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Circular Codes and Petri Nets

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Abstract

The purpose of this paper is to investigate the relationship between limited codes and Petri nets. For a given Petri net with an initial marking μ, we can naturally define an automaton $\mathcal{A}$ which has the initial marking μ as an initial state, the reachability set $Re(\mu)$ as a set of states, and the set of transitions as a set of inputs. We can define prefix codes by considering the set of firing sequences which arrive from the positive initial marking of a Petri net to a certain subset of the reachability set[10,12]. The set M of all positive firing sequences which start from the positive initial marking μ of a Petri net and reach μ itself forms a pure monoid. Our main interest is in the base D of M. The family of pure monoids contains the family of very pure monoids, and the base of a very pure monoid is a circular code. Therefore, we can expect that D may be a circular code. Here, for "small" Petri nets, we discuss under what conditions D is circular.

Key words: Petri net, Code, Prefix code, Circular code, Limited code.

1. Introduction

Let A be an alphabet, A^* the free monoid over A, and 1 the empty word. A word $v \in A^*$ is a *left factor* of a word $u \in A^*$ if there is a word $w \in A^*$ such that $u = vw$. The left factor v of u is called *proper* if $v \neq u$. A right factor and a proper right factor of a word are defined in a symmetric manner.

For a word $w \in A^*$ and a letter $x \in A$ we let $|w|_x$ denote the number of x in w. The length of w is the number of letters in w. A non-empty subset C of A^+ is said to be a *code* if for $x_1, ..., x_p, y_1, ..., y_q \in C$, $p, q \geq 1$,

$$x_1 \cdots x_p = y_1 \cdots y_q \text{ implies } p=q \text{ and } x_1=y_1, \ldots, x_p=y_p.$$

A subset M of A^* is a *submonoid* of A^* if $M^2 \subseteq M$ and $1 \in M$. Every submonoid M of a free monoid has a unique minimal set of generators

$$C = (M - \{1\}) - (M - \{1\})^2.$$

C is called the *base* of M.

This is the abstract and the details will be published elsewhere.

A submonoid M is *right unitary* in A^* if for all $u, v \in A^*$,

$$u,\ uv \in M \Longrightarrow v \in M.$$

M is called *left unitary* in A^* if it satisfies the dual condition. A submonoid M is *biunitary* if it is both left and right unitary.

Definition 1.1. Let M be a submonoid of a free monoid A^*, and C its base. If $CA^+ \cap C = \emptyset$, (resp. $A^+C \cap C = \emptyset$), then C is called a $prefix$ (resp. $suffix$) code over A. C is called a $bifix$ code if it is a prefix and suffix code.
A submonoid M of A^* is right unitary (resp. biunitary) if and only if its minimal set of generator is a prefix code (bifix code)([1,p.46]).

Definition 1.2. A Petri net is a 5-tuple, $PN = (P, A, F, W, \mu_0)$ where:
$P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places,
$A = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions,
$F \subseteq (P \times A) \cup (A \times P)$ is a set of arcs,
$W : F \to \{1, 2, \ldots\}$ is a weight function,
$\mu_0 : P \to \{0, 1, 2, \ldots\}$ is the initial marking,
$P \cap A = \emptyset$ and $P \cup A \neq \emptyset$.

We use the following notations for a pre-set and a post-set:
$\cdot t = \{p | (p, t) \in F\}$, $t\cdot = \{p | (t, p) \in F\}$,

In this paper we shall assume that a Petri net has no isolated transitions, i.e., no t such that $\cdot t \cup t\cdot = \emptyset$. A marking μ_0 can be represented by a vector:

$$\mu_0 = (\mu_0(p_1), \mu_0(p_2), \ldots, \mu_0(p_n)),\ p_i \in P,\ n = |P|.$$

For every $t \in A$ the vector Δt is defined by

$$\Delta t = (\Delta t(p_1), \Delta t(p_2), \ldots, \Delta t(p_n)),\ n = |P|,$$

where

$$\Delta t(p) = \begin{cases} -W(p,t) + W(t,p) & if\ p \in \cdot t \cap t\cdot, \\ -W(p,t) & if\ p \in \cdot t - t\cdot, \\ W(t,p) & if\ p \in t\cdot - \cdot t, \\ 0 & if\ p \notin \cdot t \cup t\cdot. \end{cases}$$

A transition $t \in A$ is said to be *enabled* in μ_0, if $W(p, t) \leq \mu_0(p)$ for all $p \in \cdot t$. A firing of an enabled transition t removes $W(p_1, t)$ tokens from each input place $p_1 \in \cdot t$, and adds $W(t, p_2)$ tokens to each output place $p_2 \in t\cdot$. Firing of an enabled transition t at μ_0 produces a new marking μ_1 such that

$$\mu_1(p) = \begin{cases} \mu_0(p) - W(p,t) & if\ p \in \cdot t - t \cdot, \\ \mu_0(p) + W(t,p) & if\ p \in t \cdot - \cdot t, \\ \mu_0(p) - W(p,t) + W(t,p) & if\ p \in t \cdot \cap \cdot t, \\ \mu_0(p) & otherwise. \end{cases}$$

If we obtain the marking μ' that results from a firing of t at μ, we write $\delta(\mu, t) = \mu'$. A word $w = t_1 t_2 \ldots t_r$, $(t_i \in A)$, of transitions is said to be a *(firing) sequence* from μ_0 if there exist markings $\mu_i, 1 \leq i \leq r$, such that $\delta(\mu_{i-1}, t_i) = \mu_i$ for all i, $(1 \leq i \leq r)$. In this case, μ_r is reachable from μ_0 by w and we write $\delta(\mu_0, w) = \mu_r$. The set of all possible markings reachable from μ_0 is denoted by $Re(\mu_0)$, and the set of all possible sequences from μ_0 is denoted by $Seq(\mu_0)$. The function $\delta : Re(\mu_0) \times T \to Re(\mu_0)$ is called a next-state function of a Petri net PN [7.p.23]. We note that the above condition for $r = 0$ is understood to be $\mu_0 \in Re(\mu_0)$.

A marking μ is said to be *positive* if $\mu(p) > 0$ for all $p \in P$. A sequence $t_1 t_2 \ldots t_n \in Seq(\mu_0)$, $(t_i \in T)$, is called a positive sequence from μ_0 if $\delta(\mu_0, t_1 t_2 \ldots t_i)$ is positive for all i, $(1 \leq i \leq n)$. The set of all positive sequences from μ_0 is denoted by $PSeq(\mu_0)$.
By $PRe(\mu_0)$ we denote the set of all possible positive markings reachable from μ_0;
$PRe(\mu_0) = \{\delta(\mu_0, w) | w \in PSeq(\mu_0)\}$.

2. Some codes related to Petri nets

For a Petri net $PN = (P, T, F, W, \mu_0)$ and a subset $X \subseteq Re(\mu_0)$ we can define a deterministic automaton $A(PN)$ as follows: $Re(\mu_0)$, T, $\delta : Re(\mu_0) \times T \to Re(\mu_0)$, μ_0, and X, are regarded as a state set, an input set, a next-state function, an initial state, and a final set of $A(PN)$, respectively. By using such automata, in [10,12] we defined four kinds of prefix codes and examined fundamental properties of these codes.

Let $PN = (P, A, F, W, \mu)$ be a Petri net. The set

$$Stab(PN) = \{w \mid w \in Seq(\mu) \ and \ \delta(\mu, w) = \mu\}$$

forms a submonoid of A^*. If $Stab(PN) \neq \{1\}$, then we denote the base of $Stab(PN)$ by $S(PN)$. Since $S(PN)A^+ \cap S(PN) = \emptyset$, $S(PN)$ is a prefix code over A.

A submonoid M of A^* is called *pure* [7] if for all $x \in A^*$ and $n \geq 1$,

$$x^n \in M \Longrightarrow x \in M.$$

A subsemigroup H of a semigroup S is *extractable* in S [9, p.191] if

$$x, y \in S, z \in H, xzy \in H \Longrightarrow xy \in H.$$

Proposition 2.1. If $Stab(PN) \neq \emptyset$, then $Stab(PN)$ is a biunitary extractable pure monoid.

Definition 2.1. Let $PN = (P, A, F, W, \mu)$ be a Petri net with a positive marking μ. Define the subset $D(PN)$ as a set of all positive sequence w of $S(PN)$.

Since $D(PN)$ is a subset of $S(PN)$, $D(PN)$ is a bifix code over A.

Proposition 2.2. If $D(PN) \neq \emptyset$, then $D(PN)^*$ is a biunitary extractable pure monoid.

Example 2.1. Let $PN = (\{p,q\}, \{a,b\}, F, W, \mu_0)$ be a Petri net defined by $W(a,p) = W(p,b) = W(q,a) = W(b,q) = 1$, $\mu_0(p) = \mu_0(q) = 2$. Then $D(PN) = \{ab, ba\}$, therefore $\{ab, ba\}^*$ is pure [1, p.324, Ex.1.3].

Proposition 2.3. If $z, xzy \in D(PN), x, y \in A^+$, then $xz^*y \in D(PN)$.

A code D is *infix* if $w, xwy \in D$ imples $x = y = 1$ [8, p.129].

Proposition 2.4. If $D(PN)$ is a non-empty finite set, then $D(PN)$ is a infix code.

3. Limited code

A submonoid M of A^* is *very pure* if for all $u, v \in A^*$,

$$u, v \in A^*, uv, vu \in M \Rightarrow u, v \in M.$$

The base of a very pure monoid is called a *circular code.*

Let $p, q \geq 0$ be two integers. If for any sequence $u_0, u_1, ..., u_{p+q}$ of words in A^*, the assumptions $u_{i-1}u_i \in M$ $(1 \leq i \leq p+q)$ imply $u_p \in M$, then a submonoid M is said to satisfy condition $C(p,q)$. If a submonoid M of A^* satisfies condition $C(p,q)$, then M is very pure [1, p.329, Proposition 2.1], and its base is called a (p,q)-*limited* code.

If a subset D of A^* is a bifix (1,1)-limited code, then for any $u_0, u_1, u_2 \in A^*$ such that $u_0u_1, u_1u_2 \in D$ we have $u_1 \in D$. Thus $u_0u_1, u_1, u_1u_2 \in D$. This imples that $u_0, u_1, u_2 \in D$, since D is bifix. Therefore D is (2,0)-,(1,1)- and (0,2)-limited.

Let $PN_0 = (\{p\}, \{a,b\}, F, W, \mu_0)$ be a Petri net such that $W(a,p) = \alpha, W(p,b) = \beta, \mu_0 = (\lambda_p), \lambda_p > 0$.

Consider the set Ω of positive markings in PN_0;

$$\Omega = \{\mu \,|\, \mu = \mu_0 + \Delta(w), w \in PSeq(\mu_0\}.$$

α and β, and let $N = \{0, 1, 2, \cdots\}$ be a set of non-negative integers. Then we have
(0) $D(PN_0)$ is dense.
(1) If $\lambda_p < g$, then $\Omega = \{\lambda_p + ng \,|\, n \in N\}$.
(2) If $\lambda_p = sg, s \geq 0, s \in N$, then $\Omega = \{ng \,|\, n \geq 1, n \in N\}$.
(3) If $\lambda_p = sg + t_p, s \geq 0, s \in N, 0 < t_p < g$, then $\Omega = \{t_p + ng \,|\, n \geq 0, n \in N\}$.

Proposition 3.1. If $\lambda_p > gcd(\alpha, \beta)$, then $D(PN_0)$ is not circular.

Proposition 3.2. $D(PN_0)$ is circular if and only if $\lambda_p \leq gcd(\alpha, \beta)$.

Let $PN_1 = (\{p,q\}, \{a,b\}, F, W, \mu_0)$ be a Petri net such that $W(a,p) = \alpha, W(p,b) = \alpha', W(q,a) = \beta, W(b,q) = \beta'$, $\mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q$.

Supose that $D(PN_1) \neq \emptyset$ and $w \in D(PN_1)$. Let $n = |w|_a$ and $m = |w|_b$, then $\Delta(w) = n\Delta(a) + m\Delta(b) = 0$ (zero vector). Consequently the linear equation

$$\begin{pmatrix} \alpha & -\alpha' \\ -\beta & \beta' \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a non-trivial solution in N. Thus $\alpha\beta' = \alpha'\beta$. Threrefore, if $D(PN_1) \neq \emptyset$, then $PN_1 = (\{p,q\}, \{a,b\}, F, W, \mu_0)$ has the following forms:
$W(a,p) = \alpha, W(p,b) = k\alpha, W(q,a) = \beta, W(b,q) = k\beta$, $k > 0$.
Here we assume that k is an integer. That is, we define a Petri net $PN_1 = (\{p,q\}, \{a,b\}, F, W, \mu_0)$ as follows

$$\Delta(a) = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}, \quad \Delta(b) = \begin{pmatrix} -k\alpha \\ k\beta \end{pmatrix},$$

where k is a positive integer.

We define an integer M_p as follows

$$M_p = \begin{cases} \dfrac{\lambda_p}{\alpha} - 1, & if\ \dfrac{\lambda_p}{\alpha}\ is\ an\ integer, \\[2ex] [\dfrac{\lambda_p}{\alpha}], & if\ \dfrac{\lambda_p}{\alpha}\ is\ \ not\ an\ integer. \end{cases}$$

where [ ] is the symbol of Gauss. Similarly we difine an integer M_q as follows, $M_q = \frac{\lambda_q}{\beta} - 1$ if $\frac{\lambda_q}{\beta}$ is an integer, and $M_q = [\frac{\lambda_p}{\beta}]$ if $\frac{\lambda_q}{\beta}$ is not an integer.

Proposition 3.3. We have
(1) If $M_p + M_q > k, M_p \geq k$ and $M_q \geq 1$, then $D(PN_1)$ is not circular.
(2) If $M_p + M_q > k, k > M_p \geq 1, M_q > 1, M_p + M_q > k$, then $D(PN_1)$ is not circular.
(3) If $M_p + M_q = k, M_p \geq 1, M_q \geq 1$, then $D(PN_1)$ is a singleton.
(4) If $M_p + M_q \geq k, M_p = 0, M_q \geq k$, then $D(PN_1)$ is (1,1)-limited.
(5) If $M_p + M_q \geq k, M_p \geq k, M_q = 0$, then $D(PN_1)$ is (1,1)-limited.

Corollary 3.1. Let n and k be arbitrary integers such that $n > k > 1$. Define the automaton

$$\mathcal{A}_{(n,k)} = (\{1, 2, \cdots, n\}, \{a, b\}, f, 1, \{1\})$$

by $f(i, a) = i + 1, 1 \leq i \leq n - 1, \quad f(j, b) = j - k, k + 1 \leq j \leq n$. Then the base of language $L(\mathcal{A}_{(n,k)})$ recognized by $\mathcal{A}_{(n,k)}$ is a (1,1)-limited code.

Proposition 3.4. Let $PN = (\{p_1, \cdots, p_n\}, \{a_1, \cdots, a_n\}, F, W, \mu_0), n \geq 2$, be a Petri net such that $W(p_i, a_i) = \alpha_i, W(a_i, p_{i+1}) = \beta_i, 1 \leq i \leq n - 1$, and $W(p_n, a_n) = \alpha_n, W(a_n, p_1) = \beta_n$. $\mu_0 = (\lambda_1, \cdots, \lambda_n), \mu_0(p_i) = \lambda_i, 1 \leq i \leq n$. Furthermore let $g_j = gcd(\beta_{j-1}, \alpha_j), 2 \leq j \leq n$. If $\lambda_1/\alpha_1 > 1$ and $\lambda_i \leq g_i$ for all $i = 2, \cdots, n$, then $D(PN)$ is (1,1)-limited.

Let $PN_2 = (\{p_1, p_2\}, \{a, b, c\}, F, W, \mu_0)$ be a Petri net such that
$W(a, p_1) = \alpha_1, W(p_1, b) = \alpha_2, W(b, p_2) = \beta_1, W(p_1, c) = \alpha_3, W(p_2, c) = \beta_2,$
$\mu_0(p_1) = \lambda_1, \mu_0(p_2) = \lambda_2$.

Lemma 3.1. Let PN_2 be a Petri net mentioned above, and let $\alpha = gcd(\alpha_1, \alpha_2, \alpha_3), \beta = gcd(\beta_1, \beta_2)$. Suppose that $D(PN_2) \neq \emptyset$ and $\lambda_1 \leq \alpha, \lambda_2 \leq \beta$. If $d \in D(PN_2)$ and v is its proper suffix, then we have one of the following:
(1) $\Delta(v)(p_1) \leq -\alpha, \ \Delta(v)(p_2) \leq -\beta$.
(2) $\Delta(v)(p_1) = 0, \ \Delta(v)(p_2) \leq -\beta$.
(3) $\Delta(v)(p_1) \leq -\alpha, \ \Delta(v)(p_2) \leq 0$.

Proposition 3.5. If $D(PN_2) \neq \emptyset$ and $\lambda_1 \leq \alpha, \lambda_2 \leq \beta$, then $D(PN_2)$ is (1,1)-limited.

Let $PN_3 = (\{p, q\}, \{a, b, c\}, W, \mu_0)$ be a Petri net such that $W(a, p) = \alpha, W(q, a) = \beta, W(p, b) =$

$\alpha+\beta, W(b,q)=\alpha+\beta, W(c,p)=\beta, W(q,c)=\alpha,\ \mu_0(p)=\lambda_p, \mu_0(q)=\lambda_q.$

Lemma 3.2. Let PN_3 be a Petri net mentioned above. Suppose that $\beta<\lambda_p\le\alpha+\beta$ and $\beta<\lambda_q\le\alpha$, then for any $u\in PSeq(PN_3)$ we have one of the following.

(1) $\Delta(u)=\begin{pmatrix} k(\alpha-\beta) \\ k(\alpha-\beta)\end{pmatrix}$, $k\ge 0$, (2) $\Delta(u)=\begin{pmatrix} k(\alpha-\beta)+l\alpha \\ k(\alpha-\beta)-l\beta\end{pmatrix}$, $k\ge 0,\ l\ge 1$,

(3) $\Delta(u)=\begin{pmatrix} k(\alpha-\beta)-l\beta \\ k(\alpha-\beta)+l\alpha\end{pmatrix}$, $k\ge 0,\ l\ge 1$.

Proposition 3.6. Suppose that $D(PN_3)\neq\emptyset$. If $\beta<\lambda_p\le\alpha+\beta$ and $\beta<\lambda_q\le\alpha$, then $D(PN_3)$ is (1,1)-limited.

References

[1] Berstel, J., and Perrin. D. Theory of Codes. Academic Press, 1985

[2] J.Hopcroft., and J. Ullman. Introduction to Automata Theory, Languages and Computation. Reading, Mass., Addison-Wesley, 1979.

[3] Ito,M., and Kunimochi,Y. "Some Petri Net Languages and Codes," Lecture Note in Computer Science 2295,(2002) pp.69-80.

[4] Lallement,G. Semigroup and Combinatorial Applications. Wiley. 1979.

[5] Murata, T. "Petri Nets:Properties,Analysis and Applications." Proceeding of the IEEE, Vol.77, No.4.(1989) pp.541-580.

[6] Peterson J. L. Petri Net Theory and the Modeling of Systems, Prentice Hall, New Jersey, 1981.

[7] Restivo, A. "On a Question of McNaughton and Papert," Information and Control 25,(1974) pp.93-101.

[8] Shyr, H. J. Free monoids and languages, Hon Min Book Company, Taichung, Taiwan, 1991.

[9] Tamura,T. Theory of Semigroups, Kyoritu-Shuppan, Tokyo,1972. (In Japanese)

[10] Tanaka, G. "Prefix codes and Petri nets," The Bulletin of the Shizuoka Institute of Science and Technology, Vol.4. (1995) pp.99-120. (In Japanese)

[11] Tanaka, G. "Prefix codes associated with input ordinal Petri nets," The Bulletin of the Shizuoka Institute of Science and Technology, Vol.5. (1996) pp.83-97. (In Japanese)

[12] Tanaka, G. "Prefix codes determined by Petri nets," Algebra Colloquium 5:3, (1998) pp.255-264.