Q^{\mathrm{st}} + p,$$

where p is considered as an element of $Cl \subset U(\mathfrak{g})\otimes Cl$:

$$p = \sum_{\alpha\in\Delta_-} p(x_\alpha)\psi_\alpha^*.$$

Lemma 1. $[p,p] = [Q^{\mathrm{st}},p] = 0$. *Therefore* $[Q,Q]=0$, *or equivalently* $Q^2=0$.

By Lemma 1 it follows that

$$(\operatorname{ad} Q)^2 = 0$$

on $U(\mathfrak{g})\otimes Cl$. Hence we can consider $(U(\mathfrak{g})\otimes Cl, \operatorname{ad} Q)$ as a homology complex by setting

$$\begin{aligned}&\deg u = 0 \quad (u\in U(\mathfrak{g})),\\ &\deg \psi_\alpha = 1, \quad \deg\psi_\alpha^* = -1 \quad (\alpha\in\Delta_-).\end{aligned}$$

Then the corresponding homology

$$H_\bullet(U(\mathfrak{g})\otimes Cl \operatorname{ad} Q) = \bigoplus_{i\in\mathbb{Z}} H_i(U(\mathfrak{g})\otimes Cl, \operatorname{ad} Q)$$

inherits the graded superalgebra structure from $U(\mathfrak{g})\otimes Cl$.

Theorem 1 (Kostant [Ko], Kostant-Sternberg [KS], cf. [A3, Theorem 2.3.2]).

(i) $H_{i\neq 0}(U(\mathfrak{g})\otimes Cl, \operatorname{ad} Q) = 0$.

(ii) *The map*

$$\begin{array}{ccc}\mathcal{Z}(\mathfrak{g}) & \to & H_0(U(\mathfrak{g})\otimes Cl, \operatorname{ad} Q)\\ z & \mapsto & z\otimes 1\end{array}$$

is an isomorphism of $\mathbb{C}$*-algebras.*

2.3. **Reduction Functor.** Let $\Lambda(\mathfrak{n}_-)$ be the Grassmann algebra of $\mathfrak{n}_-$. Then $\Lambda(\mathfrak{n}_-)$ is naturally a module over Cl. Thus, for a $\mathfrak{g}$-module M,

$$C(M) := M\otimes\Lambda(\mathfrak{n}_-)$$

is naturally a module over $U(\mathfrak{g})\otimes Cl$. Thus, $(C(M),Q)$ again has the structure of homology complex. Let

$$H_\bullet(M) := H_\bullet(C(M),Q).$$

By definition $(C(M),Q)$ is identical to the Chevalley complex for calculating the Lie algebra homology $H_\bullet(\mathfrak{n}_-, M\otimes\mathbb{C}_p)$. Hence

(2) $$H_\bullet(M) = H_\bullet(\mathfrak{n}_-, M\otimes\mathbb{C}_p).$$

On the other hand, the $U(\mathfrak{g})\otimes Cl$-module structure of $C(M)$ induces a $\mathcal{Z}(\mathfrak{g})$-module structure on $H_i(M)$, because $\mathcal{Z}(\mathfrak{g}) = H_0(U(\mathfrak{g})\otimes Cl, \operatorname{ad} Q)$. Therefore we have obtained the following functor:

(3) $$\begin{array}{cccc}H_i(?): & \mathfrak{g}\text{-Mod} & \to & \mathcal{Z}(\mathfrak{g})\text{-Mod}\\ & M & \mapsto & H_i(M).\end{array}$$

Let $\mathcal{O}$ be the BGG category [BGG] of $\mathfrak{g}$. Let $M(\lambda)\in\mathcal{O}$ the Verma module of highest weight λ, $L(\lambda)\in\mathcal{O}$ the unique irreducible quotient of $M(\lambda)$. Then it is known that the following are equivalent:

(i) The Gelfand-Kirillov dimension $\operatorname{Dim} L(\lambda)$ of $L(\lambda)$ is maximal, i.e. $\operatorname{Dim}(L(\lambda)) = \dim\mathfrak{n}_-$.

(ii) $L(\lambda) = M(\lambda)$,

(iii) λ is anti-dominant, i.e. $\lambda(\alpha^\vee) \notin \mathbb{N}$ for all $\alpha \in \Delta_+$.

The following assertion was essentially proved by Kostant [Ko] (cf. [A3, Section 2])

Theorem 2.

(i) $H_{i\neq 0}(M) = 0$ *for all* $M \in \mathcal{O}$.

(ii) $H_0(L(\lambda)) = \begin{cases} \mathbb{C}_{\gamma_\lambda} & \text{if } \operatorname{Dim} L(\lambda) = \dim \mathfrak{n}_-, \\ 0 & \text{if } \operatorname{Dim} L(\lambda) < \dim \mathfrak{n}_-. \end{cases}$

Here $\mathbb{C}_{\gamma_\lambda} = \mathcal{Z}(\mathfrak{g})/\operatorname{Ker}\gamma_\lambda$ *and* $\gamma_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ *is the central character defined as the evaluation at* $M(\lambda)$.

By Theorem 2 (i), the functor $H_0(?)$ is exact. Moreover, by Theorem 2 (ii), one can obtain each simple $\mathcal{Z}(\mathfrak{g})$-module as the image of the functor $H_0(?)$.

Remark 1. More is known for the functor $H_0(?)$. According to Soergel [S] and Backelin [Ba], it holds that

$$\operatorname{Hom}_{\mathcal{O}}(M, P) \cong \operatorname{Hom}_{\mathcal{Z}(\mathfrak{g})}(H_0(M), H_0(P))$$

provided that P is projective in $\mathcal{O}$ (cf. [A3, Section 2]).

3. Chiralization of the center

3.1. We now "chiralize" the construction of the previous section to define affine W-algebras. To this end we "chiralize" the every data used for the cohomological realization of $\mathcal{Z}(\mathfrak{g})$ in Theorem 2. Thus

- $\mathfrak{g}$ is replaced by the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$, were K is the central element and D is the degree operator;
- $\mathfrak{n}_-$ is replaced by its loop algebra $L\mathfrak{n}_- = \mathfrak{n}_-\otimes\mathbb{C}[t,t^{-1}] \subset \widehat{\mathfrak{g}}$;
- Cl is replaced by the Clifford algebra $\widehat{Cl}$ associated with $L\mathfrak{n}_- \oplus (L\mathfrak{n}_-)^*$ and its natural symmetric bilinear form, where $(L\mathfrak{n}_-)^*$ is the graded dual of $L\mathfrak{n}_-$. This algebra may be defined by the following generators and relations:

$$\begin{array}{ll} \text{generators:} & \psi_\alpha(n),\ \psi^*_\alpha(n) \quad (\alpha \in \Delta_-,\ n \in \mathbb{Z})\ , \\ \text{relations:} & \{\psi_\alpha(m), \psi^*_\beta(n)\} = \delta_{\alpha,\beta}\delta_{m+n,0}, \\ & \{\psi_\alpha(m), \psi_\beta(n)\} = \{\psi^*_\alpha(m), \psi^*_\beta(n)\} = 0; \end{array}$$

- $Q = Q^{\mathrm{st}} + p$ is replaced by $\widehat{Q} = \widehat{Q}^{\mathrm{st}} + \hat{p}$, where

$$\widehat{Q}^{\mathrm{st}} = \sum_{\substack{\alpha\in\Delta_- \\ k\in\mathbb{Z}}} x_\alpha(-k)\psi^*_\alpha(k) - \frac{1}{2}\sum_{\substack{\alpha,\beta,\gamma\in\Delta_- \\ k+l+m=0}} c^\gamma_{\alpha,\beta}\psi^*_\alpha(k)\psi^*_\beta(l)\psi_\gamma(m),$$

$$\hat{p} = \sum_{\alpha\in\Delta_-} p(x_\alpha)\psi^*_\alpha(0),$$

where $x(k) = x\otimes t^k \in \widehat{\mathfrak{g}}$.

By analogy with Theorem 1, we want to define the affine W-algebra $\mathcal{W}(\mathfrak{g})$ as

$$\text{“}\mathcal{W}(\mathfrak{g}) = H_0(U(\widehat{\mathfrak{g}})\otimes\widehat{Cl},, \operatorname{ad}\widehat{Q})\text{”}.$$

But this does not make sense, for the appearance of the infinite sum in the formula of $\widehat{Q}^{\mathrm{st}}$. Thus we need to make a suitable completion of $U(\widehat{\mathfrak{g}})\otimes\widehat{Cl}$. We also specialize the value of the central element $K \in \widehat{\mathfrak{g}}$ at a given complex number $k \in \mathbb{C}$. So let $U_k(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}})/(K - k\,\mathrm{id})$. The algebra $U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}$ is naturally graded:

$$U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl} = \bigoplus_{d\in\mathbb{Z}}(U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl})_d,$$

where the grading is taken from the relation

(4) $$\deg x(n) = \deg \psi_\alpha(n) = \deg \psi_\alpha^*(n) = n,\ \deg 1 = 0.$$

Give $U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}$ the linear topology defined by the decreasing sequence where

$$\mathcal{I}_N = \bigoplus_{d\in\mathbb{Z}}(\mathcal{I}_N)_d, \quad (\mathcal{I}_N)_d = \sum_{j\geq N}(U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl})_{d-j}(U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl})_j.$$

Let $\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}}$ be the corresponding completion:

$$\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}} = \varprojlim_N \left(U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}/\mathcal{I}_N\right).$$

Then Q is a well-defined element of the topological algebra $\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}}$, and one can define

(5) $$H_\bullet(\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}}, \mathrm{ad}\,\widehat{Q}) := \varprojlim_N H_\bullet\left(U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}/\mathcal{I}_N, \mathrm{ad}\,\widehat{Q}\right).$$

But

(6) $$\text{“}\mathcal{W}_k(\mathfrak{g}) = H_0(\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}}, \mathrm{ad}\,\widehat{Q}) \quad (k \in \mathbb{C})\text{''}.$$

is still not a correct definition of W-algebra, because what is defined by (6) is a topological algebra in the usual sense, but an affine W-algebra should be defined as a vertex algebra. So what we actually mean by (6) is the following statement:

Theorem 3 ([A3, Theorem 3.11.1]). *There is an isomorphism*

$$\mathcal{U}(\mathcal{W}_k(\mathfrak{g})) \cong H_0(\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}}, \mathrm{ad}\,\widehat{Q}),$$

where $\mathcal{U}(V) = \bigoplus_{d\in\mathbb{Z}}\mathcal{U}(V)_d$ *is the universal enveloping algebra of a vertex algebra* V *(in the sense of Frenkel and Zhu* [FZ]*).*

Remark 2. The vanishing $H_{i\neq 0}(\widetilde{U_k(\widehat{\mathfrak{g}})\otimes\widehat{Cl}}, \mathrm{ad}\,\widehat{Q}) = 0$ also holds.

We will not define the W-algebra $\mathcal{W}_k(\mathfrak{g})$ itself in this note. Instead, we take (6) as its definition because a $\mathcal{W}_k(\mathfrak{g})$-module M is by definition a $\mathcal{U}(\mathcal{W}_k(\mathfrak{g}, e))$-module (such that $\dim \mathcal{U}(\mathcal{W}_k(\mathfrak{g}, e))_n \cdot v < \infty$ for all $v \in V$ and $n \geq 0$). But it should be remarked that Theorem 3 follows from the corresponding statement for the vertex algebra $\mathcal{W}_k(\mathfrak{g})$ itself. This was proved for generic k by Feigin and Frenkel [FF2], for a general k and $\mathfrak{g} = \mathfrak{sl}_n$ by de Bore and Tjin [dBT2] and for a general k and a general $\mathfrak{g}$ by Frenkel [FB].

Remark 3. The W-algebra $\mathcal{W}_k(\mathfrak{g})$ considered here is not a simple vertex algebra in general.

Remark 4. If $k \neq -h^\vee$, then $\mathcal{W}_k(\mathfrak{g})$ has the structure of the vertex operator algebra and has the central charge

$$c(k) = \mathrm{rank}\,\mathfrak{g} - 12(\kappa|\rho^\vee|^2 - \langle\rho, \rho^\vee\rangle + |\rho|^2/\kappa), \quad (\kappa = k + h^\vee).$$

Remark 5. It is known that $\mathcal{W}_{-h^\vee}(\mathfrak{g})$ is commutative. This is one of the results of Feigin-Frenkel [FF2].

To give a more precise relationship between $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{W}_k(\mathfrak{g})$, let us introduce the notion of *Zhu algebra* $\mathrm{Zh}(V)$ of a (graded) vertex algebra V.

$$\mathrm{Zh}(V) := \mathcal{U}(V)_0/\overline{\sum_{p>0}\mathcal{U}(V)_{-p}\mathcal{U}(V)_p},$$

where $^-$ denotes the closure. By definition the following assertion is clear.

Theorem 4 (Zhu [Zhu]). *There is a one-to-one correspondence between irreducible V-modules and irreducible* $\mathrm{Zh}(V)$*-modules.*

For example, consider the universal affine vertex algebra $V_k(\mathfrak{g})$ associated with $\mathfrak{g}$ at level k. Then $\mathcal{U}(V_k(\mathfrak{g})) = \widetilde{U_k(\widehat{\mathfrak{g}})}$ and we have $\mathrm{Zh}(V_k(\mathfrak{g})) = U(\mathfrak{g})$. This reflects the fact that $\widehat{\mathfrak{g}}$ (or more precisely $V_k(\mathfrak{g})$) is a chiralization of $\mathfrak{g}$. Since $\mathcal{W}_k(\mathfrak{g})$ is a chiralization of $\mathcal{Z}(\mathfrak{g})$, it is natural to expect the following assertion:

Theorem 5 ([A3, Theorem 3.13.2]). *The Zhu algebra* $\mathrm{Zh}(\mathcal{W}_k(\mathfrak{g})$ *of* $\mathcal{W}_k((\mathfrak{g}))$ *is naturally isomorphic to* $\mathcal{Z}(\mathfrak{g})$.

By Theorems 4, 5, irreducible $\mathcal{W}_k(\mathfrak{g})$-modules are parameterized by the central characters of $\mathcal{Z}(\mathfrak{g})$. Let $\mathbf{L}(\gamma)$ denote the irreducible $\mathcal{W}_k(\mathfrak{g})$-module corresponding the central character γ. Then $\mathbf{L}(\gamma)$ is the quotient of the Verma module $\mathbf{M}(\gamma)$ with highest weight γ, which has the PBW type basis.

3.2. As in the finite dimensional case we functionally obtain the $\mathcal{W}_k(\mathfrak{g})$-modules in the following way: Let $\Lambda^{\frac{\infty}{2}}(L\mathfrak{n}_-)$ be the irreducible representation of $\widehat{Cl}$ generated by the vector $\mathbf{1}$ satisfying the following relations:

$$\psi_\alpha(n)\mathbf{1} = \psi^*_\alpha(n+1)\mathbf{1} = 0 \quad (\alpha \in \Delta_-,\ n \geq 0).$$

Denote by $\widehat{\mathcal{O}}_k$ the BGG category of $\widehat{\mathfrak{g}}$ at level k. Then

$$\widehat{C}(M) := M \otimes \Lambda^{\frac{\infty}{2}}(L\mathfrak{n}_-)$$

with $M \in \widehat{\mathcal{O}}_k$ is naturally a module over $U_k(\mathfrak{g})\otimes\widehat{Cl}$, and its action extends to the smooth action of $\widetilde{U_k(\widehat{\mathfrak{g}})}\otimes\widehat{Cl}$. In particular the action of $\widehat{Q}$ is well-defined on $\widehat{C}(M)$. Thus the homology

$$\widehat{H}_\bullet(M) := H_\bullet(\widehat{C}(M), \widehat{Q}) \tag{7}$$

well-defined and is naturally a module over $\mathcal{W}_k(\mathfrak{g})$. Note that $\widehat{H}_i(M)$ is naturally graded (cf. (4)):

$$\widehat{H}_i(M) = \bigoplus_{d\in\mathbb{C}} \widehat{H}_i(M)_d. \tag{8}$$

If k is not critical then (8) is essentially the L_0-eigenspace decomposition. Set $\mathrm{ch}\,\widehat{H}_i(M) = \sum_{d\in\mathbb{C}} q^d \dim H_i(M)_d$ whenever it is well-defined.

Remark 6. By definition we have $\widehat{H}_\bullet(M) = H_{\frac{\infty}{2}+\bullet}(L\mathfrak{n}_-, M\otimes\mathbb{C}_{\hat{p}})$, the Feigin's semi-infinite $L\mathfrak{n}_-$-homology with the coefficient in $M\otimes\mathbb{C}_{\hat{p}}$ ([Fe]).

Let $\widehat{M}(\widehat{\lambda})$ be the Verma module of $\widehat{\mathfrak{g}}$ with highest weight $\widehat{\lambda}$, $\widehat{L}(\widehat{\lambda})$ the unique simple quotient of $\widehat{M}(\widehat{\lambda})$.

Theorem 6 ([A3]). *For any $k \in \mathbb{C}$ we have the following.*

(i) $\widehat{H}_{i\neq 0}(M) = 0$ *for all objects M of $\widehat{\mathcal{O}}_k$.*

(ii) *Let $\widehat{\lambda}$ be a weight of $\widehat{\mathfrak{g}}$ at level k, λ the classical part of $\widehat{\lambda}$ (i.e. the restriction of $\widehat{\lambda}$ to $\mathfrak{h}$). Then*

$$\widehat{H}_0(\widehat{L}(\widehat{\lambda})) = \begin{cases} \mathbf{L}(\gamma_\lambda) & \text{if } \operatorname{Dim} L(\lambda) = \dim \mathfrak{n}_-, \\ 0 & \text{if } \operatorname{Dim} L(\lambda) < \dim \mathfrak{n}_-. \end{cases}$$

By Theorem 6 it follows that the functor

$$\widehat{H}_0(?) : \mathcal{O}_k \to \mathcal{W}_k(\mathfrak{g})\text{-Mod}$$

is exact for any $k \in \mathbb{C}$.

Write the formal character $\operatorname{ch} \widehat{L}(\widehat{\lambda})$ of $L(\lambda)$ as

$$\operatorname{ch} \widehat{L}(\widehat{\lambda}) = \sum_{\widehat{\mu}} m_{\widehat{\lambda},\widehat{\mu}} \operatorname{ch} \widehat{M}(\widehat{\mu}), \quad (m_{\widehat{\lambda},\widehat{\mu}} \in \mathbb{Z}).$$

Then the following assertion follows from Theorem 6:

Theorem 7 ([A3]). $\operatorname{ch} \widehat{H}_0(\widehat{L}(\widehat{\lambda})) = \sum_{\widehat{\mu}} m_{\widehat{\lambda},\widehat{\mu}} q^{\widehat{\mu}(\mathbf{D})} \prod_{i\geq 0}(1-q^{-i})^{-\operatorname{rank}\mathfrak{g}}$.

Recall that the integer $m_{\lambda,\mu}$ is known by Kashiwara-Tanisaki-Cassian [KT1, KT2, KT3, Ca] provided that $k \neq -h^\vee$. Therefore by Theorems 6 and 7 we have obtained the character formula of *all* the irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g})$ for any $k \in \mathbb{C}\backslash\{-h^\vee\}$.

Remark 7. It may be worth emphasizing that Theorems 6 and 7 remain valid even at the critical level $k = -h^\vee$, and the result for this case in particular implies the Kac-Kazhdan conjecture [KK], which was proved by Hayashi [Ha] and others [GW, FF1, Ku] by computational methods (see [A4] for details).

3.3. **Frenkel-Kac-Wakimoto Conjecture.** Note that our functor $\widehat{H}_0(?)$ kills integrable representations of $\widehat{\mathfrak{g}}$. However there are a wider class of modular invariant representations of $\widehat{\mathfrak{g}}$; they are called Kac-Wakimoto *admissible representations* [KW1, KW2].

The simple module $\widehat{L}(\widehat{\lambda})$ is called admissible if $\widehat{\lambda}$ is an admissible weight. An admissible weight is a weight $\widehat{\lambda}$ that satisfies the following:

(i) $\widehat{\lambda}$ is regular dominant;

(ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\widehat{\lambda})^\vee := \{\alpha \in \widehat{\Delta}_+^{\vee,\mathrm{re}}; \widehat{\lambda}(\alpha) \in \mathbb{Z}\}$ = the $\mathbb{Q}$-span of $\widehat{\Delta}_+^{\vee,\mathrm{re}}$.

The condition (i) implies that the corresponding Kazhdan-Lusztig polynomial is trivial. Therefore $\widehat{L}(\widehat{\lambda})$ has the Weyl-Kac type character formula:

$$\operatorname{ch} \widehat{L}(\widehat{\lambda}) = \sum_{w \in \widehat{W}(\widehat{\lambda})} (-1)^{\ell(w)} \operatorname{ch} \widehat{M}(w \circ \widehat{\lambda}),$$

where $\widehat{W}(\widehat{\lambda})$ is the integral Weyl group of $\widehat{\mathfrak{g}}$, generated by the reflections r_α with $\alpha^\vee \in \widehat{\Delta}(\widehat{\lambda})^\vee$. The condition (ii) implies that $\widehat{W}(\widehat{\lambda})$ is an infinite Coxeter group, and $\operatorname{ch} \widehat{L}(\widehat{\lambda})$ is written in terms of some theta functions ([KW1, KW2]).

If the classical part λ of an admissible weight $\widehat{\lambda}$ is anti-dominant, then $\widehat{\lambda}$ is called a *non-degenerate* admissible weight. Let $Pr_k^{\text{non-deg}}$ be the set of non-degenerate admissible weight at level k.

And as explained in Introduction, the conjecture of Frenkel, Kac and Wakimoto [FKW] follows from Theorems 6 and 7:

Corollary 1 (Frenkel-Kac-Wakimoto Conjecture [FKW]). *Let $\widehat{\lambda}$ be an non-degenerated admissible weight of $\widehat{\mathfrak{g}}$, λ the classical part of $\widehat{\lambda}$. Then*

$$\operatorname{ch} \mathbf{L}(\gamma_\lambda) = \sum_{w\in \widehat{W}(\widehat{\lambda})} (-1)^{\ell(w)} q^{(w\circ\widehat{\lambda})(D)} \prod_{i\geq 1}(1-q^{-i})^{-\operatorname{rank}\mathfrak{g}}.$$

As explained in [FKW], from Corollary 1 it follows that the (modified) characters of

$$\{\mathbf{L}(\gamma_\lambda); \lambda \text{ is the classical part of } \widehat{\lambda} \in Pr_k^{\text{non-deg}}\}$$

are modular invariant, i.e. the linear space spanned by their (modified) characters are invariant under the natural action of $SL_2(\mathbb{Z})$. In the case that $\mathfrak{g} = \mathfrak{sl}_2$, they are exactly the minimal series representations of *Vir.*

4. Generalization to other nilpotent orbits

4.1. In the above construction we started with the principal nilpotent element of $\mathfrak{g}$. However the above construction can be generalized to cases of other nilpotent elements:

Let e be a nilpotent element which corresponds to a nice parabolic subalgebra ([BW]) of $\mathfrak{g}$. Then it is straightforward to generalize the previous construction to e (cf. [dBT1, dBT2, KRW]). As a result, instead of $\mathcal{Z}(\mathfrak{g})$, we obtain the *finite W-algebra* $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ [dBT1] associated with $(\mathfrak{g}, e)$, which is the endmorphism ring of the *generalized Gelfand-Graev representation* ([Ka], cf. [Pr, GG, BG]). The corresponding affine W-algebra $\mathcal{W}_k(\mathfrak{g}, e)$ has $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ as its Zhu algebra (cf. [DK]).

We have the similar result as Theorems 6 and 7 for this case ([A5]); The difficulty is that the representation theory of $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ is not known very much in general, except for the type A cases; Recently Brundan and Kleshchev [BK] established important results on the representation theory of finite W-algebras for these cases. Thanks to their result, for the type A cases one obtains the character formula for each irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g}, e)$ (see [A5] for details).

If e does not corresponds to a nice parabolic subalgebra then the construction of W-algebras becomes more involving. The most general construction was made by Kac, Roan and Wakimoto [KRW], which applies to the Lie superalgebra case also. One of the remarkable discoveries of Kac, Roan and Wakimoto [KRW] is that almost all the superconformal algebras (such as $N = 2, 3, 4$ superconformal algebra) appears as a W-algebra associated with some Lie superalgebra $\mathfrak{g}$ and its *minimal nilpotent element.* As principal nilpotent element cases, their representation theory (such as characters of irreducible representations) can be completely described through the reduction functor (see [A2] for details).

References

[A1] Arakawa, T.; *Vanishing of cohomology associated to quantized Drinfeld-Sokolov reduction,* Int. Math. Res. Notices, 2004, No. 15, 729-767.

[A2] Arakawa, T.; *Representation Theory of Superconformal Algebras and the Kac-Roan-Wakimoto Conjecture*, Duke. Math. J. Vol. 130 (2005), No. 3, 435-478.

[A3] Arakawa, T.; *Representation Theory of $\mathcal{W}$-Algebras,* math.QA/0506056.

[A4] Arakawa, T; *A new proof of the Kac-Kazhdan conjecture,* Int. Math. Res. Notices Volume 2006, Article ID 27091, Pages 1-5.

[A5] Arakawa, T; in preparation.

[Ba] Backelin, E.; Representation of the category $\mathcal{O}$ in Whittaker categories, Int. Math. Res. Notices 1997, no. 4, 153–172.

[BW] Baur, Karin., Wallach, Nolan; *Nice parabolic subalgebras of reductive Lie algebras*, Represent. Theory 9 (2005), 1-29 (electronic).

[BD] Beilinson, Alexander; Drinfeld, Vladimir; *Chiral algebras*, American Mathematical Society Colloquium Publications, 51. American Mathematical Society, 2004.

[BGG] Bernstein, I. N., Gelfand, I. M., Gelfand, S. I.; *A certain category of $\mathfrak{g}$-modules*, Funkcional. Anal. Appl. 10 (1976), no. 2, 1–8.

[BG] Brundan, J., Goodwin, S.; *Good grading polytopes*, preprint math.QA/0510205.

[BS] *$\mathcal{W}$-symmetry*. Edited by P. Bouwknegt and K. Schoutens. Advanced Series in Mathematical Physics, 22. World Scientific Publishing Co., Inc., River Edge, NJ, 1995

[BK] Brundan, J., Kleshchev, A.; *Representations of shifted Yangians*, math.RT/0508003.

[dBT1] de Boer, J., Tjin, T.; *Quantization and representation theory of finite $\mathcal{W}$-algebras*, Comm. Math. Phys. 158 (1993), no. 3, 485–516.

[dBT2] de Boer, J., Tjin, T; *The relation between quantum $\mathcal{W}$-algebras and Lie algebras*, Comm. Math. Phys. 160(1997), 317-332.

[Ca] Casian, L; *Proof of the Kazhdan-Lusztig conjecture for Kac-Moody algebras (the characters* ch $L_{w\rho-\rho}$*)*, Adv. Math., 119 (1996) 207–281.

[DK] De Sole, A., Kac, V. ; *Finite vs affine W-algebras*, Japan J. Math., Vol. 1, Issue 1 (2006) 137-261.

[DS] Drinfeld, V. G., Sokolov, V. V.; *Lie algebra and the KdV type equations*, Soviet J. Math. 30, 1975-2036 (1985).

[EK] Elashvili, A.G., Kac, V.G.; *Classification of Good Gradings of Simple Lie Algebras*, Amer. Math. Soc. Transl. (2) vol 213 (2005), 85–104.

[Fe] Feigin, B. L.; *Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras*, Uspekhi Mat. Nauk 39 (1984), no. 2(236), 195–196.

[FF1] Feigin, B., Frenkel, E.; *A family of representations of affine Lie algebras*, (Russian) Uspekhi Mat. Nauk 43 (1988), no. 5(263), 227–228; translation in Russian Math. Surveys 43 (1988), no. 5, 221–222.

[FF2] Feigin, B., Frenkel, E.; *Quantization of the Drinfeld-Sokolov reduction*, Phys. Lett. B 246 (1990), no. 1-2, 75–81.

[FFu] Feigin, B. L., Fuchs, D. B.; *Verma modules over the Virasoro algebra*, Topology (Leningrad, 1982), 230–245, Lecture Notes in Math., 1060, Springer, Berlin, 1984.

[FB] Frenkel, E., Ben-Zvi, D.; *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, 88, 2001

[FKW] Frenkel, E., Kac, V., Wakimoto, M.; *Characters and fusion rules for W-algebras via quantized Drinfeld-Sokolov reduction*, Comm. Math. Phys. 147 (1992), no. 2, 295–328.

[FZ] Frenkel, I. B., Zhu, Y; *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. 66 (1992), no. 1, 123–168.

[GG] Gan, W. L., Ginzburg, V.; *Quantization of Slodowy slices*,

[GW] Goodman, R., Wallach, N.R.; *Higher-order Sugawara operators for affine Lie algebras*, Tras. Amer. Math. Soc. 315(1989), np. 1.

[Ha] Hayashi, Takahiro; *Sugawara operators and Kac-Kazhdan conjecture*, Invent. Math. 94(1988), no. 1, 13–52.

[K1] Kac, Victor G.; *Infinite-dimensional Lie algebras*, Cambridge University Press, 1985.

[K2] Kac, Victor G.; *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, 1997.

[KK] Kac, V. G., Kazhdan, D. A.; *Structure of representations with highest weight of infinite-dimensional Lie algebras*, Adv. in Math. 34 (1979), no. 1, 97–08.

[KP] Kac, Victor G., Peterson, Dale H.; *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. 53 (1984), no. 2, 125–264.

[KRW] Kac, V. G., Roan, S.-S., Wakimoto, M.; *Quantum reduction for affine superalgebras*, Commun. Math. Phys. 241 (2003), 307-342.

[Ku] Ku, Jong Min; *Structure of the Verma module $M(-\rho)$ over Euclidean Lie algebras*, J. Algebra 124 (1989), no. 2, 367–387.

[KW1] Kac, V. G., Wakimoto, M.; *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Nat. Acad. Sci. U.S.A. 85 (1988), no. 14, 4956–4960.

[KW2] Kac, V. G., Wakimoto, M.; *Classification of modular invariant representations of affine algebras*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 138–177, Adv. Set. Math. Phys., 7, World Sci. Publishing, Teaneck, NJ, 1989.

[KW3] Kac, V. G., Wakimoto, M.; *Quantum Reduction amd Representation Theory of Superconformal Algebras*, Adv. Math. 185 (2004) 400–458, *Corrigendum to; "Quantum reduction and representation theory of superconformal algebras", Adv. Math. 185 (2004), no. 2, 400–458; MR2060475*, Adv. Math. 193 (2005), no. 2, 453–455.

[KT1] Kashiwara, M., Tanisaki, T.; *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level II : Nonintegral case*, Duke Math. J., 84 (1996) 771–813.

[KT2] Kashiwara, M., Tanisaki, T.; *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III. Positive rational case*, Mikio Sato; a great Japanese mathematician of the twentieth century. Asian J. Math. 2 (1998), no. 4, 779–832.

[KT3] Kashiwara, M., Tanisaki, T. *Characters of irreducible modules with non-critical highest weights over affine Lie algebras. Representations and quantizations*, 275–296, China High. Educ. Press, Beijing, 2000.

[Ka] Kawanaka, N.; *Generalized Gelfand-Graev representations and Ennola duality*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), 175–206, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.

[Ko] Kostant, B.; *On Whittaker vectors and representation theory*, Invent. Math. 48 (1978), no. 2, 101–184. 329–387.

[KS] Kostant, B., Sternberg, S.; *Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras*, Ann. Physics 176 (1987), no. 1, 49–113.

[Pr] Premet, A.; *Special transverse slices and their enveloping algebras*, Adv. Math. 170 (2002), no. 1, 1–55.

[S] Soergel, W; *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. 3 (1990), no. 2, 421–445.

[Za] Zamolodchikov, A. B.; *Infinite extra symmetries in two-dimensional conformal quantum field theory*, Teoret. Mat. Fiz. 65 (1985), no. 3, 347–359.

[Zhu] Zhu, Y.; *Modular invariance of characters of vertex operator algebras*, Journal AMS 9(1996), 237-302.

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