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Graph methods in inverse semigroups *

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Abstract

Graph methods play an important role in studying the structure of inverse semigroups. We use Schützenberger graphs to show that the class of inverse semigroups has the strong HNN property and the strong amalgamation property. We also obtain an analogue of Britton's lemma.

1 Introduction

Graphs play a significant role in studying structures of algebraic systems. For groups, the Cayley graphs shed light on geometrical, structural and algorithmical problems. Similarly, Schützenberger graphs are indispensable to study the structures of inverse semigroups. In this paper, we employ Schützenberger graphs to investigate algebraic properties of HNNextensions and amalgamated free products of inverse semigroups.

Britton's lemma [1] is fundamental in studying algorithmic problems in group theory. It is basically equivalent to the normal form theorem for HNN extensions of groups. Numerous results depend on Britton's lemma as one can see many applications in [3]; the Novikov-Boone theorem, the Freiheitssatz, Higman's embedding theorem, the undecidability of Markov properties of groups, the existence of a finitely presented non-Hopfian group and the embeddability of a countable group into a group generated by two elements.

HNN extensions of inverse semigroups have been studied and applied to algorithmic and structural problems of inverse semigroups in [7, 8, 9, 10, 11]. However, no immediate generalization of Britton's lemma to the class of inverse semigroups has been known so far. In this paper, we give an analogue of Britton's lemma for inverse semigroups.

Let us briefly recall the concept of HNN extensions introduced in [11]. We suppose that S is an inverse semigroup and A and B are isomorphic inverse subsemigroups of S and ϕ is an isomorphism of A onto B. We now suppose $e_A \in A \subset e_ASe_A$, $e_B \in B \subset e_BSe_B$, where e_A and e_B are idempotents of A and B, respectively. The inverse semigroup $S\langle\phi, t\rangle$ is defiend ([11]) by the presentation

$$\mathrm{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A,\ tbt^{-1} = \phi^{-1}(b) \text{ for } \forall b \in B). \tag{1.1}$$

The element t in $S\langle\phi, t\rangle$ is called the *stable letter*. The relatioship between $S\langle\phi, t\rangle$ and the other variants of HNN extensions is discussed in [11].

*This paper is an extended abstract and the detailed version will be published elsewhere.

A class of inverse semigroups is said to have the *weak HNN property for* $S\langle\phi, t\rangle$ if the following holds. There exists an inverse semigroup T such that $S \hookrightarrow T$, $t^{-1}at = \phi(a)$ for all $a \in A$, and $tbt^{-1} = \phi^{-1}(b)$ for all $b \in B$ for some $t \in T$.

2 Schützenberger graphs and automata

We briefly review Schützenberger graphs and approximate automata introduced by Stephen [5]. Let S be an inverse semigroup and X the set of generators of S. The *Schützenberger graph* $\mathrm{S\Gamma}(S, \mathrm{X}, u)$ for the word u is given by the sets of vertices and edges defined by

$$\begin{aligned} \mathrm{Vert}(\mathrm{S\Gamma}(S,\mathrm{X},u)) &= \{s \mid s \in S,\ s\mathcal{R}u\}, \\ \mathrm{Edge}(\mathrm{S\Gamma}(S,\mathrm{X},u)) &= \{(s_1, x, s_2) \mid s_1 x = s_2,\ s_1\ \mathcal{R}\ s_2\ \mathcal{R}\ u,\ s_1, s_2 \in S,\ x \in \mathrm{X} \cup \mathrm{X}^{-1}\}, \end{aligned}$$

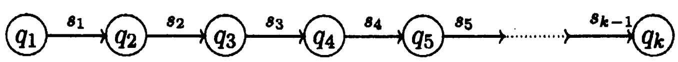
where $\mathcal{R}$ is the Green's R-relation. The initial and terminal vertex are given by

$$\mathbf{d}(s_1, x, s_2) = s_1, \quad \mathbf{r}(s_1, x, s_2) = s_2.$$

Suppose that $\mathcal{A}$ is an automaton with input alphabet $X \cup X^{-1}$. We say that $\mathcal{A}$ is an *inverse word automaton* if the tansition is consistent with the involution $x \to x^{-1}$, that is, if $(q_1) \xrightarrow{x} (q_2)$ is an edge in $\mathcal{A}$, then so is $(q_1) \xleftarrow[x^{-1}]{} (q_2)$ We can regard $\mathrm{S\Gamma}(S, \mathrm{X}, u)$ as an inverse word automaton. The initial and terminal state of the automaton $\mathrm{S\Gamma}(S, \mathrm{X}, u)$ are uu^{-1} and u, respectively. We call $\mathrm{S\Gamma}(S, \mathrm{X}, u)$ the Schützenberger automaton.

Lemma 2.1 **([5])** *For a word u in $(X \cup X^{-1})^{+}$, the language accepted by the Schützenberger automaton* $\mathrm{S\Gamma}(X, R, u)$ *consists of the words above u in S, that is,* $L(\mathrm{S\Gamma}(X, R, u)) = \{w \mid u \leq w \text{ in } S\}$.

Suppose that we are given an inverese semigroup presenation $\mathrm{Inv}(X|R)$. We consider inverse word automata whose input alphabets are $X \cup X^{-1}$. An X-labeled inverse word automaton $\mathcal{A}$ is called an *approximate* of $\mathrm{S\Gamma}(S, \mathrm{X}, u)$ if $u \in L(\mathcal{A})$ and $L(\mathcal{A}) \subset L(\mathrm{S\Gamma}(S, \mathrm{X}, u))$. In [5], a sequence of approximate automata for $\mathrm{S\Gamma}(S, \mathrm{X}, u)$ is costructed starting from the linear automaton $\mathcal{B}_0(u)$ illustrated below:



(2.1)

where we are assuming u is the word $s_1 s_2 \cdots s_k$. Clearly the linear automaton is an approximate automaton of $\mathrm{S\Gamma}(S, \mathrm{X}, u)$. There are two operations for the automata production process: *expansions* and *reductions*.

Expansions. Given an automaton $\mathcal{A}$, we construct an automaton $\mathcal{B}$ as follows. Suppose that there is a path form the state q_1 to q_2 labeled by the word r_2, there is no path from q_1 to q_2 labeled by r_1 and $r_1 = r_2$ is a defining relation belonging to R. Now $\mathcal{B}$ is obtained from $\mathcal{A}$ by adding a new path from q_1 to q_2 labeled by r_1. The automaton $\mathcal{B}$ is called an *expansion* of $\mathcal{A}$.

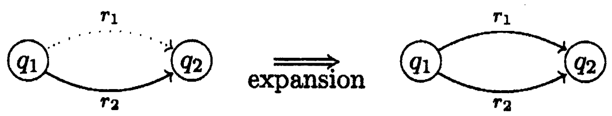


Figure 1: Expansion

Reductions. Given an automaton $\mathcal{A}$, we construct an automaton $\mathcal{B}$ as follows. Suppose that in $\mathcal{A}$ there are two edges $q \to q_1$ and $q \to q_2$ labeled by the same letter x. The new automaton is obtained from $\mathcal{A}$ by identifying these edges. Note that the states q_1 and q_2 are identified in $\mathcal{B}$. The automaton $\mathcal{B}$ is called a *reduction* of $\mathcal{A}$.

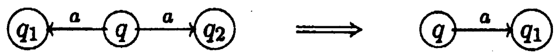


Figure 2: Reduction

We restate Theorem 5.7 and 5.9 in [5] as follows.

Lemma 2.2 *(1) Suppose that u, w are words on an inverse semigroup S with $u \leq w$. Let $\mathcal{A}$ be an approximate automaton for u. Then there exists a sequence of approximate automata $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ such that $\mathcal{A}_0 = \mathcal{A}$, w is accepted by $\mathcal{A}_n$ and each $\mathcal{A}_i$ is obtained from $\mathcal{A}_{i-1}$ by applying either an expansion or a reduction.*
(2) If $\mathcal{B}$ is obtained from an approximate automaton $\mathcal{A}$ by applying either an expansion or a reduction then $\mathcal{B}$ is also an approximate automaton.

3 Structure of approximate automata for HNN extensions

We now describe the structure of an approximate automaton for the Schützenberger graph $S\Gamma(S\langle\phi, t\rangle, u)$, where $u \in S$. The detailed structure will be discussed in the full version of the paper and we omit the detail of the structure of approximate automata.

Recall that $S\langle\phi, t\rangle$ has $S \cup \{t, t^{-1}\}$ as its set of generators. Suppose that S is generated by X relative to the relations R. Let $Y = X \cup \{t\}$. Then $S\langle\phi, t\rangle$ is generated by $Y \cup Y^{-1}$. Let $\mathcal{X}$ be a $Y \cup Y^{-1}$-labeled automaton, that is, $\mathcal{X}$ is a $Y \cup Y^{-1}$-labeled graph with the start and final state. We suppose that $\mathcal{X}$ satisfies the following properties.

1. Let $\mathcal{X}'$ be the graph obtained from $\mathcal{X}$ by deleting all the edges labeled by t or t^{-1}. Then

$$\mathcal{X}' = \bigcup_{i=0}^{n} \mathcal{X}_i \tag{3.1}$$

   where $\mathcal{X}_i$ is a connected componets. We note that each $\mathcal{X}_i$ is a maximal (t, t^{-1})-free connected subgraph of $\mathcal{X}$. We call $\mathcal{X}_i$ a *lobe of* $\mathcal{X}$.

2. One of the lobes (say $\mathcal{X}_0$) has the start and final state of $\mathcal{X}$. We call $\mathcal{X}_0$ the root lobe of $\mathcal{X}$.

3. Every lobe $\mathcal{X}_i$ $(i = 1, 2, \cdots, n)$ has its immediate ancestor.

In addition, approximate automata have several structural properties that are crucial in our study. We use those to study the structure of the inverse semigroups. The details will be given in the full version of the paper.

4 Analogue of Britton's lemma for inverse semigroups

Our object is to show that Britton's lemma in group theory can be generalized to HNN extensions of inverse semigroups to some extent.

First we recall Britton's lemma [1] for groups. Let w be a word on $X \cup X^{-1} \cup \{t, t^{-1}\}$, where a group G^* is presented by

$$\mathrm{Gr}(G, t \mid t^{-1}at = \phi(a),\ a \in A).$$

A pinch is a word of the form $t^{-1}at$ ($a \in A$) or tbt^{-1} ($b \in B$). Then Britton's lemma claims that if w represents the identity in G^* then there exists a pinch in w. A equivalent statement is that if a word w represents an element in G in G^* then there exists a pinch in w; if $w = z$ in G^*, where z represents an element in G then $wz^{-1} = 1$ in G^* and so there is a pinch in the word wz^{-1} and so it must be in w.

We now consider an analogue of Britton's lemma in the class of inverse semigroups. We recall some terminology from the theory of inverse semigroup. For a subset H of an inverse semigroup S, the set $H\omega$ [4] is the set $\{s \in S \mid h \leq s,\ h \in H\}$. A word of the form $t^{-1}st$, where $s \in A\omega$ or tst^{-1}, where $s \in B\omega$ is called a *quasi-pinch*. Suppose that w is a word on $X \cup X^{-1} \cup \{t, t^{-1}\}$. Using the structure of an approximate automaton in Section 3, we can prove the following.

Theorem 4.1 *If a word w represents an element s in $S\langle\phi, t\rangle$ where $s \in S$, then there exists a quasi-pinch in w.*

5 Amalgamation and HNN property

It is shown in [7] that the class of inverse semigroups has the weak HNN property using the weak amalgamation property. In this section we give another proof independent of the weak amalgamation property. Using the structure of approximate automaton in Section 3, we can prove the following.

Theorem 5.1 *The class of inverse semigroups has the weak HNN property for $S\langle\phi, t\rangle$.*

We say that the class of inverse semigroups has the *strong HNN property* if it satisfies the following. Let T be an inverse subsemigroup of $S\langle\phi, t\rangle$ generated by $S \cup \{t^{-1}t, tt^{-1}\}$. Then we have

$$t^{-1}Tt \cap T = B \cup \{1_B\}, \quad tTt^{-1} \cap T = A \cup \{1_A\}$$

and

$$t^{-1}St \cap S = B, \quad tSt^{-1} \cap A = A$$

in $S\langle\phi, t\rangle$.

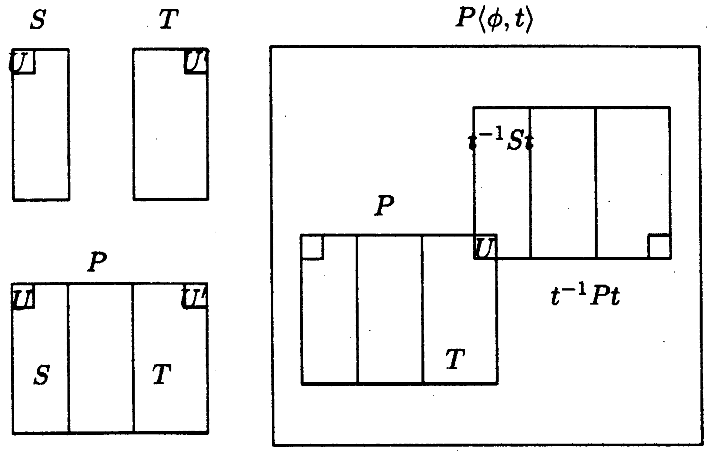
It is shown in [11] that the class of inverse semigroups has the strong HNN property using the strong amalgamation property. We can give another proof independent of the strong amalgamation property using the structure of approximate automaton in Section 3.

Theorem 5.2 *The class of inverse semigroups has the strong HNN property for $S\langle\phi, t\rangle$.*

By Theorem 5.2, we can give another proof of the strong amalgamation property as follows.

Theorem 5.3 ([2]) *The class of inverse semigroups has the strong amalgamation property.*

Proof. By the previous theorem, the class of the inverse semigroups satisfies the strong HNN property. Take any amalgam $(S, T; U)$ of inverse semigroups. Then $U \cong U_S \hookrightarrow S$ and $U \cong U_T \hookrightarrow T$. Let $P = (S \cup \{1\}) \times (T \cup \{1\})$. Then P is an inverse semigroup with the identity $(1, 1)$. Clearly $U \cong U_S \times 1 \cong 1 \times U_T$. Let ϕ be the isomorphism of $U_S \times 1$ onto $1 \times U_T$. By the strong HNNproperty, $P \hookrightarrow P(\phi, t)$ and $P \cap t^{-1}Pt = 1 \times U_T$ in $P(\phi, t)$. If we identify S and $t^{-1}(S, 1)t$, and T and $(1, T)$, then we have $S \cap T = 1 \times U_T \cong U$.



ϕ is an isomorphism of U onto U' in P
by the strong HNN property

Figure 3: Strong amalgams

See Figure 5. Thus the class of inverse semigroups has the strong amalgamation property.

We recall that the strong amalgamation property requires that if $s = h$ in $S *_{A=B} H$, where $s \in S$ and $h \in H$ then we must have $s \in A$ and $h \in B$. This poses the following question; if a word $w = x_1y_1x_2y_2 \cdots x_ny_n$ represents an element in S or H, then does one of the letters in w belong to A or B? This is clearly true for the class of groups, that is, we can prove the following theorem using the normal form theorem.

Theorem 5.4 *Let G and H be groups. If a word $w = x_1y_1x_2y_2 \cdots x_ny_n$ ($x_i \in G$, $y_i \in H$) represents an element in G or H in the amalgamated free product $G *_{A=B} H$, then $x_i \in A$ or $y_i \in B$ for some i.* □

Unfortunately this cannot be generalized to the class of inverse semigroups. However, we can prove the following using Theorem 4.1.

Theorem 5.5 *Let S and T be inverse semigroups. If a word $w = x_1y_1x_2y_2 \cdots x_ny_n$ ($x_i \in S$, $y_i \in T$) represents an element in S or T in the amalgamated free product $S *_{A=B} T$, then $x_i \in A\omega$ or $y_i \in B\omega$ for some i.*

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