数理解析研究所講究録
1503 巻 2006 年 110-120

Core Equivalence for Economy under Rough Sets Information

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Abstract. We reconsider the core equivalence theorem for a pure exchange economy under uncertainty from a rough set theoretical point of view in data mining; the traders are assumed to have a multi-modal logic of belief and to make their decision under uncertainty represented by rough sets. We propose a generalized notion of rational expectations equilibrium, called expectation equilibrium in belief for the economy, and we show an extension of the core equivalence theorem of Aumann: An allocation in the economy is ex-post core if and only if it is an expectations equilibrium allocation in belief with respect to some price system.
Keywords. Belief, Rough set, Information, Core equivalence theorem, Exchange economy under uncertainty, Expectations equilibrium in belief, Ex-post core, Multi-modal logic, Data mining.

1 Introduction

The core equivalence theorem has shown that each competitive equilibrium for an economy can be characterized as there is no allocation that weakly dominates it. The theorem is shown by Aumann [1] for an exchange economy under complete information. This paper explores the extent to which the theorem is generalized in an economy under uncertainty. We highlight the rough set theoretical aspect of the information structure for traders which represent their uncertainty.

In recent years several investigators have already generalized theorems in an economy under complete information into an economy under asymmetric information (c.f., the papers cited in Forges et al [6]). E. Einy et al [4] succeeded in extending the theorem of Aumann [1] as the equivalence theorem between the ex-post core and the rational expectations equilibria for an economy under asymmetric information. Geanakoplos [7] neatly analyzes non-partition information structure with the introduction of a new concept, ***positive balancedness.*** With this concept, he examines several classes of non-partition information and the relations among them, and characterizes Nash equilibrium and rational expectations equilibrium in those classes.

In these researches on economy either with complete information or with incomplete information, the role of traders' knowledge (belief) remains obscured:[1] The economy has not been investigated from the epistemic point of view. Here this article aims to fill that gap. This paper discusses the core equivalence theorem with emphasis on the modal logical point of view, and captures different features from their analysis. Neither the reflexivity, the transitivity nor the positive balancedness is needed in the information structure. We focus the rough set theoretical aspect of the non-partitional information structure. Specifically, we seek the role of belief of traders in a pure exchange economy under uncertainty from a rough set theoretical point of view.

We propose the notion of *pure exchange economy* based on the multi-modal *logic of belief* **B**, by which the traders use making their decision, and we introduce the extended notion of equilibrium for the economy, called an *expectation equilibrium in belief.* We establish the ex-post core equivalence theorem for the economy:

Main theorem. *In a pure exchange economy under uncertainty, assume that the traders have the multi-modal logic* **B** *and they are risk averse. Then the ex-post core coincides with the set of all expectations equilibrium allocations in belief.*

This article is organized as follows: In Section 2 we present the multi-agent modal logic **B** and show the finite model property. Section 3 introduces the economy for logic **B**. We present the generalized notion of rough set associated with the logic **B**, and we illustrate the notion by a simple example. In Section 4 we present the notion of expectations equilibrium, which is a generalized notion of rational expectations equilibrium. We establishes the ex-post core equivalence theorem, giving an outline of the proof.

2 Multi-Modal Logics

2.1 Logic of belief

Let T be a set of *traders* and $t \in T$ a trader. The language is founded on as follows: The *sentences* of the language form the least set containing each *atomic* sentence $\mathbf{P}_m (m = 0, 1, 2, \dots)$ and closed under the following operations: Nullary operators for *falsity* $\bot$ and for *truth* $\top$; unary and binary syntactic operations for *negation* $\neg$, *conditionality* $\rightarrow$ and *conjunction* $\wedge$, *disjunction* $\vee$, respectively; two unary operations for *modalities* $\Box_t$, $\Diamond_t$ for $t \in T$. Other such operations are defined in terms of those in usual ways. The intended interpretation of $\Box_t\varphi$ is the sentence that 'trader t believes a sentence φ,' and the sentence $\Diamond_t\varphi$ is interpreted as the sentence that 'a sentence φ is possible for t.'

A *multi-modal logic* L is a set of sentences containing all truth-functional tautologies and closed under substitution and modus ponens. A multi-modal

[1] See the literatures cited in the survey of Forges et al [6] treated the topics and related works from the standard view points of economic theory.

logic L' is an *extension* of L if $L \subseteq L'$. A sentence φ in a modal logic L is a *theorem* of L, written by $\vdash_L \varphi$. Other proof-theoretical notions such as *L-deducibility, L-consistency, L-maximality* are defined in usual ways. (Chellas [3].)

A *normal system* of multi-modal logic L is a multi-modal logic containing the schema (Df$_\Diamond$) and is closed under the $2n$ rules (RK$_\Box$), (RK$_\Diamond$) of inference.

$$(\mathrm{Df}_\Diamond)\quad \Diamond_t\varphi \longleftrightarrow \neg\Box_t\neg\varphi; \qquad (\mathrm{RK}_\Box)\ \frac{(\varphi_1\wedge\varphi_2\wedge\cdots\wedge\varphi_k)\longrightarrow\psi}{(\Box_t\varphi_1\wedge\Box_t\varphi_2\wedge\cdots\wedge\Box_t\varphi_k)\longrightarrow\Box_t\psi}\quad (k \geq 0)$$

Definition 1. The *logic of belief* **B** is the minimal normal system of multi-modal logic.

2.2 Belief structure, model and truth

Let Ω be a non-empty set called a *state space* and 2^Ω the field of all subsets of Ω. Each member of 2^Ω is called an *event* and each element of Ω called a *state*. A *belief structure* is a tuple $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}\rangle$ in which Ω is a state space and $B_t : 2^\Omega \to 2^\Omega$ is trader t's *belief operator*. The interpretation of the event B_tE is that 't believes E.' P_t is t's *possibility operator* on 2^Ω defined by $P_tE = \Omega \setminus B_t(\Omega \setminus E)$ for every E in 2^Ω. The interpretation of P_tE is that 'E is possible for t.'

A *model on* a belief structure is a tuple $\mathcal{M} = \langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}, V\rangle$ in which $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}\rangle$ is a belief structure and a mapping V assigns either **true** or **false** to every $\omega \in \Omega$ and to every atomic sentence $\mathbf{P}_m$. The model $\mathcal{M}$ is called *finite* if Ω is a finite set.

Definition 2. By $\models^{\mathcal{M}}_{\omega} \varphi$, we mean that a sentence φ is *true* at a state ω in a model $\mathcal{M}$. *Truth at a state* ω in $\mathcal{M}$ is defined by the inductive way as follows:

1. $\models^{\mathcal{M}}_{\omega} \mathbf{P}_m$ if and only if $V(\omega, \mathbf{P}_m) = \mathbf{true}$, for $m = 0, 1, 2, \ldots$;
2. $\models^{\mathcal{M}}_{\omega} \top$, and not $\models^{\mathcal{M}}_{\omega} \bot$;
3. $\models^{\mathcal{M}}_{\omega} \neg\varphi$ if and only if not $\models^{\mathcal{M}}_{\omega} \varphi$;
4. $\models^{\mathcal{M}}_{\omega} \varphi \longrightarrow \psi$ if and only if $\models^{\mathcal{M}}_{\omega} \varphi$ implies $\models^{\mathcal{M}}_{\omega} \psi$;
5. $\models^{\mathcal{M}}_{\omega} \varphi \wedge \psi$ if and only if $\models^{\mathcal{M}}_{\omega} \varphi$ and $\models^{\mathcal{M}}_{\omega} \psi$;
6. $\models^{\mathcal{M}}_{\omega} \varphi \vee \psi$ if and only if $\models^{\mathcal{M}}_{\omega} \varphi$ or $\models^{\mathcal{M}}_{\omega} \psi$;
7. $\models^{\mathcal{M}}_{\omega} \Box_t\varphi$ if and only if $\omega \in B_t(||\varphi||^{\mathcal{M}})$, for $t \in T$;
8. $\models^{\mathcal{M}}_{\omega} \Diamond_t\varphi$ if and only if $\omega \in P_t(||\varphi||^{\mathcal{M}})$, for $t \in T$:

Where $||\varphi||^{\mathcal{M}}$ denotes the set of all the states in $\mathcal{M}$ at which φ is true; this is called the *truth set of* φ.

We say that a sentence φ is *true in the model* $\mathcal{M}$ and write $\models^{\mathcal{M}} \varphi$ if $\models^{\mathcal{M}}_{\omega} \varphi$ for every state ω in $\mathcal{M}$. A sentence is said to be *valid in* a belief structure if it is true in every model on the belief structure. Let Γ be a set of sentences. We say that $\mathcal{M}$ is a *model for* Γ if every member of Γ is true in $\mathcal{M}$. A belief structure is said to be *for* Γ if every member of Γ is valid in it. Let **C** be a class of models on a belief structure. A multi-modal logic L is *sound with respect to* **C** if every member of **C** is a model for L. It is *complete with respect to* **C** if every sentence valid in all members of **C** is a theorem of L. A multi-modal logic L is said to have the *finite model property* if it is sound and complete with respect to the class of all finite models in $\mathbf{C}$. We denote by $\mathbf{C}_F$ the class of all finite models in $\mathbf{C}$. We can establish that

Theorem 1. *A normal system of multi-modal logic L has the finite model property; i.e.,* $\vdash_L \varphi$ *if and only if* $\models^{\mathcal{M}} \varphi$ *for all* $\mathcal{M} \in \mathbf{C}_F$. *In particular,* $\mathbf{B}$ *has the finite model property.*

Proof. Will be given in the line described in Chellas [3].

3 Economy for Multi-Modal Logic

3.1 Illustrative example

Let us consider the following situation: Three traders, Company L, Company F and Company N are willing to buy and sell the L's stocks, the F's stocks and the N's stocks each other. Thus, there are three commodities L's stocks, F's stocks and N's stocks. L has made his own stock price high by Window-dressing in order to make his market capitalization high. But F and N have not realized L's dirty tricks, and it may come out that L commits the injustice.

We shall illustrate the situation as follows:

Example 1. Let Ω be the state space consisting of the two states $\{\omega_1, \omega_2\}$: The state ω_1 represents that L does not commit the injustice, the state ω_2 represents that L commits the injustice. So L can know which is the true state of either ω_1 or ω_2 occurs when each of the two states occurs. However traders F and N don't believe the state that L commits the injustice at all. Therefore, the traders L, F, N have their information functions, $P_{\mathrm{L}}(\omega) = \{\omega\}$ for $\omega = \omega_1, \omega_2$, $P_{\mathrm{F}}(\omega_1) = \{\omega_1, \omega_2\}$, $P_{\mathrm{F}}(\omega_2) = \{\omega_2\}$, $P_{\mathrm{F}}(\omega_1) = \{\omega_1, \omega_2\}$, $P_{\mathrm{F}}(\omega_2) = \{\omega_2\}$, $P_N = P_F$.

This illustration is very interesting from the view point of rough sets. Each trader t's information structure $P_t : \Omega \to 2^{\Omega}$ assigning to each state ω in a state space Ω the information set $P_t(\omega)$ that t possesses in ω entails the two operators on 2^{Ω}: t's belief operator $B_t : 2^{\Omega} \to 2^{\Omega}$ and t's possibility operator $P_t : 2^{\Omega} \to 2^{\Omega}$. These are defined by $B_t(E) = \{\omega \in \Omega \mid P_t(\omega) \subseteq E\}$ and $P_t(E) = \Omega \setminus B_t(\Omega \setminus E)$ respectively. According to the theory of rough sets, we call an event X *exact* if $B_t(X) = P_t(X)$, and call X *rough* if it is not exact.

We can observe that P_{L} represents that trader L has the complete information with which each component $P_{\mathrm{L}}(\omega)$ is an exact set, because $B_{\mathrm{L}}(P_{\mathrm{L}}(\omega)) = P_{\mathrm{L}}(P_{\mathrm{L}}(\omega))$. Each P_{F} or P_{N} represents that trader F and N have the incomplete information, with which each component $P_t(\omega)$ is a rough set for $t = \mathrm{F}, \mathrm{N}$. This suggests that the uncertainty of traders is modeled by rough sets.

The non-partition structure P_t with the rough sets components is equivalent to the belief operator B_t satisfying 'Truth' axiom $\mathbf{T}$: $B_t(E) \subseteq E$ (what is known is true). The partition structure P_{L} with the exact sets components is equivalent to B_{L} satisfying the two axioms $\mathbf{4}$ and $\mathbf{5}$ in addition to $\mathbf{T}$: The 'positive introspection' $\mathbf{4}$: $B_{\mathrm{L}}(E) \subseteq B_{\mathrm{L}}(B_{\mathrm{L}}(E))$ (we know what we do) and the 'negative introspection'**5**: $\Omega \setminus B_{\mathrm{L}}(E) \subseteq B_{\mathrm{L}}(\Omega \setminus B_{\mathrm{L}}(E))$ (we know what we do not know). [2] One of these requirements, symmetry (or the equivalent axiom **5**), is indeed so strong that describes the hyper-rationality of traders, and thus it is particularly objectionable.

The recent idea of *bounded rationality* suggests dropping such assumptions since real people are not complete reasoners. We weaken the conditions in the information partition the information partition structure (or the equivalent postulates of knowledge), and we shall investigate the essential roles of trader's information represented by rough sets in the results. This approach can potentially yield important results in a world with imperfectly Bayesian agents. The idea has been performed in different settings. However, all those researches have been lacked the logic that represents the traders' knowledge (or belief). In this article we present the economy upon the logic of belief, and we extend the theorem of Aumann [1] into the economy.

3.2 Information structure

We shall give the generalized notion of information partition in the line of Bacharach [2].

Definition 3. The *associated information structure* $(P_t)_{t\in T}$ with a model on a belief structure $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}, V\rangle$ for a normal system of multi-modal logic L is the class of t's *possibility operator* $P_t : 2^\Omega \to 2^\Omega$ defined by $P_t(E) = \Omega \setminus B_t(\Omega \setminus E)$. t's *associated information function* $P_t : \Omega \to 2^\Omega$ is defined by $P_t(\omega) = P_t(\{\omega\}) = \Omega \setminus B_t(\Omega \setminus \{\omega\}))$. We denote by $\mathrm{Dom}(P_t)$ the set $\{\omega \in \Omega \mid P_t(\omega) \neq \emptyset\}$, called the *domain of* P_t.

The information function $P_t : \Omega \to 2^\Omega$ is called *reflexive* if $\omega \in P_t(\omega)$ for every $\omega \in \mathrm{Dom}(P_t)$, and it is said to be *transitive* if $\xi \in P_t(\omega)$ implies $P_t(\xi) \subseteq P_t(\omega)$ for any $\xi, \omega \in \mathrm{Dom}(P_t)$. Furthermore P_t is called *symmetric* if $\xi \in P_t(\omega)$ implies $P_t(\xi) \ni \omega$ for any ω and $\xi \in \mathrm{Dom}(P_t)$. It is noted that the operators B_t is uniquely determined by the information structure P_t.

Definition 4. An event X is called *exact* if $B_t(X) = P_t(X)$, and X is called *rough* if it is not exact.

Remark 1. M. Bacharach [2] introduces the *strong epistemic model* equivalent to the Kripke semantics for the modal logic **S5**. The strong epistemic model can be interpreted as the belief structure $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}\rangle$ with B_t satisfying the schemas T, 4, and 5 in Chellas [3]. We can observe that i's information set at a state ω coincides the minimal event that i knows at ω; i.e., $P_t(\omega) = \bigcap_{E\in 2^\Omega}\{E \mid \omega \in B_t E\}$. This is the definition of information structure introduced by Bacharach [2].

We let turn into the economy under uncertainty presented in Example 1, and let the notations be the same in it. We shall illustrate the situation as follows:

[2] C.f.: Bacharach [2], Fagin, Halpern et al [5].

Example 2. Let Ω be the state space consisting of the two states $\{\omega_1, \omega_2\}$: The state ω_1 represents that L does not commit the injustice, the state ω_2 represents that L commits the injustice. So L can know which is the true state of either ω_1 or ω_2 occurs when each of the two states occurs. However traders F and N don't believe the state that L commits the injustice at all. Therefore, the belief operators $(B_t)_{t=\text{L},\text{F},\text{N}}$ can be modeled as follows:

$$B_{\text{L}}(E) = E \text{ for every } E \in 2^{\Omega},$$
$$B_{\text{F}}(\{\omega_1\}) = \{\omega_1\},\ B_{\text{F}}(\{\omega_2\}) = \emptyset,\ B_{\text{F}}(\emptyset) = \emptyset, B_{\text{F}}(\Omega) = \Omega,$$
$$B_{\text{N}}E = B_{\text{F}}E.$$

Then traders L, F, N have their information structure as follows:

$$P_{\text{L}}(E) = E \text{ for every } E \in 2^{\Omega},$$
$$P_{\text{F}}(\{\omega_1\}) = \Omega,\ P_{\text{F}}(\{\omega_2\}) = \{\omega_2\},\ P_{\text{F}}(\emptyset) = \emptyset,\ P_{\text{F}}(\Omega) = \Omega,$$
$$P_{\text{N}}E = P_{\text{F}}E.$$

We can observe that the information functions $P_t : \Omega \to 2^{\Omega}$ are: $P_{\text{L}}(\omega) = \{\omega\}$ for $\omega = \omega_1, \omega_2$, $P_{\text{F}}(\omega_1) = \{\omega_1, \omega_2\}$, $P_{\text{F}}(\omega_2) = \{\omega_2\}$, $P_{\text{F}}(\omega_1) = \{\omega_1, \omega_2\}$, $P_{\text{F}}(\omega_2) = \{\omega_2\}$. These coincide with the information function appeared in Example 1.

Remark 2. M. Bacharach [2] introduces the ***strong epistemic model*** equivalent to the Kripke semantics of the multi-modal logic **S5**.[3] The strong epistemic model is a tuple $\langle \Omega, (K_t)_{t\in T}\rangle$ in which t's *knowledge operator* $K_t : 2^{\Omega} \to 2^{\Omega}$ satisfies the postulates **K**, **T**, **4** and **5** with **N**: $K_t\Omega = \Omega$. t's associated information function P_t induced by K_t makes a partition of Ω, called t's ***information partition*** P_t, which is reflexive, transitive and symmetric. This is just the Kripke semantics corresponding to the logic **S5**; the postulates for P_t: reflexivity, transitivity and symmetry are respectively equivalent to the postulates **T**, **4** and **5** for K_t. The strong epistemic model can be interpreted as the belief structure $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}\rangle$ such that B_t is the knowledge operator. It should be noted that each $P_t(\omega)$ is an exact set in this model. Different approaches of knowledge and possibility are given in Fagin et al [5].

3.3 Economy on belief

A ***pure exchange economy under uncertainty*** is a tuple $\langle T, \Sigma, \mu, \Omega, \mathbf{e}, (U_t)_{t\in T}, (\pi_t)_{t\in T}\rangle$ consisting of the following structure and interpretations: There are l commodities in each state of the state space Ω , and it is assumed that Ω is *finite* and that the consumption set of trader t is $\mathbf{R}^l_+$; (T, Σ, μ) is the measure space of the traders, Σ is a σ-field of subsets of T whose elements are called *coalitions*, and μ is a measure on Σ; $\mathbf{e} : T \times \Omega \to \mathbf{R}^l_+$ is t's *initial endowment* such that $\mathbf{e}(\cdot, \omega)$ is μ-measurable for each $\omega \in \Omega$; $U_t : \mathbf{R}^l_+ \times \Omega \to \mathbb{R}$ is t's von-Neumann and Morgenstern utility function; π_t is a subjective prior on Ω for a trader $t \in T$.

[3] The logic **S5** is denoted by **KT5** (= **KT45**) in Chellas [3].

Let L be a normal system of multi-modal logic. A pure exchange *economy for* L is a structure $\mathcal{E}^L = \langle \mathcal{E}, (B_t)_{t\in T}, (P_t)_{t\in T}, V \rangle$, in which $\mathcal{E}$ is a pure exchange economy under uncertainty, and $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}, V \rangle$ is a finite model on a belief structure for L. By the *domain* of the economy $\mathcal{E}^L$ we mean $\mathrm{Dom}(\mathcal{E}^L) = \cap_{t\in T}\mathrm{Dom}(P_t)$. We always assume that $\mathrm{Dom}(\mathcal{E}^L) \neq \emptyset$, and that $\mathrm{Dom}(P_t) \subseteq \mathrm{Supp}(\pi_t)$ for all t.[4]

Definition 5. An *economy on belief* is the pure exchange economy for the logic **B**, denoted by $\mathcal{E}^{\mathbf{B}}$. The economy is called *atomless* if (T, Σ, μ) is a non-atomic measure space.

Remark 3. An economy under asymmetric information can be interpreted as the economy $\mathcal{E}^{S5}$ for the multi-modal logic **S5**, in which the belief structure $\langle \Omega, (B_t)_{t\in T}, (P_t)_{t\in T}, V \rangle$ is given by the strong epistemic model, and that $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}}) = \Omega$.

We denote by $\mathcal{F}_t$ the field of $\mathrm{Dom}(P_t)$ generated by $\{P_t(\omega) |\ \omega \in \Omega\}$ and denote by $\Pi_t(\omega)$ the atom containing $\omega \in \mathrm{Dom}(P_t)$. We denote by $\mathcal{F}$ the join of all $\mathcal{F}_t (t \in T)$ on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$; i.e. $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$, and denote by $\{\Pi(\omega) |\ \omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})\}$ the set of all atoms $\Pi(\omega)$ containing ω of the field $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$. We shall often refer to the following conditions for $\mathcal{E}^{\mathbf{B}}$: For every $t \in T$,

- **A-1** $\sum_{t\in T} \mathbf{e}(t,\omega) > 0$ for each $\omega \in \Omega$.
- **A-2** $\mathbf{e}(t,\cdot)$ is $\mathcal{F}$-measurable on $\mathrm{Dom}(P_t)$;
- **A-3** For each $x \in \mathbf{R}^l_+$, the function $U_t(x,\cdot)$ is at least $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$, and the function: $T \times \mathbf{R}^l_+ \to \mathbf{R}$, $(t,x) \to U_t(x,\omega)$ is $\Sigma \times \mathcal{B}$-measurable where $\mathcal{B}$ is the σ-field of all Borel subsets of $\mathbf{R}^l_+$.
- **A-4** For each $\omega \in \Omega$, the function $U_t(\cdot,\omega)$ is continuous, strictly increasing and quasi-concave on $\mathbf{R}^l_+$.

4 Core Equivalence Theorem

4.1 Expectations equilibrium in belief.

An *assignment* for an economy $\mathcal{E}^{\mathbf{B}}$ on belief is a mapping $\mathbf{x} : T \times \Omega \to \mathbf{R}^l_+$ such that for each $t \in T$, the function $\mathbf{x}(t,\cdot)$ is at least $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. We denote by $Ass(\mathcal{E}^{\mathbf{B}})$ the set of all assignments for the economy $\mathcal{E}^{\mathbf{B}}$. By an *allocation* for $\mathcal{E}^{\mathbf{B}}$ we mean an assignment $\mathbf{a}$ such that $\mathbf{a}(t,\cdot)$ is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$ for all $t \in T$ and $\sum_{t\in T} \mathbf{a}(t,\omega) \leq \sum_{t\in T} \mathbf{e}(t,\omega)$ for every $\omega \in \Omega$. We denote by $Alc(\mathcal{E}^{\mathbf{B}})$ the set of all allocations for $\mathcal{E}^{\mathbf{B}}$.

[4] By the support of π we mean $\mathrm{Supp}(\pi_t) := \{\omega \in \Omega \mid \pi_t(\omega) \neq 0\ \}$.

We shall introduce the revised notion of trader's expectation of utility in $\mathcal{E}^{\mathbf{B}}$. By t's *ex-ante* expectation we mean $\mathbf{E}_t[U_t(\mathbf{x}(t,\cdot)] := \sum_{\omega\in \mathrm{Dom}(P_t)} U_t(\mathbf{x}(t,\omega),\omega)\pi_t(\omega)$ for each $\mathbf{x} \in \mathcal{A}ss(\mathcal{E}^{\mathbf{B}})$. The *interim* expectation $\mathbf{E}_t[U_t(\mathbf{x}(t,\cdot)|P_t]$ is defined by

$$\mathbf{E}_t[U_t(\mathbf{x}(t,\cdot)|P_t](\omega) = \sum_{\xi\in \mathrm{Dom}(P_t)} U_t(\mathbf{x}(t,\xi),\xi)\pi_t(\{\xi\})|P_t(\omega))$$

on $\mathrm{Dom}(P_t)$.

A ***price system*** is a non-zero function $p : \Omega \to \mathbf{R}^l_+$ which is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. We denote by $\Delta(p)$ the partition on Ω induced by p, and denote by $\sigma(p)$ the field of Ω generated by $\Delta(p)$. The ***budget set*** of a trader t at a state ω for a price system p is defined by $B_t(\omega,p) := \{\ x \in \mathbf{R}^l_+ \mid p(\omega)\cdot x \leq p(\omega)\cdot \mathbf{e}(t,\omega)\ \}$. Define the mapping $\Delta(p)\cap P_t : \mathrm{Dom}(P_t) \to 2^{\Omega}$ by $(\Delta(p)\cap P_t)(\omega) := \Delta(p)(\omega)\cap P_t(\omega)$. We denote by $\mathrm{Dom}(\Delta(p)\cap P_t)$ the set of all states ω in which $\Delta(p)(\omega)\cap P_t(\omega) \neq \emptyset$. Let $\sigma(p)\vee \mathcal{F}_t$ be the smallest σ-field containing both the fields $\sigma(p)$ and $\mathcal{F}_t$.

Definition 6. An *expectations equilibrium in belief* for an economy $\mathcal{E}^{\mathbf{B}}$ on belief is a pair $(p,\mathbf{x})$, in which p is a price system and $\mathbf{x}$ is an assignment for $\mathcal{E}^{\mathbf{B}}$ satisfying the following conditions:

EB1 $\mathbf{x}$ is an allocation for $\mathcal{E}^{\mathbf{B}}$;
EB2 For all $t \in T$ and for every $\omega \in \Omega$, $\mathbf{x}(t,\omega) \in B_t(\omega,p)$;
EB3 For all $t \in T$, if $\mathbf{y}(t,\cdot) : \Omega \to \mathbf{R}^l_+$ is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$ with $\mathbf{y}(t,\omega) \in B_t(\omega,p)$ for all $\omega \in \Omega$, then

$$\mathbf{E}_t[U_t(\mathbf{x}(t,\cdot))|\Delta(p)\cap P_t](\omega) \geq \mathbf{E}_t[U_t(\mathbf{y}(t,\cdot))|\Delta(p)\cap P_t](\omega)$$

pointwise on $\mathrm{Dom}(\Delta(p)\cap P_t)$;
EB4 For every $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$, $\sum_{t\in T}\mathbf{x}(t,\omega) = \sum_{t\in T}\mathbf{e}(t,\omega)$.

The allocation $\mathbf{x}$ in $\mathcal{E}^{\mathbf{B}}$ is called an expectations equilibrium ***allocation*** in belief for $\mathcal{E}^{\mathbf{B}}$.

We denote by $EB(\mathcal{E}^{\mathbf{B}})$ the set of all the expectations equilibria of a pure exchange economy $\mathcal{E}^{\mathbf{B}}$, and denote by $\mathcal{A}(\mathcal{E}^{\mathbf{B}})$ the set of all the expectations equilibrium allocations in belief for the economy.

4.2 Ex-post core.

An assignment $\mathbf{y}$ is called an *ex-post improvement* of a coalition $S \in \Sigma$ on an assignment $\mathbf{x}$ at a state $\omega \in \Omega$ if

Imp1 $\mu(S) > 0$; **Imp2** $\int_S \mathbf{y}(t,\omega)d\mu \leq \int_S \mathbf{e}(t,\omega)d\mu$; and
Imp3 $U_t(\mathbf{y}(t,\omega),\omega) > U_t(\mathbf{x}(t,\omega),\omega)$ for almost all $t \in S$.

Definition 7. An allocation $\mathbf{x}$ is said to be an *ex-post core* allocation of an economy on belief $\mathcal{E}^{\mathbf{B}}$ if there is no coalition having an ex-post improvement on $\mathbf{x}$ at any state $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. The *ex-post core* denoted by $\mathcal{C}^{ExP}(\mathcal{E}^{\mathbf{B}})$ is the set of all the ex-post core allocations of $\mathcal{E}^{\mathbf{B}}$.

Example 3. We let turn into the situation in Example 2, and let the notations be the same in it. Now, suppose L, F, N have risk averse utilities:

$$U_{\mathrm{L}}(x,y,z;\omega) = x^{\frac{1}{10}}y^{\frac{1}{2}}z^{\frac{2}{5}} \text{ for every } \omega \in \Omega,$$
$$U_{\mathrm{F}}(x,y,z;\omega_1) = x^{\frac{3}{10}}y^{\frac{2}{5}}z^{\frac{3}{10}},\ U_{\mathrm{F}}(x,y,z;\omega_2) = x^{\frac{1}{10}}y^{\frac{1}{2}}z^{\frac{2}{5}},$$
$$U_{\mathrm{N}}(x,y,z;\omega_1) = x^{\frac{3}{10}}y^{\frac{3}{10}}z^{\frac{2}{5}},\ U_{\mathrm{N}}(x,y,z;\omega_2) = x^{\frac{1}{10}}y^{\frac{2}{5}}z^{\frac{1}{2}}.$$

In the economy, equilibrium price $p(\omega) = (p_1, p_2, p_3)$ is given by: For $\omega = \omega_1$, $(p_1/p_2) = 13053/19760$, $(p_2/p_3) = 1040/899$, and for $\omega = \omega_2$, $(p_1/p_2) = 153/732$, $(p_2/p_3) = 244/215$. The expectations equilibrium allocation $x(t,\omega)$ is given by

$$\begin{aligned}
\boldsymbol{x}(\mathrm{L},\omega) &= (w_{\mathrm{L}}/10p_1, w_{\mathrm{L}}/2p_2, 2w_{\mathrm{L}}/5p_3),\\
\boldsymbol{x}(\mathrm{F},\omega_1) &= (3w_{\mathrm{F}}/10p_1, 2w_{\mathrm{F}}/5p_2, 3w_{\mathrm{F}}/10p_3),\\
\boldsymbol{x}(\mathrm{F},\omega_2) &= (w_{\mathrm{F}}/10p_1, w_{\mathrm{F}}/2p_2, 2w_{\mathrm{F}}/5p_3),\\
\boldsymbol{x}(\mathrm{N},\omega_1) &= (3w_{\mathrm{N}}/10p_1, 3w_{\mathrm{N}}/10p_2, 2w_{\mathrm{N}}/5p_3),\\
\boldsymbol{x}(\mathrm{N},\omega_2) &= (w_{\mathrm{N}}/10p_1, 2w_{\mathrm{N}}/5p_2, w_{\mathrm{N}}/2p_3),
\end{aligned}$$

We note that the equilibrium allocation $\mathbf{x}$ is ex-ante Pareto optimal, and is an ex-post core. Furthermore, the converse is also true.

4.3 Proof of main theorem

Now we can explicitly state the main theorem, and we shall sketch the proof.

Theorem 2. *Let $\mathcal{E}^{\mathbf{B}}$ be a pure exchange economy with belief structure satisfying the conditions* **A-1**, **A-2**, **A-3** *and* **A-4**. *Suppose that the economy is atomless. Then the ex-post core coincides with the set of all expectations equilibrium allocations in belief; i.e., $\mathcal{C}^{ExP}(\mathcal{E}^{\mathbf{B}}) = \mathcal{A}(\mathcal{E}^{\mathbf{B}})$.*

Let $\mathcal{E}^{\mathbf{B}}(\omega)$ be the economy with complete information $\langle T, \Sigma, \mu, \mathbf{e}(\cdot,\omega), (U_t(\cdot,\omega))_{t\in T}\rangle$ for each $\omega \in \Omega$. We denote by $\mathcal{C}(\mathcal{E}^{\mathbf{B}}(\omega))$ the set of all core allocations for $\mathcal{E}^{\mathbf{B}}(\omega)$, and by $\mathcal{W}(\mathcal{E}^{\mathbf{B}}(\omega))$ the set of all competitive equilibria for $\mathcal{E}^{\mathbf{B}}(\omega)$.

Proposition 1. *Notations being the same as above, we obtain*

(*i*) $\mathcal{C}^{ExP}(\mathcal{E}^{\mathbf{B}}) = \{\mathbf{x} \in \mathcal{A}lc(\mathcal{E}^{\mathbf{B}}) \mid \mathbf{x}(\cdot,\omega)) \in \mathcal{C}(\mathcal{E}^{\mathbf{B}}(\omega))$ *for all* $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})\}$.

(*ii*) $\mathcal{A}(\mathcal{E}^{\mathbf{B}}) = \{\mathbf{x} \in \mathcal{A}lc(\mathcal{E}^{\mathbf{B}}) \mid$ *There is a price system* p *such that* $(p(\omega), \mathbf{x}(\cdot,\omega)) \in W(\mathcal{E}^{\mathbf{B}}(\omega))$ *for all* $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})\}$.

Before proceeding with the proof we should note that:

Lemma 1. *Let $\mathcal{E}^{\mathbf{B}}$ be the same in Proposition 1. For every $t \in T$ and for every $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$, the event $(\Delta(p) \cap P_t)(\omega)$ has the decomposition into the disjoint union $(\Delta(p) \cap P_t)(\omega) = \cup_{k=1}^{p} \Pi(\xi_k)$. Moreover, if $\mathbf{x}$ is an assignment for $\mathcal{E}^{\mathbf{B}}$ then*

$$\mathbf{E}_t[U_t(\mathbf{x}(t,\cdot))|\Delta(p) \cap P_t](\omega) = \sum_{k=1}^{p} \frac{\pi_t(\Pi(\xi_k))}{\pi_t((\Delta(p) \cap P_t)(\omega))} U_t(\mathbf{x}(t,\xi_k),\xi_k). \tag{1}$$

$\sqcup$

Proof of Proposition 1. (i) Is given by making modifications to the proof in Theorem 3.1 in Einy et al [4].
(ii) Let $\mathbf{x} \in \mathcal{A}(\mathcal{E}^{\mathbf{B}})$ and $(p, \mathbf{x}) \in EB(\mathcal{E}^{\mathbf{B}})$. We shall show that $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^{\mathbf{B}}(\omega))$ for any $\omega \in \Omega$: Suppose to the contrary that there exist a state $\omega_0 \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$ and non-null set $S \subseteq T$ with the property: For each $s \in S$ there is an $\mathbf{a}(s, \omega_0) \in B_s(\omega_0, p)$ such that $U_s(\mathbf{a}(s, \omega_0), \omega_0) > U_s(\mathbf{x}(s, \omega_0), \omega_0)$. Define the function $\mathbf{y} : T \times \Omega \to \mathbf{R}^l_+$ by: $\mathbf{y}(t, \xi) := \mathbf{a}(t, \omega_0)$ for $\xi \in \Pi(\omega_0)$, $\mathbf{y}(t, \xi) := \mathbf{x}(t, \xi)$ otherwise. On noting that $\mathbf{y}(t, \cdot)$ is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$ and π_t is full support, we can obtain by Eq (1) that for all $s \in S$, $\mathbf{E}_s[U_s(\mathbf{x}(s, \cdot))|\Delta(p) \cap P_s](\omega_0) < \mathbf{E}_s[U_s(\mathbf{y}(s, \cdot))|\Delta(p) \cap P_s](\omega_0)$, contrary to $(p, \mathbf{x}) \in EB(\mathcal{E}^{\mathbf{B}})$.

The converse will be shown as follows: Let $\mathbf{x} \in \mathcal{A}ss(\mathcal{E}^{\mathbf{B}})$ with $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^{\mathbf{B}}(\omega))$ for all $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. Define the price system $p^* : \Omega \to \mathbf{R}^l_+$ by $p^*(\xi) := p(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$, $p^*(\xi) := p(\omega)$ for $\omega \notin \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. We can observe that $(p^*, \mathbf{x}) \in EB(\mathcal{E}^{\mathbf{B}})$: For **EB3**. Let $\mathbf{y}(t, \cdot) : \Omega \to \mathbf{R}^l_+$ be an $\mathcal{F}$-measurable function with $\mathbf{y}(t, \omega) \in B_t(\omega, p^*)$ for all $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. Since $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^{\mathbf{B}}(\omega))$ it follows that $U_t(\mathbf{x}(t, \omega), \omega) \geq U_t(\mathbf{y}(t, \omega), \omega)$ for almost all $t \in T$. Therefore by Eq (1), $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))|\Delta(p^*) \cap P_t](\omega) \geq \mathbf{E}_t[U_t(\mathbf{y}(t, \cdot))|\Delta(p^*) \cap P_t](\omega)$, as required. The other conditions in Definition 6 can be easily verified. We note by the core equivalence theorem of Aumann [1] that $\mathcal{C}(\mathcal{E}^{\mathbf{B}}(\omega)) = W(\mathcal{E}^{\mathbf{B}}(\omega))$ for each $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. ⊔

Proof of Theorem 2. Let $\mathbf{x} \in \mathcal{A}(\mathcal{E}^{\mathbf{B}})$. By Proposition 1 (ii) we obtain that for each $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$, $(p(\omega), \mathbf{x}(\cdot, \omega)) \in W(\mathcal{E}^{\mathbf{B}}(\omega))$, and thus it follows from the theorem of Aumann [1] that $\mathbf{x}(\cdot, \omega)) \in \mathcal{C}(\mathcal{E}^{\mathbf{B}}(\omega))$ for any $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. It has been verified that $\mathcal{A}(\mathcal{E}^{\mathbf{B}}) \subseteq \mathcal{C}^{ExP}(\mathcal{E}^{\mathbf{B}})$.

The converse shall be shown as follows: Let $\mathbf{x} \in \mathcal{C}^{ExP}(\mathcal{E}^{\mathbf{B}})$. It follows from Proposition 1 (i) that $\mathbf{x}(\cdot, \omega) \in \mathcal{C}(\mathcal{E}^{\mathbf{B}}(\omega))$ for every $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. By the above theorem of Aumann [1] there is a $p^*(\omega) \in \mathbf{R}^l_+$ such that $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in W(\mathcal{E}^{\mathbf{B}}(\omega))$. Let p be the price system defined by $p(\xi) := p^*(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$, $p(\omega) := p^*(\omega)$ for $\omega \notin \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. we obtain that $(p(\omega), \mathbf{x}(\cdot, \omega)) \in W(\mathcal{E}^{\mathbf{B}}(\omega))$ for every $\omega \in \mathrm{Dom}(\mathcal{E}^{\mathbf{B}})$. By Proposition 1 (ii), we have observed that $\mathcal{C}^{ExP}(\mathcal{E}^{\mathbf{B}}) \subseteq \mathcal{A}(\mathcal{E}^{\mathbf{B}})$. ⊔

References

1. R. J. Aumann: Markets with a continuum of traders, Econometrica 32 (1964) 39–50.
2. M. O. Bacharach: Some extensions of a claim of Aumann in an axiomatic model of knowledge, Journal of Economic Theory 37 (1985) 167–190.
3. B. F. Chellas: *Modal Logic: An introduction.* Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1980.
4. E. Einy, D. Moreno, and B. Shitovitz: Rational expectations equilibria and the ex-post core of an economy with asymmetric information, Journal of Mathematical Economics 34 (2000) 527–535.
5. R. Fagin, J.Y. Halpern, Y Moses and M.Y. Vardi: *Reasoning about Knowledge.* The MIT Press, Cambridge, Massachusetts, London, England, 1995.

6. F. Forges, E. Minelli, and R. Vohra: Incentive and core of an exchange economy: a survey, Journal of Mathematical Economics 38 (2002) 1–41.
7. J. Geanakoplos: Game theory without partitions, and applications to speculation and consensus. Cowles Foundation Discussion Paper No.914, Yale University, 1989. (Available in http://cowles.econ.yale.edu)